EconS 503 - Microeconomic Theory II
Homework #4 - Answer key

1. **Exercises from Tadelis:** Exercises 7 and 9 from Chapter 8.

   - See scanned answer keys at the end of this handout.

2. **Strategic pre-commitment.** Consider the following sequential-move game. In the first stage, every firm $i = \{1, 2\}$ chooses its investment in cost-reducing technologies, k_i. This investment decreases its marginal production cost from c_0 to $c_0 - \alpha k_i$, where $\alpha \in [0, 1]$ denotes the effectiveness of the investment. Intuitively, when $\alpha = 0$, investment is futile but when $\alpha = 1$ every dollar invested reduces the initial marginal cost of the firm by one dollar. In the second stage, both firms observe the investment profile (k_i, k_j) selected in the previous stage and then compete a la Cournot. Firms face inverse demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ represents aggregate output.

   (a) *Second stage.* Find the best response function in the second period, $q_i(q_j)$.

   - In the second stage, every firm i solves

     $$\max_{q_i} \ (1 - q_i - q_j)q_i - (c - \alpha k_i)\, q_i$$

     which is evaluated at its second-period cost $c - \alpha k_i$, after investing k_i dollars in cost-reducing technologies. Differentiating with respect to q_i, we obtain

     $$1 - 2q_i - q_j - (c - \alpha k_i) = 0$$

     and solving for q_i, we find firm i's best response function

     $$q_i(q_j) = \frac{1 - c + \alpha k_i}{2} - \frac{q_j}{2}$$

     This best response function is increasing in k_i, $\frac{\partial q_i}{\partial k_i} > 0$.(Since firm i becomes relatively more competitive but decreasing in q_j) Since firm j becomes more competitive. A symmetric expression applies for firm j's best response function, $q_j(q_i)$.

   (b) Find the second-period price function, $q_i(k_i, k_j)$.

   - Inserting firm j's best response function, $q_j(q_i)$, into firm i's, $q_i(q_j)$, we find

     $$q_i = \frac{1 - c + ak_i}{2} - \frac{(1 - c + \alpha k_j - qi)}{4}$$

     which, rearranging and solving for q_i, yields firm i's equilibrium output function

     $$q_i(k_i, k_j) = \frac{1 - c + 2\alpha k_i - \alpha k_j}{3}.$$

(c) *First stage.* To simplify our analysis, assume that $c = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ for the remainder of the exercise. Find firm i's best response function in the first stage, $k_i(k_j)$. Is it positively or negatively sloped? Interpret.

- We first evaluate the output functions from part (b) at parameter values $c = \frac{1}{2}$ and $\alpha = \frac{1}{4}$, obtaining

$$\begin{aligned} q_i(k_i, k_j) &= \frac{1+k_i}{6} - \frac{k_j}{12} \quad \text{and} \\ q_j(k_i, k_j) &= \frac{1+k_j}{6} - \frac{k_i}{12}. \end{aligned}$$

Therefore, the second-period profits of firm i become

$$\begin{aligned} \pi_i^{2nd} &= (1 - q_i(k_i, k_j) - q_j(k_i, k_j))q_i(k_i, k_j) - \left(\frac{1}{2} - \frac{1}{4}k_i\right) q_i(k_i, k_j) \\ &= \frac{4k_i^2 + 8k_i - 4k_ik_j + k_j^2 + 4 - 4k_j}{144} \end{aligned}$$

As a result, every firm i chooses its cost-reducing investment k_i to solve the following problem in the first period game

$$\max_{k_i} \pi_i^{2nd} - k_i$$

Differentiating with respect to k_i, we find

$$\frac{2k_i + 2 - k_j}{36} - 1 = 0$$

and, solving for k_i, we obtain firm i's best response function in the first period

$$k_i(k_j) = 17 + \frac{k_j}{2}$$

Therefore, the best response function is positively sloped, indicating that firms regard each other's cost-reducing investments as strategic complements. In other words, when firm j invests one more dollar in cost-reducing technologies, k_j, firm i responds increasing its own investment, k_i

(d) Find the equilibrium investment in cost-reducing technologies, k_i^*.

- Inserting firm j's best response function, $k_j(k_i)$, into firm i's, $k_i(k_j)$, we obtain

$$k_i = 17 + \frac{\overbrace{17 + \frac{k_j}{2}}^{k_j(k_i)}}{2} = \frac{102 + k_i}{4}$$

which, rearranging yields $3k_i = 102$. Solving for k_i, yields firm i's equilibrium investment,

$$k_i^* = 34$$

(e) Summarize the subgame perfect equilibrium of the game.

- In the subgame perfect equilibrium of the game, every firm i invests $k_i^* = 34$ in cost-reducing technologies during the first period and responds with an output function $q_i(k_i, k_j) = \frac{1+k_i}{6} - \frac{k_j}{12}$ in the second period.

3. **Entry deterrence before Bertrand competition.** Consider a market with inverse demand $p = a - Q$, where Q denotes aggregate output. The incumbent monopolist is present in the market and a potential entrant considers entering the industry at a fixed setup cost $K > 0$. For simplicity, assume that both firms face no production costs. If entry does not occur, the incumbent (firm 1) keeps its monopoly position. If entry ensues, firms compete a la Bertrand, with sales going to the firm setting the lowest price (and in the case that both firms set the same price, assume that consumers buy from firm 1).

(a) Find the SPNE of the game. Does firm 2 chooses to enter in equilibrium?

- In a Nash equilibrium, both firms will charge prices equal to their common marginal cost, that is, $p_1 = p_2 = c = 0$. Hence, if entry requires even a small initial investment $K > 0$, firm 2 will choose to stay out. Therefore, the SPNE is

$$(Out,\, p_1 = p_2 = 0).$$

(b) Let us now allow firm 2 to choose, before the beginning of the game, a pair (c_2, p_2) of a capacity c_2 and price p_2, where capacity c_2 sets an upper bound on firm 2's production. For simplicity, assume that each unit of capacity costs one dollar. If entry occurs, now firm 1 observes the pair (c_2, p_2), responds to the pair setting a price p_1 before consumers choose which firm to buy from. Find firm 1's profits if it chooses to deter entry. Find firm 1's profits from accommodating entry. Compare its profits. Under which parameter values does firm 1 choose to deter entry?

- *Fighting firm 2.* If firm 1 fights, it charges a price that coincides with that of its rival, i.e., $p_1 = p_2$, such that it captures the entire market because by assumption, when both firms set equal prices, all consumers prefer to buy from firm 1. Firm 1's profits from fighting are then

$$\begin{aligned}\pi_1^F &= q_1 p_1 \\ &= (a - p_1 - q_2)\, p_1 \\ &= (a - p_2 - 0)\, p_2 \\ &= (a - p_2)\, p_2\end{aligned}$$

where superscript F denotes that firm 1 fights. Firm 2 obtains zero profits in the Bertrand game (second stage), and thus prefers to stay out.

- *Accommodating firm 2.* The inverse demand function for firm 1 is $p_1 = a - q_1 - q_2$. Then, if firm 1 accommodates firm 2, firm 1's residual demand is

$$q_1 = a - q_2 - p_1$$

so that firm 1 now solves the following profit maximization problem:

$$\max_{p_1 \geq 0} \pi_1 (p_1|q_2) = p_1 q_1 = p_1 (a - q_2 - p_1)$$

Differentiating with respect to p_1, and solving, we obtain that the incumbent's profit-maximizing price that accomodates entry, p_1^A, is

$$p_1^A (q_2) = \frac{a - q_2}{2}$$

that serves the residual market with equilibrium output of

$$\begin{aligned} q_1^A (q_2) &= a - q_2 - p_1 \\ &= a - q_2 - \frac{a - q_2}{2} \\ &= \frac{a - q_2}{2} \end{aligned}$$

yielding equilibrium profits by accommodating entry of

$$\begin{aligned} \pi_1^A (q_2) &= p_1 (a - q_2 - p_1) \\ &= \frac{a - q_2}{2} \left(a - q_2 - \frac{a - q_2}{2} \right) \\ &= \frac{(a - q_2)^2}{4} \end{aligned}$$

where superscript A denotes that firm 1 accommodates.
Anticipating this behavior by the incumbent, firm 2's demand becomes

$$\begin{aligned} q_2 &= a - q_1^A (q_2) - p_2 \\ &= a - \frac{a - q_2}{2} - p_2 \end{aligned}$$

which, after rearranging, becomes

$$q_2 (p_2) = a - 2p_2$$

Therefore, we can alternatively express firm 1's profit by accommodating entry as

$$\pi_1^A (p_2) = \frac{(a - q_2)^2}{4} = \frac{(a - (a - 2p_2))^2}{4} = (p_2)^2$$

- *Profit comparison.* Firm 1 accomodates entry if and only if $\pi_1^A \geq \pi_1^F$, that is,

$$\begin{aligned} &(p_2)^2 \geq (a - p_2) p_2 \\ &\Rightarrow p_2 \geq a - p_2 \\ &\Rightarrow p_2 \geq \frac{a}{2} \end{aligned}$$

Therefore, firm 1 accommodates entry if the price that firm 2 charges is sufficiently high, in particular, $p_2 \geq \frac{a}{2}$, since the incumbent can earn a significant positive margin on every unit. If instead $p_2 < \frac{a}{2}$, firm 1 fights entry.

- *First stage (Firm 2).* We next find the equilibrium price that firm 2 chooses in the first stage of the game, p_2^*, and check if it satisfies $p_2 \geq \frac{a}{2}$ (inducing accomodation from the incumbent in the subsequent stage) or $p_2 < \frac{a}{2}$ (inducing entry). Specifically, firm 2 chooses the pair of prices and capacity (p_2, c_2) that solves

$$\max_{p_2, c_2 \in \mathbb{R}_+^2} p_2 q_2 - c_2 - K$$

subject to the capacity constraint, $q_2 \leq c_2$. (Note that profits include the cost of building capacity, c_2, and the fixed cost of entry, K.)
Therefore, the Lagrangian problem that firm 2 solves becomes

$$\mathcal{L} = [p_2 (a - 2p_2) - c_2 - K] - \underbrace{\lambda (q_2 - c_2)}_{\text{Capacity constraint}}$$

The FOCs are

$$\begin{aligned} a &= 4p_2 \\ 1 &= \lambda \\ \lambda (q_2 - c_2) &= 0 \end{aligned}$$

Since $\lambda = 1$, firm 2 operates at full capacity, that is, $q_2 = c_2$. Next, substituting the result from the first FOC, $a = 4p_2$, or $p_2^* = \frac{a}{4}$, into firm 2's demand function, we obtain

$$q_2^* = a - 2p_2^* = a - 2\frac{a}{4} = \frac{a}{2}$$

so that firm 2's capacity becomes $c_2^* = q_2^* = \frac{a}{2}$, yielding profits of

$$\begin{aligned} \pi_2^* &= p_2^* q_2^* - c_2^* - K \\ &= \frac{a}{4} \times \frac{a}{2} - \frac{a}{2} - K \\ &= \frac{a (a - 4)}{8} - K \end{aligned}$$

However, since the equilibrium price we found was $p_2^* = \frac{a}{4}$, condition $p_2 < \frac{a}{2}$ holds, inducing firm 1 to fight entry. In this context, firm 2 has no sales, and its profits become

$$p_2^* q_2^* - c_2^* - K = \underbrace{\frac{a}{4} 0}_{\text{No sales}} - \frac{a}{2} - K = -\frac{a}{2} - K$$

which indicate a net loss from the pair $(p_2^*, c_2^*) = \left(\frac{a}{4}, \frac{a}{2}\right)$ since it induces the incumbent to start a price war.

- Mathematically, the above profit-maximizing problem yielding pair $(p_2^*, c_2^*) = \left(\frac{a}{4}, \frac{a}{2}\right)$ considers the capacity constraint, but ignores whether this pair induces the incumbent to accomodate or fight entry. We next search for the profit-maximizing pair (p_2^*, c_2^*) among all those that induce price accomodation, i.e.,

$p_2 \geq \frac{a}{2}$. In this context, firm 2 solves the following modified Lagrangian problem:

$$\mathcal{L}' = p_2\left(a - 2p_2\right) - c_2 - K - \underbrace{\lambda\left(q_2 - c_2\right)}_{\text{Capacity constraint}} - \underbrace{\mu\left(\frac{a}{2} - p_2\right)}_{\text{Accomodation}}$$

The FOCs are

$$\begin{aligned} a - 4p_2 + \mu &= 0 \\ 1 &= \lambda \\ \lambda\left(q_2 - c_2\right) &= 0 \\ \mu\left(\frac{a}{2} - p_2\right) &= 0 \end{aligned}$$

Again, we obtain that $\lambda = 1$ implying that firm 2 operates at full capacity, that is, $q_2 = c_2$. First, suppose that the price constraint is binding, that is, $p_2 = \frac{a}{2}$, then $\mu > 0$ such that from the first FOC, $a - 4p_2 + \mu = 0$, we find

$$\mu = 4p_2 - a = 4 \times \frac{a}{2} - a = a > 0$$

Second, suppose that the price constraint is slack, that is, $p_2 > \frac{a}{2}$, then $\mu = 0$ such that from the fourth FOC, $\mu\left(\frac{a}{2} - p_2\right) = 0$, we obtain

$$p_2 = \frac{a}{4}$$

which violates the above price constraint $p_2 > \frac{a}{2}$. Therefore, the the price constraint is binding, $p_2 = \frac{a}{2}$, implying that firm 2 chooses a price $p_2^* = \frac{a}{2}$, which yields a capacity

$$c_2^* = q_2^* = a - 2p_2^* = a - 2 \times \frac{a}{2} = 0$$

and equilibrium profits

$$\pi_2^* = p_2^* q_2^* - c_2^* - K = \frac{a}{4} 0 - 0 - K = -K$$

As a result, firm 2 does not have incentives to enter. If it did, the firm would make a net loss of K by selling no units but incurring the fixed setup costs.

4. **Stackelberg with m leaders and n followers, Huck et al. (2001).** Consider a market with two firms with $m + n$ firms, inverse demand given by $P(Q) = a - Q$, and all firms facing a (constant) unit cost c.

   (a) Find the Subgame Perfect Nash Equilibrium of the following two-stage game. In the first stage, m firms (leaders) decide their output. In the second stage, the remaining n firms (followers) decide their output.

- *Second Stage.* We proceed by backward induction. Given that the leaders produced $\sum_{i=1}^{m} q_i$, the followers behave as in a standard Cournot game with (residual) demand $P = a - \sum_{i=1}^{m} q_i - \sum_{i=1}^{n} q_i$. Invoking symmetry, $\sum_{i=1}^{n} q_i = nq_f$, and each follower produces

$$q_f = \frac{a - \sum_{i=1}^{m} q_i - c}{n+1}$$

- *First Stage.* The profit of each leader in the first stage is given by:

$$\begin{aligned} \pi_i &= \left(a - \sum_{i=1}^{m} q_i - n\left(\frac{a - \sum_{i=1}^{m} q_i - c}{n+1}\right) - c\right) q_i \\ &= \left(\frac{1}{n+1}\right)\left(a - \sum_{i=1}^{m} q_i - c\right) q_i \end{aligned}$$

- Except for thefirst term, $\frac{1}{n+1}$, this profit function coordinates with that in the standard Cournot model with demand $P(Q) = a - Q$ and m firms. Then in equilibrium each leader produces $q_l = \frac{a-c}{(m+1)}$. Plugging this information in the followers' best response functions i.e., $q_f = \frac{a - m\left(\frac{a-c}{(m+1)}\right) - c}{n+1}$, each follower produces $q_f = \frac{a-c}{(m+1)(n+1)}$. Then profits of leaders and followers are given respectively by

$$\begin{aligned} \pi_l &= pq_l - cq_l = \left(a - m\frac{a-c}{(m+1)} - n\frac{a-c}{(m+1)(n+1)} - c\right)\left(\frac{a-c}{(m+1)}\right) \\ &= \left(\frac{1}{n+1}\right)\left(\frac{a-c}{m+1}\right)^2 \\ \pi_f &= pq_f - cq_f = \left(a - m\frac{a-c}{(m+1)} - n\frac{a-c}{(m+1)(n+1)} - c\right)\left(\frac{a-c}{(m+1)(n+1)}\right) \\ &= \left(\frac{1}{m+1}\right)^2\left(\frac{a-c}{n+1}\right)^2 \end{aligned}$$

(b) Study the profitability of the merger of two leaders and the profitability of the merger of two followers.

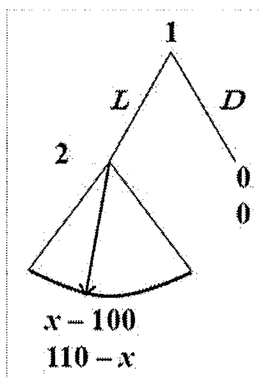
- Except for the first term $\frac{1}{n+1}$, equilibrium profit for each leader, π_l coincides with that in a standard Cournot model with m firms. The same argument applies for equilibrium profit for each follower, π_f, which except for $\left(\frac{1}{m+1}\right)^2$ coincides with the standard equilibrium profits in Cournot with n firms.
- As shown in the standard Cournot model by Salant et. al. (1983), the merger of two leaders (or of two followers) is only profitable if the market share of the merging firms is approximately larger than 80%. That is, their merger is only profitable if there were only two leaders (or two follower) to begin with. Otherwise, a merger of only two of the three (or more) firms would be unprofitable, as they represent a 75% (or less) of the industry.

subgame perfect because in the subgame following no investment, the players are not playing a Nash equilibrium. ■

7. **Debt and Repayment:** A project costing \$100 yields a gross return of \$110. A lender (player 1) is approached by a debtor (player 2) requesting a standard loan contract to complete the project. If the lender chooses *not to offer* a loan, then both parties earn nothing. If the lender chooses to *offer* a loan of \$100, the debtor can realize the projects gains, and is obliged by contract to repay \$105. For simplicity, assume that money is continuous, and that the debtor can choose to return any amount of money $x \leq 110$. Also, ignore the time value of money. Assume first that no legal system is in place that can cause the lender to repay, so that default on the loan (less than full repayment) carries no repercussions for the debtor.

   (a) Model this as an extensive form game tree as best as you can and find a subgame perfect equilibrium of this game. Is it unique?

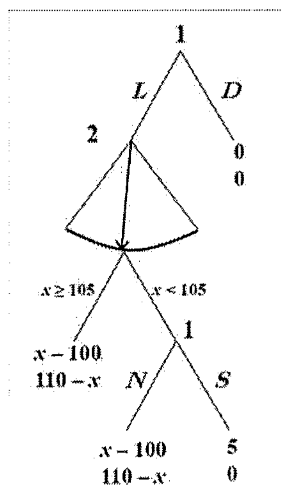
   Answer: Player 1 has two choices first, lend (L) or don't lend (D). After D both players get zero, while after L player 2 chooses a value $x \in [0, 110]$ to repay. The game can be described as follows:



   There is a unique subgame perfect equilibrium. If player 2 is offered the loan then he suffers no penalty from repaying, and his best response is to choose $x = 0$. Anticipating this behavior player 1 should choose D. ■

(b) Now assume that there is a legal system in place that allows the lender to *voluntarily* choose whether to sue or not to sue when the debtor defaults and repays an amount $x < 105$. Furthermore, assume that it is *costless* to use the legal system (it is supplied by the state), and if the lender sues a debtor that defaulted, the lender will get the \$105 repaid in full. After paying the lender, the borrower will pay a fine of \$5 to the court above and beyond the repayment. Model this as an extensive form game tree as best as you can and find a subgame perfect equilibrium of this game. Is it unique?

Answer: The game now distinguished between two conditions: $x \geq 105$ in which case it is like the game in part (a) above, and $x < 105$ in which case player 1 has a new decision node where he can choose to sue (S) or not sue (N).



Starting at the last decision node of player 1, because it is relevant only when $x < 105$, it follows that $x - 100 < 5$ implying that S dominates N. Anticipating this, player 2's best response in the repayment phase is to choose $x = 105$. This is the lowest payment that does not trigger a suit. At the root of the tree player 1 anticipates a payoff of $105 - 100 = 5 > 0$ and hence prefers to choose L. The resulting outcome yields the payoffs

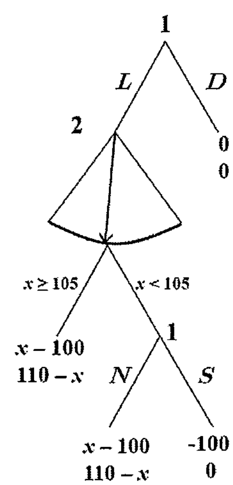
(5, 5). This backward induction argument shows that this is the unique subgame perfect equilibrium. ■

(c) Are there Nash equilibria in the game described in (b) above that are not subgame perfect equilibria?

Answer: For player 1, choosing D followed by S is a dominant strategy because it guaranties him a payoff of at least 5 (exactly 5 when $x < 105$ and $x - 100$ when $x \geq 105$.) Given this strategy, player 2's best reply is to choose $x = 105$. Hence, the only Nash equilibrium is also the subgame perfect Nash equilibrium. ■

(d) Now assume that using the legal system is *costly:* if the lender sues, he pays lawyers a *legal fee* of \$105 (this is the lawyers price which is unrelated to the contract above). The rest proceeds the same as before (if the lender sues a debtor that defaulted, the lender will get repaid in full; after paying the lender, the borrower will pay a fine of \$5 above and beyond the repayment.) Model this as an extensive form game tree as best as you can and find a subgame perfect equilibrium of this game. Is it unique?

Answer: The game is now,



Because $x - 100 \geq -100$ it follows that that S is weakly dominated by

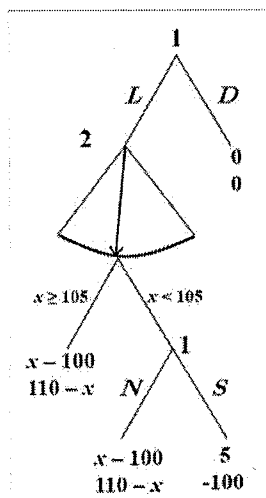
N. Anticipating this, player 2's best response in the repayment phase is to choose $x = 0$. At the root of the tree player 1 anticipates a payoff of $-100 < 0$ and hence prefers to choose D, and the outcome results in payoffs (0,0). This backward induction argument shows that this is the unique subgame perfect equilibrium. ■

(e) Are there Nash equilibria in the game described in (d) above that are not subgame perfect equilibria?

Answer: There are infinitely many. Any choice by player 2 of $x \leq 100$, for which playing D is a best response, will be a Nash equilibrium in which player 2 is not playing a best response. ■

(f) Now assume that a law change is proposed: upon default, if a debtor is sued he has to first repay the lender \$105, and then pay the legal fees of \$105 above and beyond repayment of the loan, and no extra fine is imposed. Should the lender be willing to pay for this law change? If so, how much?

Answer: The game is now as follows:



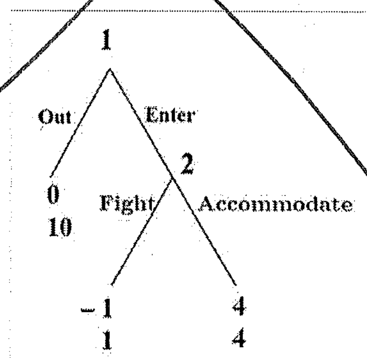
The backward induction argument follows the same logic as in part (b) resulting in the outcome $(5, 5)$. This yields player 1 an extra payoffs of 5 relative to the solution in part (d), implying that he should be willing to pay up to 5 in order to have the law implemented. ■

(g) If you were the "social planner", would you implemented the suggested law?

Answer: Yes because it results in a Pareto superior outcome of (5, 5) instead of (0, 0). ■

8. **Entry Deterrence 1:** NSG is considering entry into the local phone market in the Bay Area. The incumbent S&P, predicts that a price war will result if NSG enters. If NSG stays out, S&P earns monopoly profits valued at $10 million (net present value, or NPV of profits), while NSG earns zero. If NSG enters, it must incur irreversible entry costs of $2 million. If there is a price war, each firm earns $1 million (NPV). S&P always has the option of accommodating entry (i.e., not starting a price war). In such a case, both firms earn $4 million (NPV). Suppose that the timing is such that NSG first has to choose whether or not to enter the market. Then S&P decides whether to "accommodate entry" or "engage in a price war." What is the subgame perfect equilibrium outcome to this sequential game? (Set up a game tree.)

Answer: Letting NSG be player 1 and S&P be player 2,



Backward induction implies that player 2 will Accommodate, and player 1 will therefore enter. Hence, the unique subgame perfect equilibrium is (Enter,Accommodate). ■

9. **Entry Deterrence 2:** Consider the Cournot duopoly game with demand $p = 100 - (q_1 + q_2)$, and variable costs $c_i(q_i) = 0$ for $i \in \{1, 2\}$. The twist is

that there is now a fixed cost of production $k > 0$ that is the same for both firms.

(a) Assume first that both firms choose their quantities simultaneously. Model this as a normal form game.

Answer: This is a standard Cournot game with two players: $N = \{1, 2\}$, $S_i = \mathbb{R}_+$ (the non-negative real line) and we need to add the fixed costs to the payoff function, $v_i(q_1, q_2) = (100 - q_1 - q_2)q_i - k$ for $i \in \{1, 2\}$.

(b) Write down the firm's best response function for $k = 1000$ and solve for pure strategy Nash equilibrium. Is it unique?

Answer: Because the fixed costs do not affect the first order conditions, from section 5.2.3 we know that the two best response functions ignoring the fixed costs are,

$$q_i(q_j) = \frac{100 - q_j}{2} .$$

With fixed costs, however, each firm will produce only if it has positive profits. For example, using firm 1's best response function, its profits conditional on playing a best response are

$$\begin{aligned} v_1(q_1(q_2), q_2) &= (100 - (\frac{100 - q_2}{2} + q_2))\frac{100 - q_2}{2} - k \\ &= 2500 + \frac{q_2^2}{4} - 50q_2 - k \\ &= 1500 + \frac{q_2^2}{4} - 50q_2 . \end{aligned}$$

where the last inequality follows from $k = 1000$. Now we can compute the value of q_2 for which playing a best response by firm 1 will yield zero profits, which in turn will imply that for higher levels of q_2 firm 1 will incur a loss even when it plays a best response conditional on producing. We have,

$$1500 + \frac{q_2^2}{4} - 50q_2 \geq 0$$

which holds when $q_2 \leq 100 - 20\sqrt{10} \approx 36.75$. A symmetric argument will hold for firm 2, which yields the best response function with a fixed cost of $k = 1000$ to be,

$$q_i(q_j) = \begin{cases} \frac{100-q_j}{2} & \text{if } q_j \leq 100 - 20\sqrt{10} \\ 0 & \text{if } q_j > 100 - 20\sqrt{10} \end{cases} .$$

Using the first portion of the best response function to try and solve for a Nash equilibrium, we obtain that $q_1 = q_2 = 33\frac{1}{3} < 36.75$. Thus, when $k = 1000$, $q_1 = q_2 = 33\frac{1}{3}$ is the unique Nash equilibrium of this game. ■

(c) Now assume that firm 1 is a "Stackelberg leader" in the sense that it moves first and chooses q_1, and then after observing q_1 firm 2 chooses q_2. Also assume that if firm 2 cannot make strictly positive profits then it will not produce at all. Model this as an extensive form game tree as best as you can, and find a subgame perfect equilibrium of this game for $k = 25$. Is it unique?

Answer: similar to the analysis in section 8.3.2 we know that, ignoring fixed costs, firm 2 will choose $q_2(q_1) = \frac{100-q_1}{2}$ as derived above. With $k = 25$ it will not produce for some values of q_1 close to 100. (Similar to the analysis in part (b), q_1 must satisfy $2500 + \frac{q_1^2}{4} - 50q_1 - k > 0$ with $k = 25$. This will be satisfied when $q_1 \leq 90$.) Given firm 2's best response, firm 1 maximizes

$$\max_{q_1} \; (100 - q_1 - \frac{100 - q_1}{2})q_1 - 25 \; ,$$

which yields the first order condition $50 - q_1 = 0$ or $q_1^* = 50$. Because $q_1^* < 90$ we know that firm 2 will indeed follow $q_2(q_1) = \frac{100-q_1}{2} = 25$, profits for firm 1 are $v_1 = 25 \times 50 - 25 = 1,225$, and for firm 2 are $v_2 = 25 \times 25 - 25 = 600$. By construction, this is the unique subgame perfect equilibrium. ■

(d) How does your answer in (c) change for $k = 725$?

Answer: Now firm 2 will follow $q_2(q_1) = \frac{100-q_1}{2}$ as long as $1775 + \frac{q_1^2}{4} -$

$50q_1 \geq 0$, which holds for $q_1 \leq 100 - 10\sqrt{29} \approx 46.15$. As we saw in part (c), if firm 1 anticipates firm 2 to produce according to $q_2(q_1) = \frac{100-q_1}{2}$ then firm 1 produces $q_1^* = 50$. It turns out that if firm 1 anticipates firm 2 to stay out then it will also produce $q_1^* = 50$ which is the monopolists optimal choice for this market with only fixed costs. However, since $50 > 46.15$ this choice will indeed cause firm 2 to stay out, and the unique subgame perfect equilibrium is now $q_1^* = 50$ and

$$q_2(q_1) = \begin{cases} \frac{100-q_1}{2} & \text{if } q_1 \leq 100 - 10\sqrt{29} \\ 0 & \text{if } q_1 > 100 - 10\sqrt{29} \end{cases},$$

resulting in $q_2^* = 0$. ■

10. **Playing it safe:** Consider the following dynamic game: Player 1 can choose to play it safe (denote this choice by S), in which case both he and player 2 get a payoff of 3 each, or he can risk playing a game with player 2 (denote this choice by R). If he chooses R, then they play the following **simultaneous move** game:

Player 2

player 1	A	B
C	$8, 0$	$0, 2$
D	$6, 6$	$2, 2$

(a) Draw a game tree that represents this game. How many proper subgames does it have?

Answer: