EconS 503 - Microeconomic Theory II
Homework #3 - Answer key

1. **Exercises from Tadelis:**

   (a) Chapter 6: Exercises 7, 9, and 11.

2. **Cournot game.** Consider an industry with $N \geq 2$ firms competing a la Cournot, facing inverse demand function $p(Q)$, where $Q = \sum_{i=1}^{N} q_i$ denotes aggregate output, which is strictly decreasing and strictly concave, i.e., $p'(Q) < 0$ and $p''(Q) < 0$. Firm i faces cost function $c_i(q_i)$, which satisfies $c_i(0) = 0$ and is strictly increasing and strictly convex, i.e., $c_i'(q_i) > 0$ and $c_i''(q_i) > 0$.

   (a) Find the implicit function describing the equilibrium output of every firm i, q_i^*.

   - Every firm i solves
   $$\max_{q_i} p(Q)q_i - c_i(q_i)$$
   Differentiating with respect to q_i, yields
   $$p(Q^*) + p'(Q^*)q_i^* = c_i'(q_i^*)$$
   where $Q^* = q_1^* + q_2^* + ... + q_N^*$ denotes aggregate equilibrium output.

   (b) Consider now a regulator seeking to maximize social welfare, defined as
   $$SW = \int_0^Q p(Q)dQ - \sum_{i=1}^{N} c_i(q_i).$$
   Find the implicit function describing socially optimal output of every firm i, q_i^{SO}.

   - Differentiating with respect to every firm i's output in the above social welfare function SW, we obtain
   $$p(Q^{SO}) = c_i'(q_i^{SO})$$
   where $Q^{SO} = q_1^{SO} + q_2^{SO} + ... + q_N^{SO}$ denotes aggregate socially optimal output.

   (c) Show that $q_i^* < q_i^{SO}$.

   - We now compare the first-order condition we found in part (a) of the exercise against that from part (b). Since inverse demand has a negative slope, the second term in the first-order condition of part (a) satisfies $p'(Q^*)q_i^* < 0$. Therefore,
   $$p(Q^*) - c_i'(q_i^*) > p(Q^{SO}) - c_i'(q_i^{SO})$$
   or, after rearranging,
   $$c_i'(q_i^{SO}) - c_i'(q_i^*) > p(Q^{SO}) - p(Q^*).$$

Operating by contradiction, let us assume that output levels satisfy the opposite ranking of what we are supposed to show, $q_i^* \geq q_i^{SO}$. Then, we have that $p(Q^{SO}) > p(Q^*)$, implying that the right side of the above inequality must be positive; that is, $p(Q^{SO}) - p(Q^*) > 0$, entailing that we need the left side to satisfy

$$c_i'(q_i^{SO}) - c_i'(q_i^*) > 0.$$

(The above steps actually mean that the difference $c_i'(q_i^{SO}) - c_i'(q_i^*)$ is not only positive, but larger than the positive number $p(Q^{SO}) - p(Q^*)$, but for our proof we only need to show that the less demanding condition $c_i'(q_i^{SO}) - c_i'(q_i^*) > 0$ cannot hold in our setting, as we demonstrate below.) However, marginal cost is, by definition, increasing in firm i's output, implying that $c_i'(q_i^{SO}) > c_i'(q_i^*)$ is not compatible with the output ranking $q_i^* \geq q_i^{SO}$. We have then reached a contradiction and, as a consequence, output levels must satisfy $q_i^* < q_i^{SO}$. Intuitively, every firm i produces a larger output when the social planner chooses production levels than when the firm selects its own production.

3. **Entry that reduces aggregate output.** Consider an industry with $N \geq 2$ firms competing a la Cournot, facing inverse demand function $p(Q) = a - bQ$, and symmetric cost function $c(q_i) = cq_i + \frac{d}{2}q_i^2$, where $a > c$, $d < 0$, $d + 2b > 0$, and $d + b < 0$. (You may also assume that $\frac{a-c}{2b+d} < -\frac{c}{d}$ to avoid settings with negative costs.)

   (a) Find equilibrium output for every firm i, q_i^*.

   - Every firm i solves

$$\max_{q_i} \; [a - b(q_i + Q_{-i})]\, q_i - \left(cq_i + \frac{d}{2}q_i^2\right)$$

where $Q_{-i} = \sum_{j \neq i} q_j$ denotes the aggregate output by firm i's rivals. Differentiating with respect to q_i yields

$$a - 2bq_i - bQ_{-i} = c + dq_i$$

And solving for q_i, we find firm i's best response function

$$q_i(Q_{-i}) = \frac{a-c}{2b+d} - \frac{b}{2b+d}Q_{-i}$$

In a symmetric equilibrium, every firm produces the same output level $q_i^* = q_j^* = q^*$ for every firm $j \neq i$, so that $Q_{-i} = (N-1)q^*$. Inserting this property in the above best response function, we obtain

$$q^* \frac{a-c}{2b+d} - \frac{b}{2b+d}(N-1)q^*$$

the equilibrium output level for every firm in this Cournot game

$$q^* = \frac{a-c}{(N+1)b+d}$$

(b) Find aggregate output in equilibrium, Q^*.

- Aggregate output is

$$Q^* = Nq^* = \frac{N(a-c)}{(N+1)b+d}$$

(c) Show that Q^* decreases with entry.

- Differentiating Q^* with respect to N, we find

$$\frac{\partial Q^*}{\partial N} = \frac{(a-c)(b+d)}{[(N+1)b+d]^2}$$

  which is negative given that the denominator is positive but the numerator is negative because $a > c$ and $d + b < 0$ by assumption. Then, aggregate output decreases in the number of firms entering the industry.

- This result did not happen in the Cournot model with N firms facing linear costs. in that setting, while every firm reduces its individual output as a response to entry of new rivals, aggregate output increases. Intuitively, the new production of the newcomer more than offsets the decrease in individual output from each incumbent firm, ultimately yielding an overall increase in Q^*.
- In contrast, firms face an increasing and convex cost function in the current exercise, which leads them to reduce their individual output levels (as a response to entry) more significantly, driving Q^* downwards. Interestingly, in this context, further entry *increases* the equilibrium price, $p(Q^*)$, making consumers worse off than when fewer firms compete.

equilibrium of the original game is $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$. Notice that at this Nash equilibrium, each player is not only indifferent between E and N, but choosing T gives the same expected payoff of zero. However, choosing T with positive probability cannot be part of a mixed strategy Nash equilibrium. To prove this let player 2 play the mixed strategy $\sigma_2 = (\sigma_2(E), \sigma_2(T), \sigma_2(N)) = (\sigma_{2E}, \sigma_{2T}, 1 - \sigma_{2E} - \sigma_{2T})$. The strategy T for player 1 is at least as good as E if and only if,

$$0 \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2E} \geq \frac{1}{3}$. The strategy T for player 1 is at least as good as N if and only if,

$$4\sigma_{2E} - \sigma_{2T} - 2(1 - \sigma_{2E} - \sigma_{2T}) \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2T} \leq 1 - 3\sigma_{2E}$. But if $\sigma_{2E} \geq \frac{1}{3}$ (when T is as good as E) then $\sigma_{2T} \leq 1 - 3\sigma_{2E}$ reduces to $\sigma_{2T} \leq 0$, which can only hold when $\sigma_{2E} = \frac{1}{3}$ and $\sigma_{2T} = 0$ (which is the Nash equilibrium we found above). A symmetric argument holds to conclude that $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$ is the unique mixed strategy Nash equilibrium. ■

7. **Grad School Competition:** Two students sign up for an honors thesis with a Professor. Each can invest time in their own project: either no time, one week, or two weeks (these are the only three options). The cost of time is 0 for no time, and each week costs 1 unit of payoff. The more time a student puts in the better their work will be, so that if one student puts in more time than the other there will be a clear "leader". If they put in the same amount of time then their thesis projects will have the same quality. The professor, however, will give out only one "A" grade. If there is a clear leader then he will get the A, while if they are equally good then the professor will toss a fair coin to decide who gets the A grade. The other student gets a "B". Since both wish to continue to graduate school, a grade of A is worth 3 while a grade of B is worth zero.

   (a) Write down this game in matrix form.

Answer: Let N denote no time, O denote one week, and T denote two weeks. The matrix game is,

		Player 2		
		N	O	T
Player 1	N	$1.5, 1.5$	$0, 2$	$0, 1$
	O	$2, 0$	$0.5, 0.5$	$-1, 1$
	T	$1, 0$	$1, -1$	$-0.5, -0.5$

The payoffs are derived by the fact that a tie is an equal chance of getting 3 so each player gets 1.5 in expectation. ■

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

Answer: Each one of the three strategies can be a strict best response: N is a best response to T, O is a best response to N, and T is a best response to O. Hence, no strategy is strictly or weakly dominated. ■

(c) Find the unique mixed strategy Nash equilibrium.

Answer: Let $\sigma_i = (\sigma_{iN}, \sigma_{iO}, 1 - \sigma_{iN} - \sigma_{iO})$ denote a mixed strategy for player i. Because the game is symmetric it suffices to solve the indifference conditions for one player. For player i to be indifferent between N and O,

$$1.5\sigma_{iN} = 2\sigma_{iN} + 0.5\sigma_{iO} - (1 - \sigma_{iN} - \sigma_{iO})$$

and for him to be indifferent between N and T,

$$1.5\sigma_{iN} = \sigma_{iN} + \sigma_{iO} - 0.5(1 - \sigma_{iN} - \sigma_{iO})$$

Solving these two equations with two unknowns yields $\sigma_{iN} = \sigma_{iO} = \frac{1}{3}$ implying that the unique mixed strategy Nash equilibrium has the players mixing between all three pure strategies with equal probability. ■

8. **Market entry:** There are 3 firms that are considering entering a new market. The payoff for each firm that enters is $\frac{150}{n}$ where n is the number of firms that enter. The cost of entering is 62.

(a) Find all the pure strategy Nash equilibria.

Answer: The costs of entry are 62 so the benefits of entry must be at least that for a firm to choose to enter. Clearly, if a firm believes the other two are not entering then it wants to enter, and if it believes that the other firms are entering then it would stay out (it would only get 50). If a firm believes that only one other firm is entering then it prefers to enter and get 75. Hence, there are three pure strategy Nash equilibria in which two of the three firms enter and one stays out. ■

(b) Find the symmetric mixed strategy equilibrium where all three players enter with the same probability.

Answer: Let p be the probability that a firm enters. In order to be willing to mix the expected payoff of entering must be equal to zero. If a firm enters then with probability p^2 it will face two other entrants and receive $v_i = 50 - 62 = -12$, with probability $(1-p)^2$ it will face no other entrants and receive $v_i = 150 - 62 = 88$, and with probability $2(1-p)p$ it will face one other entrant and receive $v_i = 75 - 62 = 13$. Hence, to be willing to mix the expected payoff must be zero,$p^2 + 1 - p^2$

$$(1-p)^2 88 + 2(1-p)p13 - p^2 12 = 0$$

which results in the quadratic equation $25p^2 - 75p + 44 = 0$, and the relevant solution (between 0 and 1) is $p = \frac{4}{5}$. ■

9. **Discrete all pay auction:** In section 6.1.4 we introduced a version of an all pay auction that worked as follows: Each bidder submits a bid. The highest bidder gets the good, but *all bidders pay there bids.* Consider an auction in which player 1 values the item at 3 while player 2 values the item at 5. Each player can bid either 0, 1 or 2. The twist is that each player pays his bid regardless of whether he wins the good. If player i bids more than player j then i win's the good and both pay. If both players bid the same amount then a coin is tossed to determine who gets the good but again, both pay.

(a) Write down the game in matrix form. Which strategies survive IESDS?

Answer: Let Z denote zero, O denote one, and T denote two. The matrix game is,

Player 1 \ Player 2	Z	O	T
Z	$1.5, 2.5$	$0, 4$	$0, 3$
O	$2, 0$	$0.5, 1.5$	$-1, 3$
T	$1, 0$	$1, -1$	$-0.5, 0.5$

The payoffs are derived by the fact that a tie is an equal chance of winning so player 1 gets 1.5 and player 2 gets 2.5 in expectation. It is easy to see that for player 2, playing Z is dominated by playing T, so it is eliminated in the first stage of IESDS. In the second stage O is dominated by T for player 1 and we are left with the following reduced game that survives IESDS,

Player 1 \ Player 2	O	T
Z	$0, 4$	$0, 3$
T	$1, -1$	$-0.5, 0.5$

■

(b) Find the Nash equilibria of this game.

Answer: From the reduced game it is easy to see that there is no pure strategy Nash equilibrium. Let $\sigma_1 = (\sigma_{1Z}, \sigma_{1T})$ and $\sigma_2 = (\sigma_{2O}, \sigma_{2T})$ denote the mixed strategies for the players in the reduced game. For player 1 to be indifferent between Z and T,

$$0 = \sigma_{2O} - 0.5(1 - \sigma_{2O})$$

which yields $\sigma_{2O} = \frac{1}{3}$. For player 2 to be indifferent between O and T,

$$4\sigma_{1Z} - (1 - \sigma_{1Z}) = 3\sigma_{1Z} + 0.5(1 - \sigma_{1Z})$$

which yields $\sigma_{1Z} = 0.6$. Thus, the unique mixed strategy Nash equilibrium has the players mixing $\sigma_1 = (\frac{3}{5}, \frac{2}{5})$ and $\sigma_2 = (\frac{1}{3}, \frac{2}{3})$ in the reduced game, or $\sigma_1 = (\frac{3}{5}, 0, \frac{2}{5})$ and $\sigma_2 = (0, \frac{1}{3}, \frac{2}{3})$ in the original game. ■

10. **Continuous all pay auction:** ~~Consider an all-pay auction for a good worth~~ 1 to each of the two bidders. Each bidder can choose to offer a bid from the unit interval so that $S_i = [0, 1]$. Players only care about the expected value they will end up with at the end of the game (i.e., if a player bids 0.4 and expects to win with probability 0.7 then his payoff is $0.7 \times 1 - 0.4$).

(a) Model this auction as a normal-form game.

Answer: There are two players, $N = \{1, 2\}$, each has a strategy set $S_i = [0, 1]$, and assuming that the players are equally likely to get the good in case of a tie, the payoff to player i is given by

$$v_i(s_i, s_j) = \begin{cases} 1 - s_i & \text{if } s_i > s_j \\ \frac{1}{2} - s_i & \text{if } s_i = s_j \\ -s_i & \text{if } s_i < s_j \end{cases}$$

(b) Show that this game has no pure strategy Nash Equilibrium.

Answer: First, it cannot be the case that $s_i = s_j < 1$ because then each player would benefit from raising his bid by a tiny amount ε in order to win the auction and receive a higher payoff $1 - \varepsilon - s_i > \frac{1}{2} - s_i$. Second, it cannot be the case that $s_i = s_j = 1$ because each player would prefer to bid nothing and receive $0 > -\frac{1}{2}$. Last, it cannot be the case that $s_i > s_j \geq 0$ because then player i would prefer to lower his bid by ε while still beating player j and paying less money. Hence, there cannot be a pure strategy Nash equilibrium. ■

(c) Show that this game cannot have a Nash Equilibrium in which each player is randomizing over a finite number of bids.

Answer: Assume that a Nash equilibrium involves player 1 mixing between a finite number of bids, $\{s_{11}, s_{12,\ldots}, s_{1K}\}$ where $s_{11} \geq 0$ is the lowest bid, $s_{1K} \leq 1$ is the highest, $s_{1k} < s_{1(k+1)}$ and each bid s_{1k} is being played with some positive probability σ_{1k}. Similarly assume that player 2 is mixing between a finite number of bids, $\{s_{21}, s_{22,\ldots}, s_{2L}\}$ and each bid s_{2l} is being played with some positive probability σ_{2l}. (*i*) First observe that it cannot be true that $s_{1K} < s_{2L}$ (or the reverse by symmetry). If it were the case then player 2 will win for sure when he bids s_{2L} and pay his bid, while if he reduces his bid by some ε such that $s_{1K} < s_{2L} - \varepsilon$ then he will still win for sure and pay less, contradicting that playing s_{2L} was part of a Nash equilibrium. (*ii*) Second observe that when $s_{1K} = s_{2L}$ then the expected payoff of player 2 from bidding s_{2L} is

$$\begin{aligned} Ev_2 &= \Pr\{s_{1k} < s_{2L}\}(1 - s_{2L}) + \Pr\{s_{1k} = s_{2L}\}(\frac{1}{2} - s_{2L}) \\ &= (1 - \sigma_{1K})(1 - s_{2L}) + \sigma_{1K}(\frac{1}{2} - s_{2L}) \\ &= 1 - s_{2L} - \frac{1}{2}\sigma_K \geq 0 \end{aligned}$$

where the last inequality follows from the fact that $\sigma_{2L} > 0$ (he would not play it with positive probability if the expected payoff were negative.) Let $s'_{2L} = s_{2L} + \varepsilon$ where $\varepsilon = \frac{1}{4}\sigma_K$. If instead of bidding s_{2L} player 2 bids s'_{2L} then he wins for sure and his utility is

$$v_2 = 1 - s'_{2L} = 1 - s_{2L} - \frac{1}{4}\sigma_K > 1 - s_{2L} - \frac{1}{2}\sigma_K$$

contradicting that playing s_{2L} was part of a Nash equilibrium. ■

(d) Consider mixed strategies of the following form: Each player i chooses and interval, $[\underline{x}_i, \overline{x}_i]$ with $0 \leq \underline{x}_i < \overline{x}_i \leq 1$ together with a cumulative distribution $F_i(x)$ over the interval $[\underline{x}_i, \overline{x}_i]$. (Alternatively you can think of each player choosing $F_i(x)$ over the interval $[0, 1]$, with two values $\underline{x}_i$ and $\overline{x}_i$ such that $F_i(\underline{x}_i) = 0$ and $F_i(\overline{x}_i) = 1$.)

i. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2$ and $\overline{x}_1 = \overline{x}_2$.

Answer: Assume not. There are two cases: (a) $\underline{x}_1 \neq \underline{x}_2$: Without loss assume that $\underline{x}_1 < \underline{x}_2$. This means that there are values of $s_1' \in (\underline{x}_1, \underline{x}_2)$ for which $s_1' > 0$ but for which player 1 loses with probability 1. This implies that the expected payoff from this bid is negative, and player would be better off bidding 0 instead. Hence, $\underline{x}_1 = \underline{x}_2$ must hold. (b) $\overline{x}_1 \neq \overline{x}_2$: Without loss assume that $\overline{x}_1 < \overline{x}_2$. This means that there are values of $s_2' \in (\overline{x}_1, \overline{x}_2)$ for which $\overline{x}_1 < s_2' < 1$ but for which player 2 wins with probability 1. But then player 2 could replace s_2' with $s_2'' = s_2' - \varepsilon$ with ε small enough such that $\overline{x}_1 < s_2'' < s_2' < 1$, he will win with probability 1 and pay less than he would pay with s_2'. Hence, $\overline{x}_1 = \overline{x}_2$ must hold. ■

ii. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2 = 0$.

Answer: Assume not so that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$. This means that when player i bids $\underline{x}$ then he loses with probability 1, and get an expected payoff of $-\underline{x} < 0$. But instead of bidding $\underline{x}$ player i can bid 0 and receive 0 which is better than $-\underline{x}$, implying that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$ cannot be an equilibrium. ■

iii. Using the above, argue that if two such strategies are a mixed strategy Nash equilibrium then both players must be getting an expected payoff of zero.

Answer: As proposition 6.1 states, if a player is randomizing between two alternatives then he must be indifferent between them. Because both players are including 0 in the support of their mixed strategy, their payoff from 0 is zero, and hence their expected payoff from any choice in equilibrium must be zero. ■

iv. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\overline{x}_1 = \overline{x}_2 = 1$.

Answer: Assume not so that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$. From (iii) above the expected payoff from any bid in $[0, \overline{x}]$ is equal to zero. If one of the players deviates from this strategy and choose to bid $\overline{x}+\varepsilon < 1$ then he will win with probability 1 and receive a payoff of $1-(\overline{x}+\varepsilon) > 0$, contradicting that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$ is an equilibrium. ■

v. Show that $F_i(x)$ being uniform over $[0,1]$ is a symmetric Nash equilibrium of this game.

Answer: Imagine that player 2 is playing according to the proposed strategy $F_2(x)$ uniform over $[0,1]$. If player 1 bids some value $s_1 \in [0,1]$ then his expected payoff is

$$\Pr\{s_1 > s_2\}(1-s_1)+\Pr\{s_1 < s_2\}(-s_1) = s_1(1-s_1)+(1-s_1)(-s_1) = 0$$

implying that player 1 is willing to bid any value in the $[0,1]$ interval, and in particular, choosing a bid according to $F_1(x)$ uniform over $[0,1]$. Hence, this is a symmetric Nash equilibrium. ■

11. **Bribes:** Two players find themselves in a legal battle over a patent. The patent is worth 20 for each player, so the winner would receive 20 and the loser 0. Given the norms of the country they are in, it is common to bribe the judge of a case. Each player can offer a bribe secretly, and the one whose bribe is the largest is awarded the patent. If both choose not to bribe, or if the bribes are the same amount, then each has an equal chance of being awarded the patent. If a player does bribe, then bribes can be either a value of 9 or of 20. Any other number is considered to be very unlucky and the judge would surely rule against a party who offers a different number.

    (a) Find the unique pure-strategy Nash equilibrium of this game.

    Answer: The game is captured in the following two player matrix, where Z represents no payment, N represents a bribe of 9 and T a bribe of 20. For example, if both choose 9 then they have an equal chance of getting 20, so the expected payoff is $\frac{1}{2} \times 20 - 9 = 1$,

Player 2

Player 1	Z	N	T
Z	10, 10	0, 11	0, 0
N	11, 0	1, 1	−9, 0
T	0, 0	0, −9	−10, −10

It is easy to see that T is strictly dominated by N. In the remaining game, Z is strictly dominated by N, and hence (N, N) is the unique Nash equilibrium. ■

(b) If the norm were different so that a bribe of 15 were also acceptable, is there a pure strategy Nash equilibrium?

Answer: Now the game is as follows (where F denotes a bribe of 15),

Player 2

Player 1	Z	N	F	T
Z	10, 10	0, 11	0, 5	0, 0
N	11, 0	1, 1	−9, 5	−9, 0
F	5, 0	5, −9	−5, −5	−15, 0
T	0, 0	0, −9	0, −15	−10, −10

Using the best responses of each player it is easy to see that there is no pure strategy Nash equilibrium. ■

(c) Find the symmetric mixed-strategy Nash equilibrium for the game with possible bribes of 9, 15 and 20.

Answer: Note first that T is weakly dominated by Z, so consider the game without T,

Player 2

Player 1	Z	N	F
Z	10, 10	0, 11	0, 5
N	11, 0	1, 1	−9, 5
F	5, 0	5, −9	−5, −5

Let $\sigma_i = (\sigma_{iZ}, \sigma_{iN}, \sigma_{iF})$ denote a mixed strategy for player i where $\sigma_{iF} = 1 - \sigma_{iZ} - \sigma_{iN}$. The game is symmetric so for player 1 to be indifferent between Z and T it must be that,

$$10\sigma_{2Z} = 11\sigma_{2Z} + \sigma_{2N} - 9(1 - \sigma_{2Z} - \sigma_{2N})$$

which implies that $\sigma_{2N} = \frac{9}{10} - \sigma_{2Z}$. For player 1 to be indifferent between Z and F it must be that,

$$10\sigma_{2Z} = 5\sigma_{2Z} + 5\sigma_{2N} - 5(1 - \sigma_{2Z} - \sigma_{2N})$$

which implies that $\sigma_{2N} = \frac{1}{2}$. Hence, the unique (mixed strategy) Nash equilibrium has each player play $\sigma_i = (\frac{2}{5}, \frac{1}{2}, \frac{1}{10})$. ■

12. **The Tax Man:** A citizen (player 1) must choose whether or not to file taxes honestly or whether to cheat. The tax man (player 2) decides how much effort to invest in auditing and can choose $a \in [0, 1]$, and the cost to the tax man of investing at a level a is $c(a) = 100a^2$. If the citizen is honest then he receives the benchmark payoff of 0, and the tax man pays the auditing costs without any benefit from the audit, yielding him a payoff of $(-100a^2)$. If the citizen cheats then his payoff depends on whether he is caught. If he is caught then his payoff is (-100) and the tax man's payoff is $100 - 100a^2$. If he is not caught then his payoff is 50 while the tax man's payoff is $(-100a^2)$. If the citizen cheats and the tax man chooses to audit at level a then the citizen is caught with probability a and is not caught with probability $(1 - a)$.

   (a) If the tax man believes that the citizen is cheating for sure, what is his best response level of a?

   Answer: The tax man maximizes $a(100 - 100a^2) + (1 - a)(0 - 100a^2) = 100a - 100a^2$. The first-order optimality condition is $100 - 200a = 0$, yielding $a = \frac{1}{2}$. ■

   (b) If the tax man believes that the citizen is honest for sure, what is his best response level of a?