EconS 503 - Microeconomic Theory II
Homework #1 - Answer key

1. **Exercise #14 from Chapter 6 in Munoz-Garcia (2017), WEA with market power.** Consider an exchange economy with two consumers, A and B, whose utility functions are

$$
\begin{aligned}
u_A(x_1^A, x_2^A) &= x_1^A x_2^A \\
u_B(x_1^B, x_2^B) &= x_1^B (x_2^B)^2
\end{aligned}
$$

with endowments $e^A = (80, 150)$ and $e^B = (210, 180)$ respectively. Assume that consumer A is price setter, i.e., he makes a take-it-or-leave-it price offer to consumer B.

(a) Find the Walrasian Equilibrium allocation (WEA) in this economy.

- Consumer B takes the price ratio announced by consumer A as given, and solves his UMP

$$
\begin{aligned}
\max_{x_1^B, x_2^B} \quad u_B(x_1^B, x_2^B) &= x_1^B (x_2^B)^2 \\
\text{subject to } p_1 x_1^B + p_2 x_2^B &\leq 210p_1 + 180p_2
\end{aligned}
$$

His Lagrangian is

$$\mathcal{L} = x_1^B (x_2^B)^2 - \lambda[p_1 x_1^B + p_2 x_2^B - 210p_1 - 180p_2]$$

Taking FOCs yields

$$
\begin{aligned}
\frac{d\mathcal{L}}{dx_1^B} &= (x_2^B)^2 - \lambda p_1 = 0 \ \Rightarrow \lambda = \frac{(x_2^B)^2}{p_1} \\
\frac{d\mathcal{L}}{dx_2^B} &= 2x_1^B x_2^B - \lambda p_2 = 0 \ \Rightarrow \ \lambda = \frac{2x_1^B x_2^B}{p_2}
\end{aligned}
$$

Combining the FOCs, we obtain

$$\frac{p_1}{p_2} = \frac{x_2^B}{2x_1^B}$$

Before plugging this result into consumer's budget constraint, $p_1 x_1^B + p_2 x_2^B = 210p_1 + 180p_2$, we can divide such a constraint by p_2 to obtain

$$\frac{p_1}{p_2} x_1^B + x_2^B = 210\frac{p_1}{p_2} + 180 \Rightarrow \frac{p_1}{p_2}(x_1^B - 210) = 180 - x_2^B$$

We can now substitute $\frac{p_1}{p_2} = \frac{x_2^B}{2x_1^B}$ in the left term,

$$\frac{x_2^B}{2x_1^B}(x_1^B - 210) = 180 - x_2^B$$

which, solving for x_2^B, yields

$$x_2^B = \frac{360x_1^B}{3x_1^B - 210}$$

which constitutes the offer curve of consumer B.

- Consumer A anticipates this offer curve of consumer B, along with the following feasibility conditions

$$\begin{aligned} x_1^B &= 290 - x_1^A \quad \text{for good 1, and} \\ x_2^B &= 330 - x_2^A \quad \text{for good 2} \end{aligned}$$

From our above result of the offer curve of consumer B, $x_2^B = \frac{360x_1^B}{3x_1^B - 210}$, the feasibility condition for good 2 can be rewritten as

$$\frac{360x_1^B}{3x_1^B - 210} = 330 - x_2^A$$

which can be rearranged as

$$360x_1^B - 990x_1^B + 3x_1^B x_2^A + 69300 - 210x_2^A = 0$$

Substituting the feasibility condition for good 1, we obtain,

$$360(290 - x_1^A) - 990(290 - x_1^A) + 3(290 - x_1^A)x_2^A + 69300 - 210x_2^A = 0$$

and simplifying, yields an expression that is a function of x_1^A and x_2^A alone, that is,

$$630x_1^A - 3x_2^A x_1^A + 660x_2^A - 113,400 = 0$$

Hence, consumer A's problem becomes

$$\max_{x_1^A, x_2^A} \quad u_A(x_1^A, x_2^A) = x_1^A x_2^A$$
$$\text{subject to } 630x_1^A - 3x_2^A x_1^A + 660x_2^A - 113,400 = 0$$

with associated Lagrangian

$$\mathcal{L} = x_1^A x_2^A - \lambda[630x_1^A - 3x_2^A x_1^A + 660x_2^A - 113,400]$$

Taking FOCs yields

$$\begin{aligned} \frac{d\mathcal{L}}{dx_1^A} &= x_2^A - 630\lambda + 3x_2^A\lambda = 0 \Leftrightarrow \lambda = \frac{x_2^A}{630 - 3x_2^A} \\ \frac{d\mathcal{L}}{dx_2^A} &= x_1^A - 660\lambda + 3x_1^A\lambda = 0 \Longleftrightarrow \lambda = \frac{x_1^A}{660 - 3x_1^A} \end{aligned}$$

Setting the above FOCs equal to each other, we obtain

$$x_2^A(660 - 3x_1^A) = x_1^A(630 - 3x_2^A)$$
$$\Longrightarrow x_2^A = \frac{630}{660}x_1^A$$

Plugging this result in the constraint of consumer A, we find that

$$630x_1^A - 2.86(x_1^A)^2 + 630x_1^A - 113,400 = 0$$

Finally, solving for x_1^A yields two roots, $x_1^A = 126.08$ and $x_1^A = 314.48$, but the second root is infeasible since it exceeds the total endowment of the good. Hence, $x_1^A = 126.08$ implying that the amount of good 2 for this consumer is

$$x_2^A = \frac{630}{660}x_1^A = \frac{630}{660} \times 126.09 = 120.35$$

Using the feasibility conditions, we can obtain the equilibrium consumption bundle of individual B,

$$x_1^B = 290 - x_1^A \Rightarrow x_1^B = 163.92$$

and

$$x_2^B = 330 - x_2^A = 209.65$$

In summary, the WEA is

$$\left(x_1^A, x_2^A; x_1^B, x_2^B\right) = (126.08, 120.35; 163.92, 209.65).$$

(b) Find the Pareto optimal allocation (PEA) in this economy, and check if the WEA from part (a) is a PEA.

- For a PEA, we need

$$MRS_{1,2}^A = MRS_{1,2}^B$$

which in this setting entails

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{2x_1^B}$$

Using the feasibility conditions,

$$\begin{aligned} x_1^B &= 290 - x_1^A \\ x_2^B &= 330 - x_2^A \end{aligned}$$

Plugging x_1^B and x_2^B in terms of x_1^A and x_1^A in the $MRS_{1,2}^A = MRS_{1,2}^B$ condition, we obtain

$$\frac{x_2^A}{x_1^A} = \frac{330 - x_2^A}{2(290 - x_1^A)}$$

Rearranging, we find that the contract curve describing all PEAs is given by

$$580x_2^A - 330x_1^A - x_1^A x_2^A = 0$$

Plugging the WEA found in part (a) in this equation, we find that

$$580x_2^A - 330x_1^A - x_1^A x_2^A = 13,022.87 \neq 0$$

entailing that the WEA is not Pareto optimal, i.e., the WEA does not lie on the contract curve. Hence, the presence of market power (with one individual being the price setter) prevents the First Theorem of Welfare Economics from holding.

2. **Exercise #22 from Chapter 6 in Munoz-Garcia (2017), Production economy with CRTS.** Consider an economy with two consumers $i = \{A, B\}$, one firm (that produces good 2 using good 1 as input) and two goods $l = \{1, 2\}$. Consumer B owns the firm. Good 2 is the numeraire good (i.e., $p_2 = 1$). Consider that consumers' preferences are given by

$$u^A(x_1^A, x_2^A) = x_1^A + 4\sqrt{x_2^A} \quad \text{and} \quad u^B(x_1^B, x_2^B) = x_1^B + 2\sqrt{x_2^B}$$

while their endowments are

$$\omega^A = (4, 12) \quad \text{and} \quad \omega^B = (8, 8)$$

The production function is $y_2 = 3y_1$, and the firm operates in a perfectly competitive market facing prices $p_1 > 0$ and $p_2 > 0$. Compute the equilibrium price and allocation.

- *Consumer A.* Setting up consumer A's utility maximization problem,

$$\max_{x_1^A, x_2^A} \; x_1^A + 4\sqrt{x_2^A} + \lambda^A \left[p_1(4) + p_2(12) - p_1 x_1^A - p_2 x_2^A\right]$$

with first-order conditions

$$\begin{aligned} 1 - \lambda^A p_1 &= 0 \\ \frac{2}{\sqrt{x_2^A}} - \lambda^A p_2 &= 0 \\ p_1(4) + p_2(12) - p_1 x_1^A - p_2 x_2^A &= 0 \end{aligned}$$

combining the first two first-order conditions and rearranging yields

$$\frac{1}{p_1} = \lambda^A = \frac{\frac{2}{\sqrt{x_2^A}}}{p_2} \quad \text{or} \quad x_2^A = \frac{4p_1^2}{p_2^2}$$

Substituting this into the third first-order condition yields

$$p_1(4) + p_2(12) - p_1 x_1^A - p_2\left(\frac{4p_1^2}{p_2^2}\right) = 0 \implies x_1^A = \frac{4p_1 + 12p_2}{p_1} - \frac{4p_1}{p_2}$$

Which is positive if and only if $p_2 > \frac{p_1}{6}(\sqrt{13} - 1)$. Otherwise, consumer A would be at a corner solution where $x_1^A = 0$ and $x_2^A > 0$. In particular, he would spend all his income on good 2, that is,

$$p_2 x_2^A = p_1(4) + p_2(12) \implies x_2^A = \frac{4p_1 + 12p_2}{p_2}$$

In summary, consumer A's demand is

$$(x_1^A, x_2^A) = \begin{cases} \left(\frac{4p_1+12p_2}{p_1} - \frac{4p_1}{p_2}, \frac{4p_1^2}{p_2^2}\right) & \text{if } p_2 > \frac{p_1}{6}(\sqrt{13} - 1) \approx 0.434p_1 \\ \left(0, \frac{4p_1+12p_2}{p_2}\right) & \text{if } p_2 \leq \frac{p_1}{6}(\sqrt{13} - 1) \end{cases}$$

- *Consumer B*. Setting up consumer B's utility maximization problem,

$$\max_{x_1^B, x_2^B} \quad x_1^B + 2\sqrt{x_2^B} + \lambda^B \left[p_1(8) + p_2(8) - p_1 x_1^B - p_2 x_2^B\right]$$

with first-order conditions

$$\begin{aligned} 1 - \lambda^B p_1 &= 0 \\ \frac{1}{\sqrt{x_2^B}} - \lambda^B p_2 &= 0 \\ p_1(8) + p_2(8) - p_1 x_1^B - p_2 x_2^B &= 0 \end{aligned}$$

combining the first two first-order conditions and rearranging yields

$$\frac{1}{p_1} = \lambda^B = \frac{\frac{1}{\sqrt{x_2^B}}}{p_2} \qquad \text{or} \qquad x_2^B = \frac{p_1^2}{p_2^2}$$

Substituting this into the third first-order condition yields

$$p_1(8) + p_2(8) - p_1 x_1^B - p_2\left(\frac{p_1^2}{p_2^2}\right) = 0 \implies x_1^B = \frac{8p_1 + 8p_2}{p_1} - \frac{p_1}{p_2}$$

Which is positive if and only if $p_2 > \frac{p_1}{4}(\sqrt{6} - 2)$. Otherwise, consumer B would be at a corner solution where $x_1^B = 0$ and $x_2^B > 0$. In particular, he would spend all his income on good 2, that is,

$$p_2 x_2^B = p_1(8) + p_2(8) \implies x_2^B = \frac{8p_1 + 8p_2}{p_2}$$

In summary, consumer B's demand is

$$(x_1^B, x_2^B) = \begin{cases} \left(\frac{8p_1+8p_2}{p_1} - \frac{p_1}{p_2}, \frac{p_1^2}{p_2^2}\right) & \text{if } p_2 > \frac{p_1}{4}(\sqrt{6}-2) \approx 0.112p_1 \\ \left(0, \frac{8p_1+8p_2}{p_2}\right) & \text{if } p_2 \leq \frac{p_1}{4}(\sqrt{6}-2) \end{cases}$$

- *Equilibrium*. To solve for the equillibrium, we distinguish between 2 cases (active/inactive firm).
  - In the case in which the firm is active, we have $\frac{p_1}{p_2} = 3$ in equilibrium. Hence, $p_2 = \frac{p_1}{3}$. Howver, that entails $p_2 < \frac{p_1}{6}(\sqrt{13} - 1)$ and from our above analysis consumer A will be at a corner solution yielding demands

    $$\begin{aligned} (x_1^A, x_2^A) &= (0, 24) \quad \text{for individual } A\text{, and} \\ (x_1^B, x_2^B) &= \left(\frac{23}{3}, 9\right) \quad \text{for individual } B \end{aligned}$$

    The market clearing condition is

    $$x_1^A + x_1^B + y_1 = 12$$

    Substituting our demands, we can find the input level in equilibrium

    $$0 + \frac{23}{3} + y_1 = 12 \implies y_1 = \frac{13}{3}$$

    Finally, since $y_2 = 3y_1$, equilibrium output is $y_2 = 3 \cdot \frac{13}{3} = 13$.

– In the case in which the firm is inactive, we have that $\frac{p_1}{p_2} > 3$ in equilibrium. Recall that consumer A's consumption of good 1, when positive, is

$$x_1^A = \frac{4p_1 + 12p_2}{p_1} - \frac{4p_1}{p_2}$$

which can be rewritten as

$$x_1^A = 4\frac{p_2}{p_1}\left(\frac{p_1}{p_2} + 3 - \left(\frac{p_1}{p_2}\right)^2\right)$$

Let $X = \frac{p_1}{p_2} > 3$. Our condition becomes

$$\frac{4}{X}(-X^2 + X + 3)$$

For $X > 3$, the quadratic term in the parenthesis is negative, as is its derivative as depicted in Figure 6.20. This implies that $x_1^A < 0$ for all $\frac{p_1}{p_2} > 3$. Hence, consumer A would not consume positive amounts of good 1, $x_1^A = 0$, but instead use all his resources on good 2, i.e., $x_2^A = \frac{4p_1+12p_2}{p_2}$. Since individual B is the only consumer of good 1, $x_1^B = 12$.

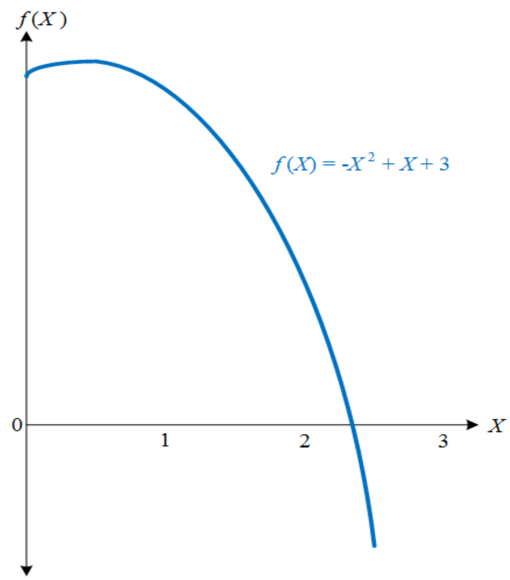

Figure 6.20. Equilibrium price ratio.

Finally, substituting in consumer B's demand yields

$$x_1^B = \frac{8p_1 + 8p_2}{p_1} - \frac{p_1}{p_2} = 12$$

and solving for $\frac{p_1}{p_2}$ yields

$$\frac{p_1}{p_2} = 2\sqrt{3} - 2 \approx 1.464 < 3$$

which contradicts our original condition, i.e. $\frac{p_1}{p_2} > 3$. Hence, there is no equilibrium in this case.

3. **Testing the first and second welfare theorems.** Consider two neighbors that trade food from their gardens (f) and groceries (g). Neighbor A has utility function $u^A(f^A, g^A) = \ln f^A + 2\ln g^A$ and neighbor B has utility function $u^B(f^B, g^B) = 2\ln f^B + \ln g^B$.

(a) Find the WEA.

- *Consumer A.* We first need the tangency condition for neighbor A, $MRS^A_{f,g} = \frac{p_f}{p_g}$, or

$$\frac{g^A}{2f^A} = \frac{p_f}{p_g},$$

which simplifies to $2f^A p^f = g^A p_g$. We can insert this into A's budget constraint

$$p_f f^A + p_g g^A = p_f 40 + p_g 25$$

to obtain

$$f^A p^f + 2f^A p^f = p_f 40 + p_g 25$$

which simplifies to $3f^A p^f = p_f 40 + p_g 25$ or

$$f^A = \frac{40}{3} + \frac{25}{3}\frac{p_g}{p_f}$$

which is neighbor A's demand for food from their garden. To find her demand for groceries, we can plug this into the tangency condition $2f^A p^f = g^A p_g$ to find that

$$2\underbrace{\left(\frac{40}{3} + \frac{25}{3}\frac{p_g}{p_f}\right)}_{f^A} p^f = g^A p_g$$

or, $\frac{80}{3}p_f + \frac{50}{3}p_g = p_g g^A$, and finally we get neighbor A's demand for groceries:

$$g^A = \frac{50}{3} + \frac{80}{3}\frac{p_f}{p_g}.$$

- *Consumer B.* Consumer B has the tangency condition:

$$\frac{2g^B}{f^B} = \frac{p_f}{p_g}$$

which simplifies to $f^B p_f = 2g^B p_g$. We can insert this into B's budget constraint

$$p_f f^B + p_g g^B = p_f 40 + p_g 25,$$

to obtain

$$g^B p_g + 2g^B p_g = p_f 40 + p_g 25$$

which simplifies to $3g^B p_g = p_f 40 + p_g 25$ or

$$g^B = \frac{25}{3} + \frac{40}{3}\frac{p_f}{p_g}$$

which is neighbor B's demand for groceries. To find her demand for food from their garden, we can plug the result of g^B into the tangency condition $f^B p_f = 2g^B p_g$ to find that

$$f^B p_f = 2\underbrace{\left(\frac{25}{3} + \frac{40}{3}\frac{p_f}{p_g}\right)}_{g^B} p_g$$

or, $f^B p_f = \frac{50}{3}p_g + \frac{80}{3}p_f$, and finally we get neighbor B's demand for food from their garden:

$$f^B = \frac{50}{3}\frac{p_g}{p_f} + \frac{80}{3}.$$

- *Equilibrium prices.* To find equilibrium prices, we can insert the demands for groceries into the feasibility condition $g^A + g^B = 25 + 25$:

$$\underbrace{\left(\frac{50}{3} + \frac{80}{3}\frac{p_f}{p_g}\right)}_{g^A} + \underbrace{\left(\frac{25}{3} + \frac{40}{3}\frac{p_f}{p_g}\right)}_{g^B} = 50$$

or,

$$\frac{75}{3} + \frac{120}{3}\frac{p_f}{p_g} = 50$$

or, $25 + 40\frac{p_f}{p_g} = 50$ and the price ratio is

$$\frac{p_f}{p_g} = \frac{25}{40}.$$

- *Equilibrium quantities.* Plugging this equilibrium price into the demand for groceries for each neighbor, obtaining that:
  - Neighbor A consumes $g^A = \frac{50}{3} + \frac{80}{3}\frac{25}{40} = 33.33$ groceries and neighbor B consumes $g^B = \frac{25}{3} + \frac{40}{3}\frac{25}{40} = 16.67$ groceries.
  - Neighbor A will consume $f^A = \frac{40}{3} + \frac{25}{3}\frac{40}{25} = 26.67$ units of food from their garden, and neighbor B will consume $f^B = \frac{50}{3}\frac{40}{25} + \frac{80}{3} = 53.33$ units of food from their garden.Find the set of PEAs.

(b) Is the WEA part of the set of PEAs? Does your result satisfy or violate the first welfare theorem?

- Yes to both questions, because efficient allocations satisfy

$$MRS^A_{f,g} = MRS^B_{f,g} = \frac{p_f}{p_g}$$

given that the MRS of consumer A is $MRS^A_{f,g} = \frac{g^A}{2f^A} = \frac{33.33}{2\times 26.67} = \frac{33.33}{53.33} = 0.62$, that of consumer B is also $MRS^B_{f,g} = \frac{2g^B}{f^B} = \frac{2\times 16.67}{53.33} = \frac{33.33}{53.33} = 0.62$, and the equilibrium price ratio is $\frac{p_f}{p_g} = \frac{25}{40} = 0.62$.

(c) Propose an allocation which, being one of the PEAs found in part (b), is different from the WEA found in part (a). How could this allocation be implemented by a social planner? Relate your results with the second welfare theorem.

- Setting the tangency condition $MRS^A_{1,2} = MRS^B_{1,2}$ yields $\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}$, or after cross multiplying, $x_2^A x_1^B = x_2^B x_1^A$. The feasibility requirement for good 1 says $x_1^A + x_1^B = 600$, or $x_1^B = 600 - x_1^A$, and similarly the feasibility requirement for good 2 says $x_2^A + x_2^B = 450$, or $x_2^B = 450 - x_2^A$. Inserting these feasibility equations into the tangency condition $x_2^A x_1^B = x_2^B x_1^A$, yields

$$x_2^A \underbrace{(600 - x_1^A)}_{x_1^B} = \underbrace{(450 - x_2^A)}_{x_2^B} x_1^A$$

which simplifies to $600x_2^A - x_1^A x_2^A = 450x_1^A - x_2^A x_1^A$, and finally to $600x_2^A = 450x_1^A$, or

$$x_2^A = \frac{450}{600} x_2^A = \frac{3}{4} x_1^A.$$

Therefore, efficient allocations satisfy

$$x_2^A = \frac{3}{4} x_1^A \quad \text{where } x_1^A \in [0, 600].$$

Here, we can pick any allocation that satisfies these conditions. For example, we could pick $x_1^A = 200$ and $x_2^A = 150$. This leaves consumer B with $x_1^B = 600 - 200 = 400$ units of good 1, and $x_2^B = 450 - 150 = 300$ units of good 2.

- *Consumer A.* Now we need to find the redistribution of initial endowment that can lead to such an allocation emerging in equilibrium. We know that for consumer A that $p_1 x_1^A = p_2 x_2^A$ and for consumer B that $p_1 x_1^B = p_2 x_2^B$. We want to now tax consumer B, $t_B < 0$, with the amount being transferred to consumer A, $t_A > 0$ so that $t_A = -t_B$. Therefore, consumer A's budget constraint after including t_A is

$$p_1 x_1^A + p_2 x_2^A = p_1 e_1^A + p_2 e_2^A + t_A$$

which, after substituting her original endowment $(e_1^A, e_2^A) = (500, 100)$ and that $p_1 x_1^A = p_2 x_2^A$, becomes

$$2p_1 x_1^A = 500p_1 + 100p_2 + t_A$$

Solving for x_1^A, we obtain

$$x_1^A = 250 + 50\frac{p_2}{p_1} + \frac{t_A}{2p_1}$$

We take this expression for consumer A and insert the specific efficient allocation that we seek to implement, that is $(x_1^A, x_2^A, x_1^B, x_2^B) = (200, 150, 400, 300)$, insert the price ratio we found in the previous problem, that is $\frac{p_2}{p_1} = \frac{4}{3}$, and normalize the price of good 2, so that $p_2 = 1$ and $p_1 = \frac{3}{4}$ in equilibrium. Doing this, we obtain:

$$200 = 250 + 50\frac{4}{3} + \frac{t_A}{2\frac{3}{4}}$$

and now we want to solve for t_A:

$$
\begin{aligned}
200 &= 250 + 50\frac{4}{3} + \frac{t_A}{2\frac{3}{4}} \\
-50 &= 66.67 + \frac{t_A}{\frac{3}{2}} \\
-116.67 &= \frac{2}{3}t_A \\
-175 &= t_A
\end{aligned}
$$

which means we tax consumer A, since $t_A < 0$.

- *Consumer* B. We apply a similar argument to consumer B, so her budget constraint as a function of the tax t_B she faces is

$$p_1 x_1^B + p_2 x_2^B = p_1 e_1^B + p_2 e_2^B + t_B$$

which, after substituting her endowment, and $p_1 x_1^B = p_2 x_2^B$, we get that

$$2p_1 x_1^B = p_1 100 + p_2 350 + t_B.$$

Solving for x_1^B, we get that

$$x_1^B = 50 + 175\frac{p_2}{p_1} + \frac{t_B}{2p_1}$$

and then inserting the specific efficient allocation that we seek to implement, that is $(x_1^A, x_2^A, x_1^B, x_2^B) = (200, 150, 400, 300)$, insert the price ratio we found in the previous problem, that is $\frac{p_2}{p_1} = \frac{4}{3}$, and normalize the price of good 2, so that $p_2 = 1$ and $p_1 = \frac{3}{4}$ in equilibrium. Doing this, we obtain:

$$400 = 50 + 175\frac{4}{3} + \frac{t_B}{2\frac{3}{4}}$$

and solving for t_B, we get that

$$
\begin{aligned}
350 &= 233.33 + \frac{t_B}{\frac{3}{2}} \\
116.76 &= \frac{2}{3}t_B \\
175 &= t_B
\end{aligned}
$$

which is subsidy to consumer B. This coincides with the tax imposed on consumer A.

4. **Stone-Geary utility function in pure exchange economy.** Consider a pure exchange economy with two individuals, A and B, whose utility functions are

$$
\begin{aligned}
u^A\left(x_1^A, x_2^A\right) &= \left(x_1^A - b_1\right)^{\frac{1}{2}}\left(x_2^A - b_2\right)^{\frac{1}{2}} \\
u^B\left(x_1^B, x_2^B\right) &= x_1^B x_2^B
\end{aligned}
$$

where $b_1, b_2 > 0$ represent the minimal amounts of goods 1 and 2 that individual A must consume in order to remain alive (such as water and shelter). Individuals A and B have endowments of $\omega^A = \left(\omega_1^A, \omega_2^A\right) = (4, 2)$ and $\omega^B = \left(\omega_1^B, \omega_2^B\right) = (2, 4)$, respectively.

(a) Set up the Lagrangian and find the individuals' Walrasian demand functions.

- *UMP of individual A.* Individual A chooses x_1^A and x_2^A to solve the following utility maximization problem,

$$\max_{x_1^A \geq b_1, x_2^A \geq b_2} u^A\left(x_1^A, x_2^A\right) = \left(x_1^A - b_1\right)^{\frac{1}{2}}\left(x_2^A - b_2\right)^{\frac{1}{2}}$$

$$\text{subject to } \; p_1 x_1^A + p_2 x_2^A = 4p_1 + 2p_2$$

Defining $\tilde{x}_1^A \equiv x_1^A - b_1$ and $\tilde{x}_2^A \equiv x_2^A - b_2$, which represent the above-subsistence consumption levels of individual A, we can rewrite his budget constraint as

$$p_1 \tilde{x}_1^A + p_2 \tilde{x}_2^A = p_1(4 - b_1) + p_2(2 - b_2)$$

Therefore, the Lagrangian function of individual A becomes

$$L_A = \left(\tilde{x}_1^A\right)^{\frac{1}{2}}\left(\tilde{x}_2^A\right)^{\frac{1}{2}} + \lambda_A\left[p_1(4 - b_1) + p_2(2 - b_2) - p_1\tilde{x}_1^A - p_2\tilde{x}_2^A\right]$$

The first order conditions of individual A's Lagrangian are

$$\frac{\partial L_A}{\partial \tilde{x}_1^A} = \frac{1}{2}\left(\frac{\tilde{x}_2^A}{\tilde{x}_1^A}\right)^{\frac{1}{2}} - \lambda_A p_1 \leq 0$$
$$\frac{\partial L_A}{\partial \tilde{x}_2^A} = \frac{1}{2}\left(\frac{\tilde{x}_1^A}{\tilde{x}_2^A}\right)^{\frac{1}{2}} - \lambda_A p_2 \leq 0$$
$$\frac{\partial L_A}{\partial \lambda_A} = p_1(4 - b_1) + p_2(2 - b_2) - p_1\tilde{x}_1^A - p_2\tilde{x}_2^A \geq 0$$

with the associated Kuhn-Tucker conditions of

$$\tilde{x}_1^A \frac{\partial L_A}{\partial \tilde{x}_1^A} = 0$$
$$\tilde{x}_2^A \frac{\partial L_A}{\partial \tilde{x}_2^A} = 0$$
$$\lambda_A \frac{\partial L_A}{\partial \lambda_A} = 0$$

Assuming interior solutions, the first order conditions hold with equality, so that by equating $\frac{\partial L_A}{\partial \tilde{x}_1^A} = \frac{\partial L_A}{\partial \tilde{x}_2^A} = 0$, we obtain

$$\frac{\frac{1}{2}\left(\frac{\tilde{x}_2^A}{\tilde{x}_1^A}\right)^{\frac{1}{2}}}{\frac{1}{2}\left(\frac{\tilde{x}_1^A}{\tilde{x}_2^A}\right)^{\frac{1}{2}}} = \frac{\lambda_A p_1}{\lambda_A p_2}$$

which, after rearranging, yields

$$p_1 \tilde{x}_1^A = p_2 \tilde{x}_2^A$$

Substituting $p_1 \tilde{x}_1^A = p_2 \tilde{x}_2^A$ into the budget constraint, we have

$$2p_1 \tilde{x}_1^A = p_1 (4 - b_1) + p_2 (2 - b_2)$$

which is rearranged to give individual A's Walrasian demand of good 1,

$$\tilde{x}_1^A = \frac{4 - b_1}{2} + \frac{p_2 (2 - b_2)}{2p_1}$$

and, similarly, we can obtain individual A's Walrasian demand of good 2,

$$\tilde{x}_2^A = \frac{p_1 (4 - b_1)}{2p_2} + \frac{2 - b_2}{2}$$

- *UMP of individual B.* Individual B chooses x_1^B and x_2^B to solve the following utility maximization problem,

$$\max_{x_1^B, x_2^B \geq 0} u^B \left(x_1^B, x_2^B\right) = x_1^B x_2^B$$

subject to

$$p_1 x_1^B + p_2 x_2^B = 2p_1 + 4p_2$$

The Lagrangian function of individual B becomes

$$L_B = x_1^B x_2^B + \lambda_B \left[2p_1 + 4p_2 - p_1 x_1^B - p_2 x_2^B\right]$$

The first order conditions of individual B's Lagrangian are

$$\frac{\partial L_B}{\partial x_1^B} = x_2^B - \lambda_B p_1 \leq 0$$
$$\frac{\partial L_B}{\partial x_2^B} = x_1^B - \lambda_B p_2 \leq 0$$
$$\frac{\partial L_B}{\partial \lambda_B} = 2p_1 + 4p_2 - p_1 x_1^B - p_2 x_2^B \geq 0$$

with the associated Kuhn-Tucker conditions of

$$x_1^B \frac{\partial L_B}{\partial x_1^B} = 0$$
$$x_2^B \frac{\partial L_B}{\partial x_2^B} = 0$$
$$\lambda_B \frac{\partial L_B}{\partial \lambda_B} = 0$$

Assuming interior solutions, the first order conditions hold with equality, so that by equating $\frac{\partial L_B}{\partial x_1^B} = \frac{\partial L_B}{\partial x_2^B} = 0$, we obtain

$$\frac{x_2^B}{x_1^B} = \frac{\lambda_B p_1}{\lambda_B p_2}$$

which, after rearranging, yields

$$p_1 x_1^B = p_2 x_2^B$$

Substituting $p_1 x_1^B = p_2 x_2^B$ into the budget constraint, we have

$$2p_1 x_1^B = 2p_1 + 4p_2$$

which is rearranged to give individual B's Walrasian demand of good 1,

$$x_1^B = 1 + 2\frac{p_2}{p_1}$$

and, similarly, we can obtain individual B's Walrasian demand of good 2,

$$x_2^B = \frac{p_1}{p_2} + 2$$

(b) Find the set of Pareto efficient allocations (PEAs). (*Hint*: Your answer should be in terms of b_1 and b_2).

- The feasibility constraints in this pure exchange economy are

$$\underbrace{\left(\tilde{x}_1^A + b_1\right)}_{=x_1^A} + x_1^B = 4 + 2$$
$$\underbrace{\left(\tilde{x}_2^A + b_2\right)}_{=x_2^A} + x_2^B = 2 + 4$$

which are rearranged to give

$$\tilde{x}_1^A = 6 - b_1 - x_1^B$$
$$\tilde{x}_2^A = 6 - b_2 - x_2^B$$

The contract curve, which defines the set of Pareto efficient allocations, is the locus of tangency of indifference curves between individuals A and B, satisfying

$$MRS_{12}^A = \frac{MU_1^A}{MU_2^A} = \frac{MU_1^B}{MU_2^B} = MRS_{12}^B$$

which is rearranged to give

$$\frac{\tilde{x}_2^A}{\tilde{x}_1^A} = \frac{x_2^B}{x_1^B}$$

Substituting the feasibility constraints into the above expression, we obtain

$$\frac{6 - b_2 - x_2^B}{6 - b_1 - x_1^B} = \frac{x_2^B}{x_1^B}$$

which, after rearranging, yields the contract curve as follows,

$$x_2^B = \frac{6 - b_2}{6 - b_1} x_1^B$$

(c) Find the Walrasian equilibrium allocation (WEA). (*Hint*: Your answer should be in terms of b_1 and b_2).

- Substituting the Walrasian demands for good 1 into $\tilde{x}_1^A + x_1^B = 6 - b_1$, we obtain

$$\underbrace{\left(\frac{4-b_1}{2} + \frac{p_2\left(2-b_2\right)}{2p_1}\right)}_{=\tilde{x}_1^A} + \underbrace{\left(1 + 2\frac{p_2}{p_1}\right)}_{=x_1^B} = 6 - b_1$$

which is rearranged to yield the equilibrium price ratio, as follows.

$$\begin{aligned}\left(6-b_2\right)\frac{p_2}{p_1} + \left(6-b_1\right) &= 2\left(6-b_1\right)\\ \Rightarrow \frac{p_1}{p_2} &= \frac{6-b_2}{6-b_1}\end{aligned}$$

Substituting $\frac{p_1}{p_2} = \frac{6-b_2}{6-b_1}$ into the Walrasian demand functions of individual A, the Walrasian equilibrium allocation (WEA) of this individual becomes

$$\begin{aligned}\tilde{x}_1^A &= \frac{4-b_1}{2} + \frac{\left(2-b_2\right)}{2}\cdot\frac{6-b_1}{6-b_2}\\ &= \frac{\left(4-b_1\right)\left(6-b_2\right) + \left(6-b_1\right)\left(2-b_2\right)}{2\left(6-b_2\right)}\\ &= \frac{18-4b_1-5b_2+b_1b_2}{6-b_2}\\ \tilde{x}_2^A &= \frac{\left(4-b_1\right)}{2}\cdot\frac{6-b_2}{6-b_1} + \frac{2-b_2}{2}\\ &= \frac{\left(4-b_1\right)\left(6-b_2\right) + \left(6-b_1\right)\left(2-b_2\right)}{2\left(6-b_1\right)}\\ &= \frac{18-4b_1-5b_2+b_1b_2}{6-b_1}\end{aligned}$$

Given $\tilde{x}_1^A \equiv x_1^A - b_1$ and $\tilde{x}_2^A \equiv x_2^A - b_2$, we can rewrite the above expressions as

$$\begin{aligned}x_1^A &= \tilde{x}_1^A + b_1\\ &= \frac{18-4b_1-5b_2+b_1b_2}{6-b_2} + b_1\\ &= \frac{18+2b_1-5b_2}{6-b_2}\\ x_2^A &= \tilde{x}_2^A + b_2\\ &= \frac{18-4b_1-5b_2+b_1b_2}{6-b_1} + b_2\\ &= \frac{18-4b_1+b_2}{6-b_1}\end{aligned}$$

Similarly, the Walrasian equilibrium allocation (WEA) of individual B is

$$\begin{aligned} x_1^B &= 1 + 2 \cdot \frac{6 - b_1}{6 - b_2} \\ &= \frac{18 - 2b_1 - b_2}{6 - b_2} \\ x_2^B &= \frac{6 - b_2}{6 - b_1} + 2 \\ &= \frac{18 - 2b_1 - b_2}{6 - b_1} \end{aligned}$$

(d) Evaluate the contract curve and WEA at the following three different subsistence levels: (i) $(b_1, b_2) = (4, 2)$, (ii) $(b_1, b_2) = (3, 3)$, and (iii) $(b_1, b_2) = (2, 4)$. In which case(s) is individual A unable to survive?

- *First case.* Substituting $(b_1, b_2) = (4, 2)$ into the Walrasian equilibrium allocation,

$$\begin{aligned} x_1^{A*} &= \frac{18 + 2 \cdot 4 - 5 \cdot 2}{6 - 2} = 4 \\ x_2^{A*} &= \frac{18 - 4 \cdot 4 + 2}{6 - 4} = 2 \\ x_1^{B*} &= \frac{18 - 2 \cdot 4 - 2}{6 - 2} = 2 \\ x_2^{B*} &= \frac{18 - 2 \cdot 4 - 2}{6 - 4} = 4 \\ \frac{p_1}{p_2} &= \frac{6 - 2}{6 - 4} = 2 \end{aligned}$$

Summarizing, the WEA of

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (4, 2; 2, 4; 2)$$

which means that individuals do not exchange their goods, and individual A can survive by consuming endowment ω^A. The contract curve in this context is

$$x_2^B = \frac{6 - 2}{6 - 4} x_1^B = 2x_1^B$$

- *Second case.* Substituting $(b_1, b_2) = (3, 3)$ into the Walrasian equilibrium allocation, we find

$$\begin{aligned} x_1^{A*} &= \frac{18 + 2 \cdot 3 - 5 \cdot 3}{6 - 3} = 3 \\ x_2^{A*} &= \frac{18 - 4 \cdot 3 + 3}{6 - 3} = 3 \\ x_1^{B*} &= \frac{18 - 2 \cdot 3 - 3}{6 - 3} = 3 \\ x_2^{B*} &= \frac{18 - 2 \cdot 3 - 3}{6 - 3} = 3 \\ \frac{p_1}{p_2} &= \frac{6 - 3}{6 - 3} = 1 \end{aligned}$$

Intuitively, individual A (B) exchanges 1 unit of good 1 (2) for 1 unit of good 2 (1) to yield the WEA

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}\right) = (3, 3; 3, 3; 1),$$

such that individual A can remain alive with this trade. The contract curve in this setting is

$$x_2^B = \frac{6-3}{6-3} x_1^B = x_1^B$$

- *Third case.* Substituting $(b_1, b_2) = (2, 4)$ into the Walrasian equilibrium allocation, we obtain

$$\begin{aligned}
x_1^{A*} &= \frac{18 + 2 \cdot 2 - 5 \cdot 4}{6-4} = 1 \\
x_2^{A*} &= \frac{18 - 4 \cdot 2 + 4}{6-2} = \frac{7}{2} \\
x_1^{B*} &= \frac{18 - 2 \cdot 2 - 4}{6-4} = 5 \\
x_2^{B*} &= \frac{18 - 2 \cdot 2 - 4}{6-2} = \frac{5}{2} \\
\frac{p_1}{p_2} &= \frac{6-4}{6-2} = \frac{1}{2}
\end{aligned}$$

Summarizing, the WEA is

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}\right) = (1, 3.5; 5, 2.5; 0.5)$$

It is easy to check that, at this allocation, individual A's utility is negative, entailing that he cannot survive. In part (e) of the exercise, we examine a wealth redistribution program to keep this individual alive.
The contract curve in this context is

$$x_2^B = \frac{6-4}{6-2} x_1^B = \frac{1}{2} x_1^B$$

(e) Consider now a tax transfer so individual A survives in the case(s) you identify in part (b) where he suffers from a negative utility at the WEA. Identify the tax/transfer that the government can impose, and the resulting WEA. (For compactness, let us normalize $p_2 = 1$ so that $p \equiv p_1 = \frac{p_1}{p_2}$.)

- Suppose the government levies a tax t on individual B to provide it to individual A as a transfer. In this context, the budget constraint of individual A becomes

$$p\tilde{x}_1^A + \tilde{x}_2^A = p(4 - b_1) + (2 - b_2) + t$$

Substituting $p\tilde{x}_1^A = \tilde{x}_2^A$ and the price ratio $p = \frac{6-b_2}{6-b_1}$ into the budget constraint of individual A, we obtain

$$\begin{aligned}
\tilde{x}_1^A &= \frac{4-b_1}{2} + \frac{2-b_2+t}{2p} \\
&= \frac{4-b_1}{2} + \frac{2-b_2+t}{2} \cdot \frac{6-b_1}{6-b_2} \\
&= \frac{2\left(18-4b_1-5b_2+b_1b_2\right)+\left(6-b_1\right)t}{2\left(6-b_2\right)} \\
\tilde{x}_2^A &= \frac{p\left(4-b_1\right)}{2} + \frac{2-b_2+t}{2} \\
&= \frac{4-b_1}{2} \cdot \frac{6-b_2}{6-b_1} + \frac{2-b_2+t}{2} \\
&= \frac{2\left(18-4b_1-5b_2+b_1b_2\right)+\left(6-b_1\right)t}{2\left(6-b_1\right)}
\end{aligned}$$

Substituting the subsistence level of the third case we analyzed in part (d) of the exercise, $(b_1, b_2) = (2, 4)$, into the above expressions, yields

$$\begin{aligned}
\tilde{x}_1^A &= \frac{2\cdot(-2)+4t}{4} = t-1 \\
\tilde{x}_2^A &= \frac{2\cdot(-2)+4t}{8} = \frac{t-1}{2}
\end{aligned}$$

Therefore, to ensure individual A can remain alive, we need

$$\begin{aligned}
\tilde{x}_1^A &\geq 0 \\
\tilde{x}_2^A &\geq 0
\end{aligned}$$

which is equivalent to

$$t = 1$$

Therefore, the equilibrium allocation of individual A is

$$\begin{aligned}
x_1^{A*} &= \tilde{x}_1^A + b_1 = 0 + 2 = 2 \\
x_2^{A*} &= \tilde{x}_2^A + b_2 = 0 + 4 = 4
\end{aligned}$$

- The budget constraint of individual B becomes now

$$px_1^B + x_2^B = 2p_1 + 4p_2 - t$$

Substituting $px_1^B = x_2^B$ into the budget constraint of individual B, we have

$$\begin{aligned}
x_1^B &= 1 + \frac{4-t}{2p} \\
x_2^B &= p + \frac{4-t}{2}
\end{aligned}$$

Further substituting $p = \frac{1}{2}$ and $t = 1$ into the above expressions, we obtain the equilibrium allocation of individual B, as follows

$$x_1^{B*} = 1 + \frac{4-1}{2 \cdot \frac{1}{2}} = 4$$
$$x_2^{B*} = \frac{1}{2} + \frac{4-1}{2} = 2$$

Therefore, the Walrasian equilibrium allocation (WEA) becomes

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (2, 4; 4, 2; 0.5)$$

which is supported by a tax-transfer, $t^* = 1$, from individual B to individual A.