Cournot competition with $n$ asymmetric firms

1. **Cournot competition with $n$ firms facing asymmetric costs.** Consider an industry of $n \geq 2$ firms competing a la Cournot. Firms face an inverse demand curve $p(Q) = a - Q$, where $Q \geq 0$ denotes aggregate output. Every firm $i$ has a marginal cost of production $c_i$, where $a > c_i \geq 0$.

   (a) Set up firm $i$’s profit-maximization problem and find its first-order condition.

   - Every firm $i$ chooses its output $q_i$ to solve
     \[
     \max_{q_i \geq 0} [a - (q_i + Q_{-i})] q_i - c_i q_i
     \]
     where $Q_{-i}$ denotes the aggregate output of firm $i$’s rivals. Differentiating with respect to $q_i$, yields
     \[
     a - 2q_i - Q_{-i} - c_i = 0
     \]
     which we can rearrange as
     \[
     a - c_i = 2q_i + Q_{-i}.
     \]

   (b) Find equilibrium output. *[Hint: Invoke symmetry in the first-order condition that you found in part (a). Then sum over all $n$ firms.]*

   - In a symmetric equilibrium, we have that $q_i = q_j = q$, which entails $Q_{-i} = (n - 1)q$. Inserting this result into the above first-order condition, we obtain
     \[
     a - c_i = 2q + (n - 1)q
     \]
     Summing over the first-order conditions of all $n$ firms, yields
     \[
     na - \sum_{i=1}^{n} c_i = 2 \sum_{i=1}^{n} q + (n - 1) \sum_{i=1}^{n} q
     \]
     Denoting, for compactness, $C = \sum_{i=1}^{n} c_i$ for the aggregate costs, and $\sum_{i=1}^{n} q = Q$ for aggregate output, the above expression becomes
     \[
     na - C = (n + 1)Q
     \]
     which yields an aggregate output in equilibrium of
     \[
     Q = \frac{na - C}{n + 1}
     \]

---

1Felix Munoz-Garcia, Associate Professor in Economics, Address: 103H Hulbert Hall, Washington State University, Pullman, WA 99164, USA. E-mail: fmunoz@wsu.edu.
Therefore, the equilibrium of each individual firm is
\[ q_i = \frac{Q}{n} = \frac{na - C}{n(n+1)}, \]
and equilibrium profits are
\[ \pi_i = \left( a - c_i - \frac{na - C}{n+1} \right) \frac{na - C}{n(n+1)}. \]

(c) **First example.** Consider a setting with \( n = 2 \) firms (firm 1 and 2) facing inverse demand function \( p(Q) = 1 - Q \), and marginal production costs \( c_1 \) and \( c_2 \), where \( 1 > c_i \geq 0 \) for every firm \( i = \{1, 2\} \). Evaluate your results from part (b) to find the equilibrium output for each firm, aggregate output, and profits. Then evaluate your results in the case that marginal production costs coincide, \( c_1 = c_2 = c \), where \( 1 > c \geq 0 \).

- **Asymmetric costs, \( c_1 \neq c_2 \).** In this context, the sum of marginal costs costs is \( C = c_1 + c_2 \), and demand parameters are \( a = b = 1 \). Therefore, aggregate output becomes
  \[ Q = \frac{2 - (c_1 + c_2)}{2 + 1} = \frac{2 - (c_1 + c_2)}{3} \]
  individual output is
  \[ q_i = \frac{Q}{2} = \frac{2 - (c_1 + c_2)}{6} \]
  and profits become
  \[ \pi_i = \frac{1 + c_j - 2c_i}{3} \frac{2 - (c_1 + c_2)}{6} \]
- **Symmetric costs, \( c_1 = c_2 = c \).** In this setting, the above results become
  \[ Q = \frac{2(1 - c)}{3} \]
  individual output is
  \[ q_i = \frac{Q}{2} = \frac{1 - c}{3} \]
  and profits become
  \[ \pi_i = \frac{(1 - c)^2}{9} \]

(d) **Second example.** Consider a setting with \( n \geq 2 \) firms facing inverse demand function \( p(Q) = 1 - Q \), and symmetric marginal production cost \( c \), where \( 1 > c \geq 0 \). Assuming that \( k \) firms merge, benefiting from a lower marginal cost \( c - x \), while the \( n - k \) unmerged firms still face marginal cost \( c \). Find the aggregate output in equilibrium when \( k \) firms merge, and compare it against aggregate output before the merger. For which parameter values the merger produces an increase in aggregate output?
• **Before the merger.** With \( n \) firms in the industry, all facing marginal cost \( c \), the sum of marginal costs is \( C = nc \). Therefore, expression \( Q = \frac{na - C}{n+1} \), we can then write aggregate output in this setting as

\[
Q^{NM} = \frac{n - nc}{n + 1} = \frac{n(1 - c)}{n + 1}
\]

since \( a = 1 \), where superscript \( NM \) denotes “no merger.”

• **After the merger.** If \( k \) out of \( n \) firms merge, leaving \( n - k \) firms unmerged, then there are \( (n - k) + 1 \) firms in the industry. In this context, the sum of marginal costs is

\[
C = \underbrace{c - x}_{\text{Merged firm}} + \underbrace{(n - k)c}_{\text{Unmerged firms}} = (n - k + 1)c - x.
\]

Using expression \( Q = \frac{na - C}{n+1} \), we can then write aggregate output in this setting as

\[
Q^M = \frac{[(n - k) + 1] - [(n - k + 1)c - x]}{[(n - k) + 1] + 1}
= \frac{(n - k + 1)(1 - c) + x}{n - k + 2}
\]

since \( a = 1 \), where superscript \( M \) denotes “merger.”

• **Output comparison.** Aggregate output after the merger increases if \( Q^M \geq Q^{NM} \), which entails

\[
\frac{(n - k + 1)(1 - c) + x}{n - k + 2} \geq \frac{n(1 - c)}{n + 1}.
\]

Rearranging, we obtain

\[
\theta \equiv \frac{x}{1 - c} \geq \frac{k + 1}{n + 1}.
\]

Intuitively, the merger increases aggregate output (and, as a consequence, consumer surplus) if the cost-reduction effect relative to firms’ margin (left-hand side, \( \theta \)) is sufficiently large.

As an illustration, we can fix the total number of firms at \( n = 10 \), and evaluate cutoff \( \frac{k + 1}{n + 1} \) at \( k = 2 \), obtaining that

\[
\frac{2 + 1}{10 + 1} = 0.27.
\]

Intuitively, the cost-reduction effect, relative to per-unit margins (as measured by \( \theta \)), must be larger than 27\% for the merger to increase consumer surplus. Mergers between more firms (higher \( k \)) produce an even larger ratio \( \frac{k + 1}{n + 1} \), thus increasing the minimum cost-reduction effect, \( \theta \), required for the merger to increase consumer surplus.