Strategic Complementarities in Oligopoly

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1 Introduction

Oligopoly theory is closely connected with game theory. Indeed, oligopoly competition is the leading example of strategic interaction and it should suffice to mention that the equilibrium concept of Cournot is just the modern Nash equilibrium. Modeling strategic interaction presents formidable problems as the founders of oligopoly theory (Cournot, Bertrand, Edgeworth, Chamberlin, Robinson and Hotelling) made clear. The oligopoly problem was to establish where would prices settle when market conditions were neither monopoly nor perfect competition. Technical problems in the analysis include lack of quasi-concavity and smoothness of payoffs, indivisibilities, and complex strategy spaces. A Nash equilibrium may not exist, at least in pure strategies. Or, instead, there may be multiple equilibria: How do players coordinate on one of them? How can policy intervention ensure that changing a parameter will have the desired effect? Classical comparative statics analysis provides ambiguous results in the presence of multiple equilibria and imposes strong regularity conditions. These regularity conditions become particularly strong when applied to games with complex functional strategy spaces, such as dynamic or Bayesian games.

Complementarities are intimately linked to multiple equilibria and have a deep connection with strategic situations, and the concept of strategic complementarity is at the center stage of game-theoretic analyses. Examples abound from price games with differentiated products, R&D races, technology adoption, and store and brand location.

Lattice-theoretic methods provide the appropriate toolbox to deal with the problems encountered in oligopoly theory, in particular when complementarities are involved. The theory of monotone comparative statics and supermodular games exploits both order and monotonicity properties (Topkis (1978, 1979) and further developed and applied to economics by Vives (1985a, 1990a) and Milgrom and Roberts (1990a)). By now it has proved useful not only in oligopoly theory but in all fields of economics from macroeconomics and finance to development and international trade. It continues to be extended at the frontier of research— for example, to dynamic games and games of incomplete information.

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Monotone comparative statics analysis provides conditions under which optimal solutions to optimization problems move monotonically with a parameter. This approach exploits order and monotonicity properties, in contrast to classical convex analysis. A central piece of attention are games of strategic complementarities, where the best response of a player to the actions of rivals is increasing in their level, and monotone comparative statics results that allow to extend the approach to more general games. In this chapter I provide an introduction to this methodology and then apply it to the study of strategic interaction in the presence of complementarities in oligopoly games.

The achievements of this approach are as follows. First, it provides a framework for thinking rigorously about complementarities, identifying key parameters in the environment to look at (e.g., what are the critical properties of the payoffs and action spaces). Second, it simplifies the analysis, clarifying the drivers of the results (e.g., what regularity conditions are really needed to obtain the desired comparative static results). Third, it encompasses the analysis of multiple equilibria situations by ranking equilibria and helping understand how potential equilibria move with the parameters of interest. Finally, it easily incorporates complex strategy spaces, including indivisibilities and functional spaces, such as those arising in dynamic games and games of incomplete information. More specifically, the approach:

- ensures the existence of equilibrium in pure strategies (without requiring quasiconcavity of payoffs or smoothness assumptions) in games of strategic complementarities and beyond;

- allows a global analysis of the equilibrium set, which has an order structure with largest and smallest elements, equilibrium has useful stability properties, and there is an algorithm to compute extremal equilibria.

- permits the use of monotone comparative statics analysis with minimal assumptions by either focusing on extremal equilibria or considering best–response dynamics after the perturbation; and

- results extend beyond the class of games with strategic complementarities.

However, we should be aware also that the lattice-theoretic approach is not a panacea and cannot be applied to everything. Indeed, the approach builds on a set of assumptions.

The chapter provides an introduction to the tools of supermodular games and a range of applications to industrial organization. Section 2 provides an introduction to the theory and basic results. Section 3 provides applications to oligopoly and comparative statics in the context of Cournot, Bertrand, R&D, advertising, multidimensional and multimarket competition. Section 4 deals with dynamic games studying entry, characterizing strategic incentives in two stage and Markov games. Applications to menu and adjustment costs are provided. Section 5 studies games of incomplete information, characterizing equilibria in pure strategies and comparative statics properties, with applications (among others) to games of voluntary disclosure and auctions. The Appendix provides a brief recollection of the most important definitions and results of the lattice-theoretic method.

2 The framework: supermodular games and monotone comparative statics

In this section we provide a brief introduction to the tools and main results of the theory of monotone comparative statics and supermodular games. Those tools are based on lattice-theoretic results that exploit order and monotonicity properties of action sets and payoffs. Assumptions are put on strategy sets and payoffs so that best responses are increasing and move monotonically with the parameters under study (following Topkis (1978). Tarski's (1955) fixed point theorem delivers then existence of equilibrium and ord). properties of the equilibrium set. This section provides the background to follow the rest of the chapter and the Appendix contains technical definitions and intermediate results.[1]

For simplicity, I provide a definition of a supermodular game in a smooth context. Consider a game $(A_i, \pi_i; i \in N)$, where N is the set of players, $i = 1, ..., n$; for player $i \in N$, A_i is the strategy set, a compact cube in Euclidean space, $a_i \in A_i$, and π_i his payoff (defined on the cross product of the strategy spaces of the players A). Let a_{-i} denote the strategy profile $(a_1, ..., a_n)$ excepting the th element, $a_{-i} \in \prod_{j \neq i} A_j$. Let a_{ih} denote the hth component of the strategy a_i of player i. The game $(A_i, \pi_i; i \in N)$ is *smooth supermodular* if, for all i, $\pi_i(a_i, a_{-i})$ is twice continuously differentiable,

1. supermodular in a_i for fixed a_{-i} or $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$; and

2. with increasing differences in (a_i, a_{-i}) or $\partial^2 \pi_i / \partial a_{ih} \partial a_{jk} \geq 0$ for all $j \neq i$ and for all h and k.

The game is smooth *strictly* supermodular if the inequality in (2) is strict.

Condition (1) is a complementarity property in own strategies: the marginal payoff to any strategy of player i is increasing in the other strategies of the player. Condition (2) is a strategic complementarity property in rivals' strategies a_{-i}: the marginal payoff to any strategy of player i is increasing in any strategy of any rival player.

In a supermodular game, general strategy spaces can be allowed, including indivisibilities as well as functional strategy spaces, such as those arising in dynamic or Bayesian games (as we will see in sections 4.3 and 5). Regularity conditions (such as concavity and interior solutions) can be dispensed with.[2]

In a supermodular game each player i has a largest, $\overline{\Psi}_i(a_{-i}) = \sup \Psi_i(a_{-i})$, and a smallest, $\underline{\Psi}_i(a_{-i}) = \inf \Psi_i(a_{-i})$, best reply and they are increasing in the strategies of the other players. If the game is strictly supermodular, then any selection from the best–reply correspondence is increasing.[3]

[1] See Chapter 2 of Vives (1999) and Topkis (1998) for a more thorough treatment.

[2] In the general formulation of a supermodular game, strategy spaces need only be "complete lattices", only continuity (not differentiability) of payoffs is needed, and properties (1) and (2) are stated in non-differential terms. A continuity requirement is needed to ensure the existence of best replies. See the Appendix for the general definitions of lattices, supermodularity, increasing differences, and supermodular game.

[3] The basic monotone comparative statics result states that the set of optimizers of a function $u(x, t)$ that is parameterized by t, supermodular in x, and with increasing differences in x and t has a largest and a smallest element and that both are increasing in t. See Lemma 1 in the Appendix for a precise statement of the result.

The (weaker) concept of *game of strategic complementarities* (GSC), under our maintained assumptions, is a game where: (a) strategy sets are compact cubes (or "complete lattices"); (b) the best reply of any player has extremal (largest and smallest) elements; and (c) those extremal elements are increasing in the strategies of rivals. Similarly, a game of strict strategic complementarities would have, in addition, any selection from the best reply of any player is increasing in the strategies of the rivals.[4] The results stated below will hold, replacing (strictly) supermodular game by GSC (game of strict SC).

The following results in sections 2.1-2.4 hold in a supermodular game. Let $\overline{\Psi} = \left(\overline{\Psi}_1, ..., \overline{\Psi}_n\right)$ and $\underline{\Psi} = (\underline{\Psi}_1, ..., \underline{\Psi}_n)$ denote the extremal best–reply maps. Consider as a maintained example a Bertrand oligopoly with differentiated gross substitutable products, with each firm producing a different variety and constant marginal costs.

2.1 Existence and characterization of the equilibrium set

There always exist extremal equilibria in a supermodular game: a largest equilibrium $\overline{a} = \sup\left\{a \in A : \overline{\Psi}(a) \geq a\right\}$ and a smallest equilibrium $\underline{a} = \inf\left\{a \in A : \underline{\Psi}(a) \leq a\right\}$ of the equilibrium set (Topkis (1979)).[5]

In the Bertrand oligopoly when the payoffs fulfil the complementary conditions (to be discussed in Section 3.1) then it follows that extremal price equilibria do exist.

Symmetric games. Consider a symmetric supermodular game (exchangeable against permutations of the players), then[6]:

- The extremal equilibria $\overline{a}$ and $\underline{a}$ are symmetric. Hence, if there is a unique symmetric equilibrium then the equilibrium is unique (since $\overline{a} = \underline{a}$). This result proves useful, for example, to show uniqueness in standard versions of symmetric Bertrand oligopoly models).

- All equilibria are symmetric equilibria if the game is strictly supermodular and the strategy spaces of the players are one–dimensional (or, more generally, completely ordered).

Welfare. In a supermodular game, if the payoff to a player has positive spillovers (i.e., it is increasing in the strategies of the other players) then the largest (resp., smallest) equilibrium point is the Pareto best (resp., worst) equilibrium. This is at the base of finding equilibria that can be Pareto ranked in games with strategic complementarities (Milgrom and Roberts (1990a), Vives (1990a)). For example, in the Bertrand oligopoly case, the profits associated with the largest price equilibrium are also the highest for every firm.

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[4] This definition was used in Vives (1985a). See the Appendix for a more formal definition along those lines.

[5] The result is shown by applying Tarski's fixed point theorem (which implies that an increasing function from a compact cube into itself has a largest and a smallest fixed point; see Appendix) to the extremal selections of the best-reply map $\overline{\Psi}$ and $\underline{\Psi}$, which are monotone by the strategic complementarity assumptions. There is no reliance on quasi–concave payoffs and convex strategy sets to deliver convex-valued best replies, as is required when showing existence using Kakutani's fixed point theorem. Furthermore, the equilibrium set of a supermodular game is a complete lattice (see Vives (1985a), Vives (1990a), Problem 2.5 in Vives (1999), and Zhou (1994)).

[6] See Vives (1985a) and Vives (1999).

2.2 Stability and rationalizability

In a supermodular game with continuous payoffs:

1. Simultaneous response best-reply dynamics (Vives (1990a))

   - approach the "box" $[\underline{a}, \overline{a}]$ defined by the smallest and the largest equilibrium points of the game;
   - converge monotonically downward (upward) to an equilibrium when starting at any point in the intersection of the upper (lower) contour sets of the largest (smallest) best replies of the players $A^+ \equiv \left\{a \in A : \overline{\Psi}(a) \leq a\right\}$ ($A^- \equiv \left\{a \in A : \underline{\Psi}(a) \geq a\right\}$).

2. The extremal equilibria $\underline{a}$ and $\overline{a}$ correspond to the largest and smallest serially undominated strategies (Milgrom and Roberts (1990a)).

This result implies that all relevant strategic action is happening in the box $[\underline{a}, \overline{a}]$ defined by the smallest and largest equilibrium points.[7] Results extend to a large class of adaptive dynamics, of which best–reply dynamics are a particular case. A corollary is that if the equilibrium is unique then it is globally stable and dominance solvable. An example is the Bertrand oligopoly market with linear, constant elasticity, or logit demands, where the equilibrium is unique.

2.3 Comparative statics

Consider a n-player supermodular oligopoly game with payoff for firm i, $\pi_i(a_i, a_{-i}; t)$, parameterized by a vector $t = (t_1, ..., t_n)$. If π_i has increasing differences in (a_i, t) (i.e. $\partial^2\pi_i/\partial a_{ih}\partial t_j \geq 0$ for all h and j) then with an increase in t:

(i) the largest and smallest equilibrium points increase; and

(ii) starting from any equilibrium, best–reply dynamics lead to a (weakly) larger equilibrium following the parameter change.[8]

Increasing actions by one player reinforce the desire of all other players to increase their actions, and the increases are mutually reinforcing (i.e., they exhibit positive feedback). We can think in terms of multiplier effects as pointed out in Vives (2005a). Indeed, a multiplier effect in the parameter t_j obtains if the equilibrium reaction of each player to a change in the parameter is strictly larger than the reaction of the player keeping the strategies of the other players constant. This will happen, for example, in a smooth strictly supermodular game with one–dimensional strategy spaces for which $\partial^2\pi_i/\partial a_i\partial t_j \geq 0$ with strict inequality for at least one firm if we consider extremal equilibria (or following best–reply adjustment dynamics after a parameter change).[9]

As an example consider the Bertrand oligopoly fulfiling the complementarity conditions. There, extremal equilibrium price vectors will be increasing in an excise tax t since $\partial^2\pi_i/\partial p_i\partial t > 0$ whenever demand is strictly downward sloping.

[7] For example, rationalizable outcomes (Bernheim (1984), Pearce (1984)) and supports of mixed–strategy and correlated equilibria must lie in the box $[\underline{a}, \overline{a}]$.

[8] The result holds for a class of adaptive dynamics, including fictitious play and gradient dynamics. Furthermorel, continuous equilibrium selections that do not increase monotonically with t predict unstable equilibria (Echenique (2002)). The comparative statics result is generalized in Milgrom and Shanon (1994).

[9] See Peitz (2000) for sufficient conditions for a price game to display multiplier effects.

Multiplier effects can be related to the *LeChatelier-Samuelson principle* in a strategic environment. This principle states that the response of an agent to a shock will be smaller in the short-run than in the long-run when other related actions can also be adjusted. Alexandrov and Bedre-Defolie (2016) show, indeed, that the principle holds for extremal equilibria of supermodular games as in the result both for idiosyncratic and common shocks; and in other games under more restrictive conditions for idiosyncratic shocks.[10]

In games with strategic complementarities, we have a multidimensional global version of Samuelson's (1979) *correspondence principle*. This principle links unambiguous comparative statics with stable equilibria and is obtained with standard calculus methods applied to interior and stable one-dimensional models. In GSC, unambiguous monotone comparative statics obtain if we concentrate on stable equilibria.

2.4 Duopoly with strategic substitutability

Consider a duopoly ($n = 2$) where there is (a) strategic complementarity in own strategies, with π_i supermodular in a_i or $\partial^2\pi_i/\partial a_{ih}\partial a_{ik} \geq 0$ for all $k \neq h$, and (b) strategic substitutability in rivals' strategies, with π_i with decreasing differences in (a_i, a_j) or $\partial^2\pi_i/\partial a_{ih}\partial a_{jk} \leq 0$ for all $j \neq i$ and for all h and k. Then the transformed game with new strategies $s_1 = a_1$ and $s_2 = -a_2$ is (smooth) supermodular (Vives (1990a)). it follows that all the results stated previously apply to this duopoly game as well. However, the extension to the strategic substitutability case for n players does not apply since the transformation does not work for $n > 2$.

A typical example is a Cournot duopoly with gross substitutes, where typically - but not always- best replies are decreasing. In this case, if for some players payoffs are increasing in the strategies of rivals and for other players they are decreasing, then the largest equilibrium is best for the former and worst for the latter. We have that the preferred equilibrium for a firm is the one in which its output is largest and the output of the rival lowest.

2.5 Extensions to non-supermodular games

Totally ordered strategy spaces For totally ordered strategy spaces (e.g. one-dimensional, say a subset of the real line) existence of symmetric equilibrium in n-player games can be obtained relaxing the monotonicity requirement of best responses (which characterizes supermodular games). The result follows from Tarski's intersection point theorem[11] between a quasi-increasing function and a quasi-decreasing function when they both have the same domains and ranges (which are totally ordered) and the first starts above and ends below the second. A quasi-increasing function can not have jumps down and, under the assumptions, will have necessarily an intersection with a quasi-decreasing function (which can not have jumps up). The result can be used to show existence of symmetric Cournot equilibrium since the identity function (the 45^o line) is quasi-decreasing (the theorem is then a fixed-point theorem for quasi-increasing functions). The first result has been successively rediscovered in economics to show existence of equilibrium in a class of symmetric Cournot games starting with the work of McManus (1962, 1964), Roberts and Sonnenschein (1976), and Milgrom and Roberts (1994).

[10] See also Milgrom and Roberts (1996).

[11] Tarski (1955). See the appendix and Section 2.3.1 in Vives (1999).

The result can also be used to show existence of equilibrium in two-player asymmetric games by applying the fixed-point theorem to the composition of the best replies assuming one is quasi-increasing and the other continuous and increasing (noting that the composition of the two functions will be itself quasi-increasing). This situation, where one player displays continuous strategic complementarity and the other limited strategic substitutability, is considered by Amir and De Castro (2015).[12]

Aggregative games Results can be extended to aggregative games where the payoff to a player depends on his strategy and an aggregator (typically an additive separable function) of the strategies of all the players. A key tool is the cumulative best reply (or backwards response correspondence) of a player introduced by Selten (1970) and used by Bamon and Frayssé (1985) and Novsheck (1985) to show existence of a Cournot equilibrium when outputs are strategic substitutes and best reply correspondences are decreasing (see Theorem 2.7 in Section 2.3.2 of Vives (1999), Kukushkin (1994), and the chapter by Jensen (2016) in this Handbook). For example, consider a symmetric n-player game where for each player the strategy space is a compact interval of the realms and the payoff depends only on his own strategy and the aggregate strategy of the rivals. Then if the best reply of a player has no jumps down, symmetric equilibria exist.[13] In this approach uniqueness of equilibrium is obtained with the requirement that best reply correspondences (depending on a linear aggregate of the strategies of rivals) have slopes strictly larger than -1 (see Theorem 2.8 in Section 2.3.2 of Vives (1999)). Under smoothness and regularity conditions ("nice aggregative games", see Jensen (2016), which require concavity assumptions) existence of equilibrium and monotone comparative statics results are obtained without substitutability or complementarity requirements.

Large games The results obtained so far apply also to large games (e.g. nonatomic games with a continuum of players) with some technical caveats. In this case existence of equilibrium can be shown under standard continuity and compactness requirements without requiring quasi-concavity or supermodularity of payoffs because of the convexifying effect of the continuum of players formulation (see Schmediler (1973)). For example, consider our definition of a GSC and note that it applies to games with an infinite number of players (be it countable or uncountable).

2.6 The scope of the theory

If not everything is a game of strategic complementarities, where are the bounds of the theory?

If we take the view that the order of the strategy spaces is part of the description of the game or that there is a “natural” order in the strategy spaces, then there are many games that are not of strategic complementarities (as we will see in the next section). In many games, best responses are nonmonotone, e.g. they are increasing in a part of the

[12]The results apply also to the dual case where the best reply of one player is continuous and decreasing and of the other quasi-decreasing.

[13]The argument is simple. Let $\pi_i(a_i, a_{-i}) = \pi\left(a_i, \sum_{j\neq i} a_j\right)$ for any i, as in a Cournot game with homogeneous product and identical cost functions. Existence of symmetric equilibria follows then from the stated result if the best-reply Ψ_i of a player (identical for all i due to symmetry) has no jumps down. This is in fact true if costs are convex in the Cournot game. Symmetric equilibria are given by the intersection of the graph of $a_i = \Psi_i\left(\sum_{j\neq i}\right) a_j$ with the line $a_i = \left(\sum_{j\neq i} a_j\right) / (n-1)$.

strategy space and decreasing in another. However, if take the view that the order of the strategy sets of the players is a modeling choice at the convenience of the researcher (and this is what we have done to extend the reach of the theory to duopolies with strategic substitutes) the answer may be different. In fact, if we allow the construction of this order ex post, with knowledge of the equilibria of the game, then most games *are* of strategic complementarities. To put it another way, complementarities alone, in the weak stated sense, do not have much predictive power unless coupled with additional structure (Echenique (2004a)). However, this procedure, with a priori knowledge of the equilibria and the defined order; may not be "natural".

3 Static oligopoly games and comparative statics

This section provides a brief review of some of the basic applications to static models of oligopoly competition. It surveys Cournot and Bertrand markets, including comparative statics results, patent races, and multidimensional competition (including extensions of the methods to games that do not display global complementarities). The analysis highlights the power and applicability of the approach.

3.1 Cournot and Bertrand markets

Leading oligopoly models fit, in natural but not universal specifications, the assumptions made in supermodular games. This is the case for a Cournot oligopoly with complementary products. In this case, the strategy sets are compact intervals of quantities and the complementarity assumptions are natural. A second case is a Bertrand oligopoly with differentiated substitutable products, with each firm producing a different variety. The demand for variety i is given by $D_i(p_i, p_{-i})$, where p_i is the price of firm i and p_{-i} denotes the vector of the prices charged by the other firms. A linear demand system with gross substitutes will satisfy the complementarity assumptions.

Considering increasing transformations of the payoffs the application of the theory is extended (since this operation does not change the equilibrium set of the game). The game is *log-supermodular* if $\pi_i \geq 0$ and if $\log \pi_i$ fulfills the complementarity conditions (1) and (2) of Section 2. In the Bertrand oligopoly example, $\pi_i = (p_i - c_i) D_i(p_i, p_{-i})$, where c_i is the constant marginal cost, is log-supermodular in (p_i, p_{-i}) whenever $\partial^2 \log D_i / \partial p_i \partial p_j \geq 0$. This holds when the own-price elasticity of demand η_i is decreasing in p_{-i}, as with constant elasticity, logit, or constant expenditure demand systems.[14] However, not all Bertrand games with product differentiation are supermodular games. Examples include games with payoffs which are not single-peaked as well as with avoidable fixed costs, and the Hotelling model where firms are located close to each other. In those cases, at some point best replies jump down and a price equilibrium (in pure strategies) fails to exist.[15] Even with goods that are gross substitutes, prices may not be strategic complements since the own-price elasticity of demand need not decrease in the prices charged by rivals.[16] An

[14]See Chapter 6 of Vives (1999).

[15]See Roberts and Sonnenschein (1977), Friedman (1983), and Vives (1999, Sec. 6.2). However, in the modified Hotelling game in Jacques Thisse and Vives (1992) best responses may be discontinuous but are increasing.

[16]A price increase by rival j may lead to an *increase* in the own-price elasticity of demand for firm i because it makes consumers of brand i who do not have a strong preference for any product —that is, who are more price sensitive, more willing to switch brands (see Berry, Levinsohn, and Pakes (1999) for

instance were strategic price substitutability among prices may arise is in the presence of strong network externalities.[17] Furthermore, even a linear Bertrand oligopoly game with continuous best replies and more than two firms need not be supermodular or satisfy single-crossing conditions when demands have kinks and some firms may not produce.[18]

The lattice-theoretic methods can be further extended to non-supermodular price games. Consider, for example, a Bertrand duopoly with differentiated gross substitute products where firm 1 has concave costs (increasing returns) and supermodularity fails with competition being of the "strategic quasi-substitutability" type (that is, demands are such so that the best reply of firm 2 is continuous and decreasing, and that of firm 1 is quasi-decreasing). In this case an equilibrium exists using Tarski's intersection point theorem (see Appendix). Another example is provided by the mixed price-quantity duopoly of Singh and Vives (1984) where firm 1 is a price setter and firm 2 a quantity setter. Assume constant marginal costs and suppose that demands are such that the payoff of firm 1 in quantities submodular and the payoff of firm 2 in prices supermodular and quasiconcave in own price. Then the mixed duopoly displays "strategic quasi-complementarity" (with the best reply of firm 1 quasi-increasing and the best reply of firm 2 continuous and increasing), and an equilibrium exists (generalizing the results of Singh and Vives (1984) with continuous best replies).[19]

3.1.1 Comparison of Cournot and Bertrand equilibria

Consider the n-firm Bertrand oligopoly case with firm i producing q_i of variety i at cost $C_i(q_i)$. In the Bertrand game firms compete in prices and $\pi_i = p_i D_i(p_i, p_{-i}) - C_i(D_i(p_i, p_{-i}))$. If firms compete in quantities in the same market then profits for firm i are given by $P_i(q_i, q_{-i}) q_i - C_i(q_i)$, where $P_i(q_i, q_{-i})$ is the inverse demand for firm i. The lattice-theoretical approach makes precise in what sense Bertrand equilibria are more competitive than Cournot equilibria and what drives the result. With gross substitute, or complementary, products if the price game is supermodular and quasiconcave (that is, π_i is quasiconcave in p_i for all i) then at any interior Cournot equilibrium prices are higher than the smallest Bertrand equilibrium price vector. A dual result holds also. With gross substitute, or complementary, products, if the quantity game is supermodular and quasiconcave, then at any interior Bertrand equilibrium outputs are higher than the smallest Cournot equilibrium quantity vector (Vives (1985b, 1990a)).

3.1.2 Comparative statics in Cournot markets

The standard Cournot game displays strategic substitutability and, therefore, the game is supermodular only in the duopoly case (by changing the sign of the strategy space of one player), as discussed in Section 2.4. We can obtain also results with n firms with the lattice-theoretic approach even if the game is not supermodular. The standard approach (Dixit (1986)) assumes quasiconcavity of payoffs, downward sloping best replies, and that the equilibrium analyzed is unique and stable to derive comparative statics results. The classical approach has several problems. First of all, it requires unnecessary regularity

some empirical support for this effect).

[17] For example, in the logit model with network externalities (Anderson, de Palma, and Thisse (1992, Ch.7)), increasing the price set by a rival raises the value for consumers of the network of firm i, so it may pay this firm to cut prices in order to enlarge this lead if network externalities are large enough.

[18] See Cumbul and Virág (2014).

[19] See Section 2.5 and Amir and De Castro (2015).

conditions to deliver results. Second, it is silent when payoffs are not quasiconcave. Third, it is problematic for comparative static analysis when there are multiple equilibria. For example, if the uniqueness condition for symmetric equilibria does not hold and there are multiple symmetric equilibria, changing n may either cause the equilibrium considered to disappear or introduce more equilibria.[20]

3.1.3 Monopolistic competition

Monopolistic competition is characterized by firms which are negligible with respect to the overall market but still retain market power on the differentiated product supplied. The appropriate modeling of such a situation is with a continuum of firms each supplying a product. Whenever the complementarity assumptions are fulfilled the price game will be supermodular and the results of Section 2 will apply (and, indeed, even with heterogeneous firms, e.g. Yang and Qi (2014)). It is worth noting that in monopolistic competition, and with no uncertainty, Cournot and Bertrand equilibria deliver the same outcome (that is, quantity or price competition are equivalent, see Section 6.6 in Vives (1999)).[21]

3.2 Patent races

Consider an n-firm oligopoly engaged in a memoryless patent race. All firms have access to the same R&D technology. An innovating firm obtains prize V and losers obtain nothing. If a firm spends x continuously then the (instantaneous) probability of innovating is given by $h(x)$ (where h is a smooth concave function with $h(0) = 0$, and $h' > 0$, $\lim_{x\to\infty} h'(x) = 0, h'(0) = \infty$, a region of increasing returns for small x may be allowed). With no innovation, the normalized profit of firms is null. We have then that the expected discounted profits (at rate r) of firm i investing x_i if rival j invests x_j is

$$\pi_i = \frac{h(x_i) V - x_i}{h(x_i) + \sum_{j\neq i} h(x_j) + r}.$$

The best response of a firm by $x_i = R\left(\sum_{j\neq i} h(x_j) + r\right)$ is well defined under the assumptions. Restricting attention to symmetric Nash equilibria of the game, under a stability condition at a symmetric equilibrium x^*, $R'((n-1)h(x^*))(n-1)h'(x^*) < 1$, x^* increases with n (Lee and Wilde (1980)). However, this approach requires assumptions to ensure a unique and stable symmetric equilibrium and cannot rule out the existence of asymmetric equilibria. Alternatively, the following mild assumptions, $h(0) = 0$ and h is strictly increasing in $[0, \overline{x}]$, with $h(x)V - x < 0$ for $x \geq \overline{x} > 0$, ensure that the game is strictly log-supermodular. It follows then from Section 2.1 that equilibria exist and are symmetric. It follows that at extremal equilibria the expenditure intensity x^* is increasing in n (strictly if h is smooth with $h' > 0$ and $h'(0) = \infty$). Furthermore, starting at any equilibrium, an increase in n will raise the research intensity, with out-of-equilibrium adjustment according to best–reply dynamics. This will be so even if some equilibria disappear or new ones appear as a result of increasing n.

[20] See Amir (1996a), Vives (1999), Amir and Lambson (2000) and the chapter by Amir in this Handbook for the results.

[21] See Thisse and Uschev (2016) in this Handbook for a survey of monopolistic competition models. Vives (1985b) and Gabaix et al. (2016) provide approximations of margins in large monopolistically competitive markets.

3.3 Multidimensional competition

The lattice-theoretic approach can readily handle multidimensional strategy spaces. I consider Cournot competition with cost reduction, advertising and pricing, and multi-market oligopoly.

3.3.1 Cournot competition with cost reduction

Consider an n-firm Cournot market for a homogenous product with smooth inverse demand $P(\cdot)$, $P' < 0$. Firm i can invest z_i to reduce its constant marginal cost of production c_i according to a smooth function $c_i = c(z_i)$ with $c(z) > 0$, $c'(z) < 0$, and $c''(z) > 0$ for all $z > 0$. The profit to firm i is given by

$$\pi_i = P(Q) q_i - c(z_i) q_i - z_i,$$

where q_i is the output of the firm and Q is total output. Firms simultaneously choose output and cost reduction effort. Using lattice-theoretic methods we do not need to invoke regularity conditions to obtain the existence of equilibrium and comparative statics results on the number of firms, as long as we restrict attention to extremal equilibria. Under the assumptions plus some mild boundary conditions interior extremal equilibria (q^*, z^*) exist and q^* and z^* are strictly decreasing (increasing) in n if Cournot best replies are strictly decreasing (increasing) (Vives (2008)).

3.3.2 Advertising, prices, and quantities

I examine complementarities between advertising and other strategic variables, considering first a price game and then a quantity and cost reduction game, both with differentiated products.

In the price game the demand of firm i $D_i(p; t_i)$ increases on advertising effort t_i, $\partial D_i / \partial t_i > 0$ with cost $F_i(t_i)$ and $F_i' > 0$, so that $\pi_i = (p_i - c_i) D_i(p; t_i) - F_i(t_i)$. Suppose that goods are gross substitutes, $\partial D_i / \partial p_j \geq 0$ for $j \neq i$, and that demand is downward sloping, $\partial D_i / \partial p_i < 0$. The action of the firm is $a_i = (p_i, t_i)$, lying in a compact rectangle. A sufficient condition for π_i to be strictly supermodular in a_i is that $\partial^2 D_i / \partial p_i \partial t_i \geq 0$ since

$$\frac{\partial^2 \pi_i}{\partial p_i \partial t_i} = (p_i - c_i) \frac{\partial^2 D_i}{\partial p_i \partial t_i} + \frac{\partial D_i}{\partial t_i} > 0.$$

The condition requires advertising to increase customers' willingness to pay. If $\partial^2 D_i / \partial p_i \partial p_j \geq 0$ for $j \neq i$ (noting that $\partial D_i / \partial p_i \partial t_j = 0$, $j \neq i$), π_i has increasing differences in $((p_i, t_i), (p_{-i}, t_{-i}))$. Under these assumptions, the game is supermodular and the largest (smallest) equilibrium displays high (low) prices and high (low) advertising levels. In a symmetric model and with a linear demand system, multiple equilibria obtain, with t_i increasing the demand intercept if F' is concave enough. Under these conditions high advertising levels are associated with high prices.

Immordino (2009) considers a Cournot oligopoly with differentiated product launching. Consumers become aware of the products via advertising and firms decide on simultaneously on production, advertising expenditure and cost-reducing investment. It is assumed that consumers with higher willingness to pay are more likely to be receptive to advertising; that marginal consumer awareness is increasing in advertising effort; that the consumer awareness of product i decreases in the intensity of advertising of firm j, and the marginal effectiveness of the advertising of firm i is decreasing in the advertising effort of firm j. Furthermore, the marginal cost of advertising is decreasing by using more specialized media. It is shown that in a strategic substitutes duopoly where firm i benefits from the improvements in advertising technology, but not firm j, all strategic variables at extremal equilibria increase in targeted advertising for firm i and decrease for firm j. In an oligopoly with complementary products (and where consumer awareness of product i increases in the intensity of advertising of firm j, and the marginal effectiveness of the advertising of firm i is increasing in the advertising effort of firm j) and strategic complements, all variables at extremal equilibria increase with a move towards targeted advertising.

3.3.3 Multimarket oligopoly

The approach allows the study of multiproduct firms and even of price games which are neither supermodular nor log-supermodular. I provide three applications: multimarket oligopoly pricing, two-sided markets, and the pricing of components.

Multimarket oligopoly. In a standard multiproduct logit oligopoly pricing model best responses are increasing and there is a unique Bertrand equilibrium despite the fact that payoffs are single–peaked (not quasi-concave) and neither supermodular or log-supermodular in own actions or prices (Spady (1984)). However, strategic complementarity across prices of different firms holds. A similar, and more general, result is obtained by Nocke and Schutz (2015) who, using a discrete/continuous choice framework with iid type 1 extreme-value taste shocks, introduce a class of demand systems for multiproduct firms which nests the cases of multinomial logit and CES.[22] The demand for product $k \in N$, where N is the set of differentiated products, is given by $D_k(p) = \frac{-h_i'(p_i)}{\sum_{j \in N} h_j(p_j)}$ where $p \in R_{++}^N$, $h_j' < 0$ and h_j is log convex. The set of firms is a partition N. Suppose that firm i produces goods in the set N_i. With constant (and positive) marginal costs this defines an aggregative pricing game since the profit of firm i depends only on $(p_k)_{k \in N_i}$ and on $H \equiv \sum_{j \in N} h_j(p_j)$. Under the assumption that the relative degree of convexity of h_j is non-decreasing in price[23] a Bertrand equilibrium exists (and it is unique under stronger conditions). The result is obtained even though profits are not quasiconcave in own prices, but they are single-peaked, and the price game is not supermodular. Monotone comparative static results can be derived on extremal equilibria (with largest and smallest H). For example, with an increasing outside option H^0 at extremal equilibria profits and prices of all firms decrease and consumer surplus increases (with expansion of the set of products sold). With CES demands an algorithm to compute the price equilibrium with multiproduct firms is provided. The results allow to characterize the dynamic optimality of myopic merger policy.

A multimarket mixed oligopoly featuring products demand complements within the firm and substitutes across firms provides another example. This situation obtains in two-sided markets, where two groups of market participants benefit from interaction via a platform or intermediary, or when final products are combinations of components.

Two-sided exclusive intermediation. Consider two groups of participants in platforms[24] where each participant joins one of the two existing intermediaries. The utility

[22] The authors also show that these demand systems are integrable with quasi-linear preferences.

[23] In the monopolistic competition case where H is taken as given this corresponds to a non-decreasing perceived price elasticty of demand.

[24] Examples are numerous and include readers/viewers and advertisers in media markets, cardhold-

derived by a member of a group from joining a particular intermediary is increasing in the number of members of the other group joining the same intermediary. With linear demands arising from Hotelling-type preferences for the intermediaries, we have that prices charged by intermediaries are strategic complements across firms but strategic substitutes within the firm. The multimarket oligopoly game is therefore not a supermodular game. However, best replies will be increasing as long as the demand complementarity among the products of the same platform/intermediary is not very strong. With linear demands and small and symmetric network effects, best replies are increasing and there is a unique symmetric equilibrium.

Pricing of components. Consider now a situation where each of a finite set of end products uses one or more components and where no two products share a component. Each component is produced by a separate monopolist who sets its price, and the price of a product is the sum of the prices of its components. The price game is not supermodular since the prices of the different components are strategic substitutes. However, Quint (2014) provides conditions on the distribution of consumer valuations for a discrete-choice demand system to yield demand for each product which is log-concave in price, and has log-increasing differences in own and another product's price. This leads to the consideration of an auxiliary game in product prices, with an equilibrium directly linked to the equilibrium of the pricing components game, which is supermodular and from which we can derive comparative static properties in terms of costs, qualities and entry of new products, as well as derive the effects of mergers between firms. The results apply to retail competition, licensing of intellectual property, and patent pools.

4 Dynamic games

This section examines dynamic games building on the stated comparative statics results. I examine entry, a generalization of the taxonomy of strategic behavior of Fudenberg and Tirole (1985), conditions under which increasing or decreasing dominance occurs in oligopoly, and Markov games and Markov perfect equilibria (MPE). I characterize conditions for dynamic strategic complementarity and the link between static and dynamic complementarities, and the existence of MPE.

4.1 Entry

Consider a two-stage game where first firms decide whether to enter or not in the market, paying an entry cost, and then compete in quantities, and study subgame-perfect equilibria. Amir et al. (2014) extend the Mankiw and Whinston (1986) excess entry results in symmetric Cournot oligopoly with free entry to allow for limited increasing returns to scale using lattice-theoretic methods. The authors assume that inverse demand is downward sloping ($P' < 0$) and costs are strictly increasing ($C' > 0$), and both smooth, with $-P' + C'' > 0$, and price is below average cost for high enough outputs. Under

ers/consumers and merchants/retailers in payment systems such as credit cards, consumers and shops in shopping malls, authors and readers in academic journals, borrowers and depositors in banking, "subscription to a network" and "number of calls made to a network" in telecom markets, and in general buyers and sellers put together with the help of intermediaries (in real estate, financial products, or auction markets). The interaction between the two sides gives rise to complementarities or externalities between groups that are not internalized by end users. See Armstrong (2006) for a survey of two-sided competition.

the assumptions Cournot extremal equilibria exist for any n and the authors show that there is excessive entry in the sense that at most there is one firm too few in the market solution (with respect the structural second best where the number of firms is decided by the planner) whenever there is business stealing. This is always the case when outputs are globally strategic substitutes. When $-P' + C'' < 0$ then only one firm should enter but the market solution will let enter (weakly) more and with no entry cost it would let entry of an infinite number of firms.

Anderson et al. (2016) study free entry in aggregative oligopoly games with potentially asymmetric firms (including potential asymmetric entry costs).[25] They make the observation that a Bertrand pricing game is aggregative if demands satisfy the IIA property, e.g., CES or logit, (the converse assertion is not true). The authors derive neutrality results (where the aggregate stays the same) across market structures and the corresponding policy implications for merger analysis.

Mrazova and Neary (2016) study selection effects with *heterogeneous* firms in the decision of whether and how to enter a market using lattice-theoretic methods. They find that "first order" selection effects (in terms of firms entering or not) are very robust while "second order" effects (in terms of the entry mode, exporting or FDI, conditional on entry) are much less so. More efficient firms select the entry mode with lower market-access costs if firm's profits are supermodular in production and market access costs but need not do so otherwise. The authors derive microfoundations for supermodularity to hold in a range of standard models and show how supermodularity may fail with FDI when demands are less convex than CES, with fixed costs increasing with productivity, and with threshold effects in R&D (that is, when the average cost function is first convex and then concave in investment).

4.2 Taxonomy of strategic behavior

The taxonomy of strategic behavior provided by Fudenberg and Tirole (1984) in the context of a two-stage game between an incumbent (firm 1) and an entrant (firm 2) illustrates the use of the approach. At the first stage the incumbent can make an observable investment t. The incumbent can influence the market outcome at the second stage by affecting the equilibrium behavior of the rival at the second stage. At the (market) stage payoffs are, respectivley, $\pi_1(a_1, a_2; t)$ and $\pi_2(a_1, a_2)$ where a_i is the market action of firm i. We want to sign the strategic effect taking as benchmark behavior where the incumbent when deciding about t only takes into account the direct effect of the investment on his payoff. This corresponds to the open-loop equilibrium of the two-stage game, which is equivalent to the game with simultaneous choice by the incumbent of t and a_1.

The standard approach assumes that at the second stage there are well-defined best-response functions for both firms, and that there is a unique and (locally) stable Nash equilibrium that depends smoothly on t, $a^*(t)$. A taxonomy of strategic behavior (see Table 1) can be provided depending on whether competition is of the strategic substitutes $\left(\frac{\partial^2 \pi_2}{\partial a_1 \partial a_2} < 0\right)$ or complements $\left(\frac{\partial^2 \pi_2}{\partial a_1 \partial a_2} > 0\right)$ variety and on whether investment makes firm 1 soft $\left(\frac{\partial \pi_1}{\partial a_2} \frac{\partial^2 \pi_1}{\partial t \partial a_1} > 0\right)$ or tough $\left(\frac{\partial \pi_1}{\partial a_2} \frac{\partial^2 \pi_1}{\partial t \partial a_1} < 0\right)$.[26] The top dog strategy obtains if

[25] See Corchón (1994) for an early analysis and the chapter by Polo in this Handbook.

[26] Indeed, if $\frac{\partial \pi_i}{\partial a_j} < 0$, $j \neq i$, an increase in the market action of firm j hurts firm i, and if $\frac{\partial^2 \pi_1}{\partial t \partial a_1} > 0$ an increase in t will shift the best response function of firm 1 out and this will represent an aggressive move.

competition is of the strategic substitutes type and investment makes firm 1 tough, then the incumbent wants to overinvest to push the entrant down his best response curve. Cournot competition and investment in cost reduction are an example. The puppy dog strategy obtains if competition is of the strategic complements type and investment makes firm 1 tough, then the incumbent wants to underinvest to move the entrant up his best response curve. Price competition with differentiated products and investment in cost reduction provide an example. We can define similarly the strategies "lean and hungry" and "fat cat".

Table 1

Taxonomy of strategic behavior

Strategic\	Investment makes player 1: Tough	Soft
Substitutes	Overinvest (top dog)	Underinvest (lean and hungry)
Complements	Underinvest (puppy dog)	Overinvest (fat cat)

The taxonomy follows from minimal assumptions, the character of competition and investment, as applied to extremal equilibria in the lattice theoretic version of the result (Section 7.4.3, Vives (1999)) with no need to impose strong restrictions to obtain a unique and stable equilibrium at the market stage. Indeed, if the market game is supermodular and $\partial^2\pi_1/\partial a_1\partial t \geq 0$ then extremal equilibria are increasing in t. If the game is of strategic substitutes (submodular) then extremal duopoly equilibrium strategies for firm 1(2) are increasing (decreasing) in t if $\partial^2\pi_1/\partial a_1\partial t \geq 0$. The results are reversed if $\partial^2\pi_1/\partial a_1\partial t \leq 0$.The taxonomy follows for extremal equilibria: $sign\frac{\partial a_2^*}{\partial t} = sign\left(\frac{\partial^2\pi_2}{\partial a_1\partial a_2}\frac{\partial^2\pi_1}{\partial t\partial a_1}\right)$ when a_2^* is an extremal equilibrium. What if at the market stage firms are sitting on a non-extremal equilibrium? Then if out of equilibrium adjustment is governed by best reply dynamics the sign of the impact of a change in t is the same as with an extremal equilibrium.

4.3 Markov games

We explore in this section dynamic complementarities and their relation to static ones in discrete time Markov games. A Markov strategy depends only on state variables that condense the direct effect of the past on the current payoff. Denote by $\pi_i(x, y)$ the current payoff of player i, where x is the current action profile vector and y is the state which evolves according to the law of motion $y = f(x^-, y^-)$, with x^- and y^- (respectively) the lagged action profile vector and the lagged state. A Markov perfect equilibrium (MPE) is a subgame-perfect equilibrium in Markov strategies. That is, an MPE is a set of strategies optimal for any firm, and for any state of system, given the strategies of rivals.

Let us speak of "contemporaneous" strategic complementarity (SC) when the value function at an MPE $V_i(y)$ displays SC (V_i has increasing differences in (y_i, y_{-i})). "Intertemporal" SC obtains when a player raising his state variable today increases the state variable of her rival tomorrow. "Intertemporal" strategic substitutability (SS) obtains when a player raising his state variable today decreases the state variable of her rival tomorrow. I restrict attention to a class of simple dynamic Markov games that admits two-stage games, simultaneous move games with adjustment costs, and alternating moves games. Consider a n-player game in which the actions of player i in any period lie in A_i, a compact cube of Euclidean space; $\pi_i(x, y)$ is the current payoff for player i, continuous in both $y \in A$, the action profile in the previous period (state variables) and in $x \in A$, the current action profile.

I take in turn the issues of contemporaneous SC in two–stage games and intertemporal SC or SS in infinite–horizon games. I end the section with results on the existence of MPE in stochastic games.

4.3.1 Two-stage games

Let $y \in A$ be the action profile in the first stage and $x \in A$ the action profile in the second stage. The contemporaneous SC property obtains under two conditions: (a) if at the second stage, for any actions y in the first stage, payoffs $\pi_i(x, y)$ display SC and (b) if the SC property is preserved when payoffs are folded back at the first stage in a subgame-perfect equilibrium. Suppose that $\pi_i(x, y)$ displays increasing differences (or is supermodular) in any pair of variables. Let $x^*(y)$ be an extremal equilibrium in the second stage (they exist at the second–stage for any y because the second stage game is supermodular). $V_i(y) \equiv \pi_i(x^*(y), y)$ is the first–period reduced form payoff for player i. Vives (2009) shows that $V_i(y)$ is supermodular in y provided that for any player i: π_i is increasing and convex in each component of x_j, $j \neq i$, and each component of $x_j^*(y)$ is supermodular in y. The result can be generalized to Markov finite–horizon multistage games with observable actions (e.g., Fudenberg and Tirole (1991)), where the payoff to each player displays increasing differences in any two variables.[27]

An example of the result is provided by the linear demand Bertrand oligopoly with advertising when advertising levels are chosen in a first stage and are observable. Under the assumptions made (Section 3.3.2), profits are supermodular in any pair of arguments, and the first–stage value function at extremal equilibria is supermodular (that is, advertising expenditures are strategic complements). Indeed, the assumptions are fulfilled in the classical linear gross substitutes products Bertrand competition model with constant marginal costs when either advertising or investment in product quality raises the demand intercept of the firm exerting the effort (Vives (1985a)) or increases the willingness to pay for the product of the firm by lowering the absolute value of the slope of demand $|\partial D_i / \partial p_i|$ (Vives (1990b)). In this case, for a given advertising effort there is a unique price equilibrium at the second stage.[28]

The result can be extended to a duopoly case in which, for all i, $\pi_i(x, y)$ has increasing differences in $(x_i, -x_j)$, $(y_i, -y_j)$, and $(x_i, (y_i, -y_j))$, $j \neq i$. An example is provided by a linear demand and cost Cournot duopoly in which outputs are strategic substitutes and y_i is the cost–reduction effort by firm i. Let $\pi_i = P_i(x_1, x_2)x_i - C_i(x_i, y_i)$ with $\partial^2 C_i / \partial x_i \partial y_i \leq 0$. Then the assumptions are fulfilled because $\partial^2 \pi_i / \partial x_i \partial y_i \geq 0$, and $\partial^2 \pi_i / \partial x_i \partial y_j = \partial^2 \pi_i / \partial y_i \partial y_j = 0$ for $j \neq i$. We then have that cost reduction investments are strategic substitutes at the first stage.[29]

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[27] Nonetheless, the result cannot be extended to the case where each payoff function $\pi_i(x, y)$ fulfills the ordinal complementarity conditions or the single–crossing property in any pair of variables (Echenique (2004b)).

[28] If firms invest in cost reduction, the second–stage SC is transformed into a first–stage SS. The same happens with investments in models of vertical quality differentiation when the market is covered (Shaked and Sutton (1982)).

[29] With linear demand there is a unique equilibrium at the second stage (see Vives (1990b) for a computed example where investment reduces the slope of marginal costs and for a reinterpretation in

With some further restrictions we can find conditions for increasing or decreasing dominance, that is whether an initial dominance is reinforced by subsequent market actions (Athey and Schmutzler (2001)). Examples are provided by the Bertrand differentiated oligopoly model with learning by doing or, alternatively, with production adjustment costs, or even with switching costs. Similar results can be obtained in Cournot model with network demand externalities (Katz and Shapiro (1986)).

4.3.2 Infinite-horizon games

Consider an infinite–horizon simultaneous move game with discount factor δ, and let $V_i(y)$ be the continuous value function associated to player i at a stationary MPE. Player i solves

$$\max_{x_i} \{\pi_i(x,y) + \delta V_i(x)\}.$$

Assume $x^*(y)$ is the unique contemporaneous Nash equilibrium given y. We have that $x^*(y)$ is increasing in y (i.e., we have intertemporal SC: x_i^* increases with y_j for $j \neq i$) if for all i

1. $\pi_i(x,y) + \delta V_i(x)$ has increasing differences in (x_i, x_{-i}) and
2. π_i has increasing differences in (x_i, y),

In order for (1) to hold it is sufficient that both π_i and V_i have increasing differences in (x_i, x_{-i}).

Similarly, we have the corresponding result for a duopoly with strategic substitutability. We have that x_i^* increases in $(y_i, -y_j)$ (i.e., we have intertemporal SS: x_i^* decreases with y_j for $j \neq i$) if for all i,

1. $\pi_i(x,y) + \delta V_i(x)$ has increasing differences in $(x_i, -x_j)$, $j \neq i$, and
2. π_i has increasing differences in $(x_i, (y_i, -y_j))$.

We check the fulfilment of the conditions in an adjustment cost model (see Vives (2005a) for the alternating move duopoly). With simultaneous moves and adjustment costs, the payoff to player i is given by

$$\pi_i(x,y) = u_i(x) - F_i(x,y),$$

where $u_i(x)$ is the current profit in the period and $F_i(x,y)$ is the convex adjustment cost in going from past actions (y) to current actions (x) with $F_i(x,x) = 0$, $i = 1,2$; that is, when actions are not changed, there is no adjustment cost. We can interpret actions as either prices or quantities and correspondingly let price or production bear the adjustment cost. Models with price adjustment costs, or "menu costs", are commonly used in macroeconomics. Our conditions are fulfilled in a linear-quadratic specification. With price competition (and static SC) and menu costs, the marginal profit for firm i is increasing in the price y_i charged by the firm in the previous period and is independent of the price y_j charged by the rival in the previous period. Furthermore, the value function V_i displays SC. With quantity competition (static SS) and production adjustment costs,

terms of firms that invest in expanding their own market).

the marginal profit for firm i is increasing in the production y_i of the firm in the previous period and independent of the production y_j of the rival in the previous period. The value function displays SS in the duopoly case.

In these two cases, static SC or SS is transformed into intertemporal SC or SS. However, this need not be always the case. Jun and Vives (2004) fully characterize the linear and stable MPE in a symmetric differentiated duopoly model with quadratic payoffs and adjustment costs in a continuous time infinite–horizon differential game. They find also that contemporaneous (dynamic) SC or SS are inherited from static SC or SS but intertemporal SC or SS obtains depending on what variable bears the adjustment cost. If production is costly to adjust then intertemporal SS obtains, whereas if price is costly to adjust then intertemporal SC obtains. In particular, for the mixed case of price competition with production adjustment costs, the static SC is transformed into intertemporal SS.[30] Having intertemporal SC or SS matters because it governs strategic incentives at the MPE with respect to non-strategic behavior at the open-loop equilibrium. Indeed, with intertemporal SC (SS), steady–state prices at the MPE are above (below) the stationary open–loop equilibrium prices. This provides a generalization of the taxonomy of strategic behavior in two-stage games of Section 4.2 to the full-blown infinite–horizon game.

4.3.3 Existence of MPE in stochastic games

Existence of MPE in deterministic dynamic games has been shown only in particular models such as the linear-quadratic and general results in stochastic games have been difficult to come by and rely on strong assumptions (particularly on transition probabilities) precluding deterministic transitions when actions spaces are uncountably infinite.[31] Lattice-theoretic methods are of help when there is enough monotonicity in the problem under study.

The existence of (stationary) MPE of stochastic games with complementarities in discrete time and infinite horizon is studied by Curtat (1996) under strong assumptions. He considers multidimensional action spaces and a multidimensional state evolving according to a transition probability as a function of the current state and action profile. Payoffs are smooth and display per–period complementarities and positive spillovers (the payoff to a player is increasing in the actions of rivals and the state); the transition distribution function is smooth, displays complementarities, and is stochastically increasing in actions and states. Furthermore, the payoff to a player as well as the transition distribution function fulfil a strict dominant diagonal condition.[32] These assumptions allow to collapse the multiperiod problem to a reduced form static game (with continuation value functions increasing in the state variable) which can be shown to be supermodular. An equilibrium can then be found with value functions increasing in the state. An example fulfilling the assumptions is a dynamic version of a Cournot oligopoly with complementary products and learning by doing, where a high level of accumulated output by one firm yields stochastically higher levels of cumulated experience and lower production costs to the firm (learning by doing) and to the rivals (spillovers).

[30] The reason —as in the learning curve model with price competition— is that a firm wants to make the rival small today in order to induce it to price softly tomorrow. A cut in price today will therefore bring a price increase by the rival tomorrow.

[31] See the discussion of the literature in Duggan (2012).

[32] The continuity assumptions on the transition probability are akin to Amir (1996b) and Nowak (2007) who also proved the existence of stationary MPE in games possessing strategic complementarities with uncountable state and action spaces.

Balbus et al. (2014) consider an n-player discounted infinite horizon stochastic game in discrete time. They allow for multidimensional action spaces and a multidimensional state (compact cube in Euclidean space with smallest point at 0) evolving according to a transition probability as a function of the current state and action profile. Payoffs are continuous and display per–period complementarities and positive spillovers (the payoff to a player is increasing in the actions of rivals); the transition distribution function is continuous, displays complementarities and is stochastically increasing in actions, and fulfils a strong mixing assumption, with a positive probability of setting the state to zero. The authors show existence of a largest and a smallest stationary Markov Nash equilibrium. The assumptions relax smoothness (Lipschitz continuity) conditions of Curtat (1996) as well as the increasing differences assumptions between actions and states. That is, they do not require *monotone* Markov equilibrium to obtain existence. The results can be applied to study supermodular price competition with durable goods (and convex costs). Here the state is a demand shock and the assumptions on transition probabilities mean that there is a positive probability that the market disappears and that high prices today result in a high probability of future positive demand, and there is no need of the monotonicity assumption for at a high demand state today to translate stochastically into a high demand state tomorrow. Under the assumptions extremal strategies display intertemporal strategic complementarity.

Sleet (2001) considers a version of the adjustment cost model of the previous section in an infinite–horizon discrete game with a continuum of heterogeneous players and symmetric payoffs. This a dynamic monopolistic competition model with menu costs where firms interact repeatedly over an infinite horizon and each firm receives an idiosyncratic demand or cost shock every period. The demand for the product of a firm may depend on the average price charged in the market or on a price index. The assumptions are fulfilled with linear or constant elasticity demands, quadratic or constant elasticity production costs (subject to a multiplicative shock), and quadratic costs of price adjustment.

5 Uncertainty and private information: Bayesian oligopoly games

The lattice-theoretical approach allows for general strategy spaces and payoff functions and therefore is apt for games of incomplete information. I present in this section a framework for Bayesian games and three approaches to characterize equilibria in pure strategies together with applications: supermodular games (Vives (1990a)), single-crossing properties (Athey (2001)), and "monotone supermodular" games (Van Zandt and Vives (2007)). The last two approaches deliver conditions for equilibria to be monotone in type, a desirable property in applications.

5.1 A framework for Bayesian games

Let T_i be the set of possible types t_i of player i,a subset of Euclidean space. The types of the players are drawn from a common prior distribution μ on $T = \prod_{i=0}^{n} T_i$, where T_0 is interpreted as unobserved residual uncertainty. In a game of incomplete information, the type of a player embodies all the decision-relevant private information. The action space of player i is a compact cube of Euclidean space A_i, and his payoff is given by the (measurable and bounded) function $\pi_i : A \times T \to \mathbb{R}$. The (ex post) payoff to firm

i when the profile of actions is $a = (a_1, ..., a_n)$ and the realized types $t = (t_1, ..., t_n)$ is thus $\pi_i(a; t)$. Action spaces, payoff functions, type sets, and the prior distribution are common knowledge. The Bayesian game is fully described by $(A_i, T_i, \pi_i; i \in N)$.

A (pure) strategy for player i is a (measurable) function $\sigma_i : T_i \to A_i$ that assigns an action to every possible type of the player. Let Σ_i denote the strategy space of player i (and identify strategies σ_i and τ_i if they are equal with probability 1). Denote the expected payoff to player i, when agent j uses strategy σ_j, by $U_i(\sigma) = E\pi_i(\sigma_1(t_1), ..., \sigma_n(t_n); t)$ where $\sigma = (\sigma_1, ..., \sigma_n)$. A Bayesian Nash equilibrium is a Nash equilibrium of the game $(\Sigma_i, U_i, i \in N)$ where the strategy space and payoff function of player i are denoted Σ_i and U_i, respectively.[33]

The formulation of the Bayesian game encompasses common and private values as well as perfect or imperfect signals. With pure private values, allowing for correlated types, say costs of firms, we have $\pi_i(a; t_i)$. With a common value case, say a demand shock where firm i observes component t_i only, we may have that $\pi_i(a; t) = v_i(a; \Sigma_i t_i)$. For an example of imperfect signals, suppose firms observe with noise their cost parameters. In this case t_0 could represent the n-vector of firms' cost parameters and t_i the private cost estimate of firm i, allowing for correlation among the cost parameters as well as the error terms in the private signals.

5.2 Equilibrium existence in pure strategies

To show existence of pure-strategy equilibria in games of incomplete information with a continuum of types and/or actions has proved difficult. Known sufficient conditions for existence include typically conditionally independent types, finite action spaces, and atomless distributions for types.[34] Under these assumptions existence of mixed strategy equilibria is shown first and then equilibria are purified. The lattice-theoretic approach has provided results:

1. for supermodular games with general action and type spaces (Vives (1990a));

2. for games satisfying single-crossing properties in which each player uses a strategy monotone (increasing) in type in response to monotone (increasing) strategies of rivals (Athey (2001), McAdams (2003, 2006), Reny (2011)); and

3. for "monotone" supermodular games with general action and type spaces (Van Zandt and Vives (2007)).

In the first approach, existence of pure–strategy Bayesian equilibria follows directly from supermodularity of the underlying family of games defined with the ex post payoffs for given realizations of the types of the players. A key observation is that supermodularity of this underlying family of games is inherited by the Bayesian game.[35] Existence of

[33] Denote by $\beta_i : \Sigma_{-i} \to \Sigma_i$ player i's best-reply correspondence in terms of strategies. Then a Bayesian Nash equilibrium is a strategy profile σ such that $\sigma_i \in \beta_i(\sigma_{-i})$ for $i \in N$. We can define a natural order in the strategy space $\Sigma_i : \sigma_i \leq \sigma_i'$ if $\sigma_i(t_i) \leq \sigma_i'(t_i)$, in the usual component-wise order, with probability 1 on T_i.

[34] See Radner and Rosenthal (1982) and Milgrom and Weber (1985)). Khan and Sun (1995) show existence of pure-strategy equilibria when types are independent, payoffs continuous, and action sets countable.

[35] Let π_i be supermodular in a_i and have increasing differences in (a_i, a_{-i}). Then $U_i(\sigma)$ is supermodular in σ_i and has increasing differences in (σ_i, σ_{-i}), because supermodularity and increasing differences are

extremal pure strategy Bayesian equilibria then follows from the general versions of the results in Section 2 (see also Vives (1990a; 1999, Sec. 2.7.3)). This existence result holds for multidimensional action spaces and requires no distributional restrictions. Applications of this approach can be found beyond oligopoly games in Diamond's (1982) search model, and natural resource exploration games with private information (Hendricks and Kovenock (1989) and Milgrom and Roberts (1990a)).

5.2.1 Single-crossing properties

In this approach, conditions are imposed so that an equilibrium in monotone increasing strategies in types can be found. Suppose that both action A_i and types sets T_i for any player i are compact subsets of the real line and that types have a joint density μ that is bounded, atomless, and log-supermodular (i.e., types are affiliated). Suppose also that $\pi_i(a,t)$ is continuous and supermodular in a_i and has increasing differences in (a_i, a_{-i}) and (a_i, t) or, alternatively, that $\pi_i(a,t)$ is nonnegative and log-supermodular in (a,t). Then the Bayesian game has a pure-strategy equilibrium in increasing strategies (Athey (2001)). Note that under the assumptions the first approach outlined already delivers existence of a pure-strategy equilibrium.[36] An example of the result in the differentiated Bertrand oligopoly has firm i with random marginal cost t_i with both $D_i(p_i, p_{-i})$ and the joint density of $(t_1, ..., t_n)$ log-supermodular. Then if the strategies of rivals, $p_j(\cdot)$, $j \neq i$, are increasing in types, $E(\pi_i \mid t_i) = (p_i - t_i)E(D_i(p_i, p_{-i}(t_{-i}) \mid t_i)$ is log-supermodular in (p_i, t_i) and the best–reply map of player i is increasing in t_i.

The approach can be used also in games that are not of strategic complementarities and with discontinuous payoffs. The existence of monotone equilibria in pure strategies can be shown for first-price auctions with heterogeneous (weakly) risk–averse bidders characterized by private affiliated values or common value and conditionally independent signals (Athey (2001)); as well as for uniform-price auctions featuring multiunit demand, interdependent values and independent types (McAdams (2003, 2006)).[37] Reny (2011) extends those results using a fixed-point theorem by Eilenberg and Montgomery (1946) which replaces the requirement in Kakutani or Glicksberg of convex-valued correspondences used by Athey (2001) by contractible-valued ones. [38] Reny (2011) weakens

preserved by integration. Furthermore, strategy spaces in the Bayesian game Σ_i can be shown to have the appropriate order structure (i.e., they are complete lattices). Then the game $(\Sigma_i, U_i, i \in N)$ is a GSC and for all $\sigma_{-i} \in \Sigma_{-i}$, $\beta_i(\sigma_{-i})$ contains extremal elements $\bar{\beta}_i(\sigma_{-i})$ and $\underline{\beta}_i(\sigma_{-i})$.

[36] The proof of these results relies on the standard Kakutani fixed point theorem, based on convex-valued correspondences since with discrete action spaces and under the prevailing assumptions, best–response correspondences are convex valued. A key step in the proof is to show that if the rivals of player i use increasing strategies then the payoff to player i fulfills an appropriate single–crossing property (e.g., is log-supermodular or has increasing differences) in action and type. This ensures that a player uses a strategy that is increasing in his type as a best response to increasing strategies of rivals. The existence result for discrete action spaces can then be used to show existence with a continuum of actions via a purification argument.

[37] McAdams (2006) uses a discrete bid space and atomless types to show existence of monotone equilibria with risk neutral bidders checking that the single-crossing condition in Athey (2001) used in the single-object case extends to multi-unit auctions.

[38] A set is contractible if, within itself, it can be continuously deformed to a single point. Convex sets are, indeed, contractible but the converse is not true. The author finds conditions for the existence of a monotone (pure) equilibrium whenever monotone (pure) best replies are non-empty and join-closed in response to rivals using monotone best replies. Those conditions require, among other technical conditions, payoffs to be bounded, jointly measurable in actions and types, continuous in actions for every type, and the marginal distribution of types to be atomless.

previous conditions on interim payoff functions for the monotone best reply condition to hold. The conditions allow for infinite-dimensional type and action spaces, general joint distributions over types, general partial orders on both action and type spaces (this is useful since single-crossing may fail for one partial order but hold for another), and dispense with single crossing (although it remains very useful).

The results are applied to prove existence of monotone equilibrium in uniform-price multiunit auctions with weakly risk averse bidders and interdependent values (and where bids are restricted to a finite grid) and to oligopoly pricing using judicious partial orders over types. The oligopoly application considers n firms competing with differentiated products with random constant marginal costs and random demand. The firms are partitioned into two groups with goods being substitutes within each group. Firms have private information about both cost and demand conditions and marginal costs are affiliated and information about demand may be correlated across firms. It is shown that a pure-strategy price equilibrium exists with prices monotone in costs (which is the coordinate in which strict single-crossing holds, and also in the demand signal according to the defined partial order).

5.2.2 Monotone supermodular games

For "monotone" supermodular games with multidimensional action spaces and type spaces a strong result is provided by Van Zandt and Vives (2007). Let $\Delta(T_{-i})$ be the set of probability distributions on T_{-i} and let player i's posteriors be given by the (measurable) function $p_i : T_i \rightarrow \Delta(T_{-i})$, consistent with the prior μ. The following properties define a monotone supermodular game:

1. *Supermodularity and complementarity between action and type* with π_i supermodular in a_i, and with increasing differences in (a_i, a_{-i}) and in (a_i, t).

2. *Monotone posteriors* with $p_i : T_i \rightarrow \Delta(T_{-i})$ increasing with respect to the partial order on $\Delta(T_{-i})$ of first–order stochastic dominance (a sufficient but not necessary condition is that μ be affiliated).

The result is that in a monotone supermodular game there is a largest and a smallest Bayesian equilibrium and each one is in monotone strategies. There might be other equilibria that are in nonmonotone strategies but, if so, they will be "sandwiched" between the largest and the smallest one. Furthermore, extremal equilibria are increasing in posteriors (according to first-order stochastic dominance) and can be obtained through the iterative application of the best reply map β.

The assumptions on action and type spaces can be considerably weakened (beyond Euclidean spaces), and there is no need to assume a common prior, but the result cannot be extended to log-supermodular payoffs. Yang and Qi (2014) provide an extension of the results to nonatomic games.[39]

Monotone supermodular games fit a variety of problems.[40] We provide here applications to strategic information revelation and endogenous information acquisition.

[39] Under their assumptions monotone equilibria form a complete lattice.

[40] Van Zandt and Vives (2007) present an application to the discrete setup of an adoption game on a graph with local network effects, and so called "global games" are typically monotone supermodular (see Section 7.2 in Vives (2005a)).

Comparative statics and strategic information revelation. If payoffs display positive spillovers (π_i is increasing in a_{-i}), then increasing the posteriors increases the equilibrium expected payoffs. This is a consequence of extremal equilibria being increasing in posteriors and it implies that the expected payoff of each player in an extremal equilibrium is increasing in the posteriors of the other players. The result can easily be strengthened to "strictly increasing" under certain regularity assumptions (including some smooth strict complementary conditions and requiring π_i to be strictly increasing in a_j).

Okuno-Fujiwara et al. (1990) have provided conditions under which fully revealing equilibria obtain in duopoly games of voluntary disclosure of information when information is verifiable. The conditions involve regularity assumptions such as one–dimensional actions, concavity of payoffs, uniqueness and interiority of equilibrium, and independent types for the players.[41] Once we realize that their framework is within the realm of monotone supermodular games, it is only needed that the marginal payoff of an action of a player is strictly increasing in the actions of rivals and in the types of players. The results extend to n-player GSC games and to a duopoly with strategic substitutability, multidimensional actions, affiliated types, and possibly multiple noninterior extremal equilibria. Mensch (2016) provides an extension of the existence of monotone equilibrium results in Reny (2011) to dynamic games (such as stopping games) and uses them to weaken the assumptions to obtain a full separating equilibrium in voluntary disclosure games.[42]

Endogenous information acquisition Amir and Lazzati (2016) study covert endogenous information acquisition in the framework of common value monotone supermodular games. The authors use the supermodular stochastic order to arrange the information structures (joint distribution of state of the world and signals) and show that better information increases expected payoffs. If a convexity assumption is added, implying that increasing the quality of information raises informativeness with increasing returns, then the value of information for a player is convex in its quality. This leads to extreme behavior of agents with choices of a highest or a lowest quality signal. The results contrast with models with linear-quadratic payoffs and Gaussian information structures with concave values of information (Vives (2008)).[43]

[41] The basic intuition for the result is that in equilibrium inferences are skeptical: if a player reports a set of types others believe the worst (i.e., others believe that the player is of the most unfavorable type in the reported set). This unravels the information.

[42] The author constructs an auxiliary static game to deal with the endogeneity of beliefs in the dynamic game preserving at the same time the continuity of payoffs. Then he uses the methods in Athey (2001) and Reny (2011) to show that there exist monotone best replies to monotone strategies by rivals.

[43] Myatt and Wallace (2016) provide an analysis of information use and acquisition in a strategic complements price-setting oligopoly with differentiated products with a continuum of goods, where a finite number of suppliers have access to multiple sources of information about the uncertain common demand level. Myatt and Wallace (2015) examine similar issues in the context of a Cournot model. Bonatti et al. (2016) examine learning and signaling in a dynamic Cournot game with firms having private information on costs and observing the (noisy) market price.

5.3 Complementarities in uniform-price divisible good auctions and behavioral traders

Progress in the characterization of equilibria in uniform-price divisible good auctions when there is incomplete information and market power has been made in a linear-Gaussian specification. Kyle (1989) considers a Gaussian model of a divisible good double auction where some bidders are privately informed and others are uninformed. Vives (2011) shows how increased correlation in the values of the traders increases their market power and how private information generates market power over and above of the full information level. Bergemann et al. (2015) generalize the information structure in Vives (2011) keeping the symmetry assumption. Rostek and Weretka (2012) partially relaxes the symmetry assumption in Vives (2011) and replaces it with a weaker "equicommonality" assumption on the matrix correlation among the agents' values. Manzano and Vives (2016) consider the case of two types of bidders.

Consider the case of supply function competition to fix ideas. This corresponds to competition in the wholesale market for electricity in many countries. It is worth noting that restricting attention to linear supply functions and with complete information, there is strategic complementarity in the supply slopes (e.g. Klemperer and Meyer (1989) and Akgün (2004)) despite the game not being supermodular. The reason is that if rivals of a firm increase the slope of supply then the residual demand left for the firm becomes steeper and induces this firm to set also a steeper supply. Interestingly, with uncertainty and incomplete information where prices convey information (say about an uncertain common cost component), there is an inference effect that moderates, and may even reverse, the strategic complementarity in slopes. This is so since when costs are positively correlated, the inference effect moderates the reaction to the price (a high price means high costs), the more so, the more rivals react to the price. The reason is that a higher reaction to the price by rivals induces a trader to also give a higher weight to the price in the estimation of his cost and hence it increases the magnitude of the inference effect. However, the equilibrium happens at a point where there are strategic complementarities in slopes (see Bayona et al. (2016)).

When traders or firms have trouble retrieving the information from the price, be it because they neglect the correlation between random variables or because of other types of bounded rationality, then the outcome is more competitive since high prices are not interpreted as signals of high costs. If there is a proportion of such naive traders in the market then sophisticated traders, who take into account information in prices as well as the presence of naive traders who bid relatively flat supply schedules, will respond also with flatter supply schedules and the outcome will be more competitive than predicted by the Bayesian equilibrium in supply functions with fully rational traders. This is in fact what happens in the experiment conducted in Bayona et al. (2016) and which leads to the behavior of naïve and sophisticated sellers being not so distinct. The general phenomenon was first noted by Camerer and Fehr (2006) in the context of games characterized by strategic complementarities and in the presence of sophisticated and boundedly rational subjects. When actions are strategic complements then sophisticated players align their actions with those of naive players not providing a check of the effects of naive strategies on outcomes. Instead, when actions are strategic substitutes then sophisticated players counteract the actions of naive ones on aggregate behavior. The result is that under strategic substitutes a small proportion of sophisticated agents may be sufficient to lead to aggregate outcomes not far from the equilibrium predictions with fully rational agents, while under strategic complements, a not so large proportion of naive agents may generate outcomes far from the equilibrium rational agent predictions. This means that rational agent equilibrium analysis in games of strategic complementarities may be less robust than in games of strategic substitutability.

6 Concluding remarks

In this chapter I have provided a selective survey of the theory and applications of the lattice-theoretic approach in the study of oligopoly games. The approach has been shown fruitful well beyond the domain of games of strategic complementarities. Indeed, in many situations patterns of complementarity and substitutability are present but still the approach is useful and delivers existence and characterization of equilibrium results as well as comparative statics analysis. The approach has proved useful in fact in all domains of economic theory and is being progressively incorporated in the standard toolbox of economics, including empirical studies. The research agenda ahead is challenging in terms of continue pushing the frontier of the theory with a view toward applications including

heterogeneous agents and fully dynamic games with incomplete information, as well as developing more fully the empirical analysis.[44]

7 Appendix: Brief summary of lattice-theoretic methods[45]

7.1 Definitions

A binary relation $\geq$ on a nonempty set X is a partial order if $\geq$ is reflexive, transitive, and antisymmetric. An upper bound on a subset $A \subset X$ is $z \in X$ such that $z \geq x$ for all $x \in A$. A greatest element of A is an element of A that is also an upper bound on A. Lower bounds and least elements are defined analogously. The greatest and least elements of A, when they exist, are denoted $\max A$ and $\min A$, respectively. A supremum (resp., infimum) of A is a least upper bound (resp., greatest lower bound); it is denoted $\sup A$ (resp., $\inf A$).

A *lattice* is a partially ordered set $(X, \geq)$ in which any two elements have a supremum and an infimum. A lattice $(X, \geq)$ is *complete* if every nonempty subset has a supremum and an infimum. A subset L of the lattice X is a *sublattice* of X if the supremum and infimum of any two elements of L belong also to L.

Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. A function $f \colon X \to T$ is *increasing* if, for x, y in X, $x \geq y$ implies that $f(x) \geq f(y)$.

A function $g \colon X \to \mathbb{R}$ on a lattice X is *supermodular* if, all x, y in X, $g(\inf(x, y)) + g(\sup(x, y)) \geq g(x) + g(y)$. It is *strictly supermodular* if the inequality is strict for all pairs x, y in X that cannot be compared with respect to $\geq$ (i.e., neither $x \geq y$ nor $y \geq x$

[44]Progress on the empirical front in dynamic oligopoly has been made, among others, by Ericson and Pakes (1995), Bajari et al. (2007), Fershtman and Pakes (2012), and Ifrach and Weintraub (2016). An example of empirical analysis of complementarities is provided by Miravete and Pernias (2006), and of markets with multiple equilibria by Sweeting (2006), Ciliberto and Tamer (2009), and Galichon and Henry (2011). A promising approach based on monotone comparative statics is provided by Echenique and Komunjer (2009).

[45]More complete treatments can be found in Vives (1999, Ch. 2) and Topkis (1998).

holds). A function f is (*strictly*) *submodular* if $-f$ is (strictly) supermodular; a function f is (*strictly*) *log-supermodular* if $\log f$ is (strictly) supermodular.

Let X be a lattice and T a partially ordered set. The function $g\colon X \times T \to R$ has (*strictly*) *increasing differences* in (x,t) if $g(x',t) - g(x,t)$ is (strictly) increasing in t for $x' > x$ or, equivalently, if $g(x,t') - g(x,t)$ is (strictly) increasing in x for $t' > t$. Decreasing differences are defined analogously. If X is a convex subset of $\mathbb{R}^n$ and if $g\colon X \to R$ is twice-continuously differentiable, then g has increasing differences in (x_i, x_j) if and only if $\partial^2 g(x)/\partial x_i \partial x_j \geq 0$ for all x and $i \neq j$.

7.2 Results

Supermodularity is a stronger property than increasing differences: if T is also a lattice and if g is (strictly) supermodular on $X \times T$, then g has (strictly) increasing differences in (x,t). The two concepts coincide on the product of linearly ordered sets: if X is such a lattice, then a function $g\colon X \to \mathbb{R}$ is supermodular if and only if it has increasing differences in any pair of variables.

The complementarity properties are robust in the sense that they are preserved under addition or integration, pointwise limits, and maximization (with respect to a subset of variables, preserving supermodularity for the remaining variables).[46]

Lemma 1 ***Monotonicity of optimal solutions.*** *Let X be a compact lattice and let T be a partially ordered set. Let $u\colon X \times T \to \mathbb{R}$ be a function that (a) is supermodular and continuous on the lattice X for each $t \in T$ and (b) has increasing differences in (x,t). Let $\varphi(t) = \arg\max_{x \in X} u(x,t)$. Then:*

1. *$\varphi(t)$ is a non-empty compact sublattice for all t;*
2. *φ is increasing in the sense that, for $t' > t$ and for $x' \in \varphi(t')$ and $x \in \varphi(t)$, we have $\sup(x',x) \in \varphi(t')$ and $\inf(x',x) \in \varphi(t)$; and*
3. *$t \mapsto \max \phi(t)$ and $t \mapsto \min \phi(t)$ are well-defined increasing functions.*

Remark If u has strictly increasing differences in (x,t), then all selections of φ are increasing.

Remark If $X \subset \mathbb{R}^m$, solutions are interior, and $\partial u/\partial x_i$ is strictly increasing in t for some i, then all selections of φ are strictly increasing (Edlin and Shannon (1998)).

Theorem 2 *(Tarski (1955)) Let A be a complete lattice (e.g., a compact cube in $\mathbb{R}^m$). Then an increasing function $f : A \to A$ has a largest $\sup\{a \in A : f(a) \geq a\}$ and a smallest $\inf\{a \in A : a \geq f(a)\}$ fixed point.*

[46] Supermodularity and increasing differences can be weakened to define an "ordinal supermodular" game, relaxing supermodularity to the weaker concept of quasi–supermodularity and increasing differences to a single–crossing property (see Milgrom and Shanon (1994)). However, such properties (unlike supermodularity and increasing differences) have no differential characterization and need not be preserved under addition or partial maximization operations.

Supermodular game The game $(A_i, \pi_i; i \in N)$ is *supermodular* if, for all i, the following statements hold:

- A_i is a compact lattice.
- $\pi_i(a_i, a_{-i})$ is continuous:
    1. is supermodular in a_i; and
    2. has increasing differences in (a_i, a_{-i}).

Game of strategic complementarities Given a set of players N, strategy spaces A_i, and (nonempty) best–reply maps Ψ_i, $i = 1, ..., n$, we define a *game of strategic complementarities* (GSC) as one in which, for each i, A_i is a complete lattice and Ψ_i is increasing and has well-defined extremal elements.

Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is *quasi-increasing* if for every $x \in X$, $\limsup_{y\uparrow x} f(y) \leqslant f(x) \leqslant \liminf_{y\downarrow x} f(y)$; f is *quasi-decreasing* if $-f$ is quasi-increasing. The following is a real-valued version of Theorem 3 in Tarski (1955).

Theorem 3 (Tarski's Intersection Point Theorem) *If $f : [a, b] \longrightarrow \mathbb{R}$ is quasi-increasing, $g : [a, b] \longrightarrow \mathbb{R}$ is quasi-decreasing, $f(a) \geqslant g(a)$ and $f(b) \leqslant g(b)$, then the set $\{x \in [a, b] : f(x) = g(x)\}$ is non-empty, and has as largest element sup $\{x \in [a, b] : f(x) \geqslant g(x)\}$ and as smallest element inf $\{x \in [a, b] : f(x) \leqslant g(x)\}$.*

Corollary: *Let $X = [a, b]$. Then a quasi-increasing function $f : X \to X$ has a largest* $\overline{x} \equiv \sup\{x \in X : f(x) \geq x\}$ and a smallest $\underline{x} \equiv \inf\{x \in X : x \geq f(x)\}$ fixed point.

The result is easy to grasp considering a function f: $[0, 1] \to [0, 1]$ which when discontinuous jumps up but not down. The function must then cross the 45^o line at some point. Indeed, suppose that it starts above the 45^o line (otherwise, 0 is a fixed point); then it either stays above it (and then 1 is a fixed point) or it crosses the 45^o line.

Comparative statics analysis is trivial when the function f is (strictly) increasing in a parameter t (for t in a partially ordered set T). Then $\overline{x}(t)$ and $\underline{x}(t)$ are (strictly) increasing in t. This follows since $\overline{x}(t) = sup\ \{x \in X\text{: } f(x;t) \geq x\}$, $\underline{x}(t) = inf\ \{x \in X\text{: } f(x;t) \leq x\}$, and f is (strictly) increasing in t. It is worth to remark that as t varies, the number of equilibria may change, but still the largest and the smallest equilibrium will be increasing in t.

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