1. **Mergers with synergies.** Consider an industry of \( n \geq 2 \) firms competing a la Cournot. Let us analyze the following sequential-move game where, in the first stage, every firm chooses whether or not to merge with other \( k - 1 \) firms (so \( k \) firms merge and the remaining \( n - k \) firms do not); and, in the second stage, firms compete a la Cournot. The \( k \) firms that merge coordinate their production decision to maximize their joint profits while the remaining \( n - k \) firms do not coordinate. Firms face an inverse demand curve \( p(Q) = 1 - Q \), where \( Q \geq 0 \) denotes aggregate output. Firms are symmetric in their marginal cost of production \( c \), where \( 1 > c \geq 0 \).

In this setting, we allow for the \( k \) firms that merge to benefit from a cost-reduction effect, so their marginal cost of production decreases to \( c \times x \). In contrast, the firms that did not merge still face a marginal cost \( c \).

(a) As a benchmark, find the profits of every firm \( i \) before the merger.

- If no merger occurs, every firm solves

\[
\max_{q_i \geq 0} (1 - q_i - Q_{-i})q_i - cq_i
\]

where \( Q_{-i} \) denotes the aggregate output of firm \( i \)'s rivals. Differentiating with respect to \( q_i \), and solving for \( q_i \), we find firm \( i \)'s best response function

\[
q_i(Q_{-i}) = \frac{1 - c}{2} - \frac{1}{2} Q_{-i}
\]

which is negatively sloped in its rivals' aggregate output, \( Q_{-i} \). In a symmetric equilibrium, \( q_i = q_j = q \) for every two firms \( i \neq j \), which entails \( Q_{-i} = (N - 1)q \). Therefore, the above best response function simplifies to

\[
q = \frac{1 - c}{2} - \frac{1}{2} (N - 1)q
\]

and, solving for \( q \), we obtain the equilibrium output of every firm \( i \) when no merger occurs:

\[
q = \frac{1 - c}{n + 1}.
\]

Then, equilibrium profits become

\[
\pi^{NM} = \left( \frac{1 - c}{n + 1} \right)^2
\]

where the superscript \( NM \) indicates "no merger."

(b) Find the output that the merged and unmerged firms produce.

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- After the merger, we need to solve a Cournot problem where firms face asymmetric costs. The merged entity solves

$$\max_{q_M \geq 0} \left(1 - q_M - Q_{-i}\right)q_M - (c - x)q_M$$

where $q_M$ denotes the production level of the merged entity and $Q_{-i}$ represents the aggregate output of all outside firms that did not merge. Differentiating with respect to $q_M$, yields

$$1 - 2q_M - Q_{-i} - c + x = 0$$

and solving for $q_M$, we obtain the best response function

$$q_M(Q_{-i}) = \frac{1 - c + x}{2} - \frac{1}{2}Q_{-i}.$$  

- For the unmerged firms, each chooses its individual production $q_i$ to solve

$$\max_{q_i \geq 0} \left(1 - q_i - q_M - Q_{-i}^{U_i}\right)q_i - cq_i$$

where $Q_{-i}^{U_i}$ denotes the aggregate production level of all other $(n - k) - 1$ firms that did not merge. Differentiating with respect to $q_i$, we find

$$1 - 2q_i - q_M - Q_{-i}^{U_i} - c = 0$$

and solving for $q_M$, we obtain the best response function

$$q_i(q_M, Q_{-i}^{U_i}) = \frac{1 - c}{2} - \frac{1}{2} (q_M + Q_{-i}^{U_i})$$

At this point, we can invoke symmetry: in a symmetric equilibrium all unmerged firms produce the same output, $q_i = q_j = q_{UM}$ for all $n - k$ firms, where subscript $UM$ denotes unmerged firms. Therefore, $Q_{-i}^{U_i}$ can be expressed as $Q_{-i}^{U_i} = [(n - k) - 1]q_{UM}$. Inserting this result in the above best response function, we find that

$$q_{UM} = \frac{1 - c}{n - k + 1} - \frac{1}{n - k + 1}q_M$$

In addition, $Q_{-i} = (n - k)q_{UM}$, which inserted in the merged entity’s best response function, yields

$$q_M = \frac{1 - c + x}{2} - \frac{n - k}{2}q_{UM}.$$  

- Combining the above two best response functions, we can solve for the equilibrium values of $q_M$ and $q$,

$$q_M = \frac{1 - c + (n - k + 1)x}{n - k + 2} \quad \text{and} \quad q_{UM} = \frac{1 - c - x}{n - k + 2}.$$
(c) For which value of the cost-reduction effect, $x$, the unmerged firms produce a positive output?

- Setting $q_{UM} \geq 0$, we obtain

$$\frac{1 - c - x}{n - k + 2} \geq 0$$

rearranging, yields

$$\frac{x}{1 - c} < 1.$$  

Intuitively, the cost-reduction effect, $x$, as a percentage of per-unit margins, $1 - c$, must be relatively small. Otherwise, the cost-reduction effect that the merged entity experiences forces the exit of those firms that did not merge. For compactness, let $\theta \equiv \frac{x}{1 - c}$, so we can write the above inequality as $\theta < 1$.

(d) Find the profits for the merged and unmerged firms.

- Inserting the equilibrium values of $q_M$ and $q_{UM}$ into their corresponding profits, we obtain

$$\pi_M = \left( \frac{1 - c + (n - k + 1)x}{n - k + 2} \right)^2,$$

$$\pi_{UM} = \left( \frac{1 - c - x}{n - k + 2} \right)^2$$

(e) For which values of $n$ and $k$ can a merger be sustained in equilibrium? Interpret.

- A merger between $k$ out of $n$ firms is profitable if the post-merger profits exceed the pre-merger profits (for all firms that merged, as a whole), that is,

$$\pi_M \geq k\pi^{NM}$$

or

$$\left( \frac{1 - c + (n - k + 1)x}{n - k + 2} \right)^2 \geq k \left( \frac{1 - c}{n + 1} \right)^2$$

which we can start simplifying as

$$\frac{1 - c + (n - k + 1)x}{n - k + 2} \geq \sqrt{k} \frac{1 - c}{n + 1}$$

or further rearrange as

$$\theta \equiv \frac{x}{1 - c} \geq \frac{(n - k + 2)\sqrt{k} - (n + 1)}{(n - k + 1)(n + 1)} \equiv \theta_M.$$  

Intuitively, for the merger to be profitable, the cost-reduction effect, as captured by $\theta$ in the left-hand side of the inequality, must be sufficiently strong (higher than $\theta_M$, where subscript $M$ denotes that firms merge).
For instance, when the merger does not produce cost reduction effects, $x = 0$, the left-hand side collapses to zero, and then the merger is only profitable if $0 \geq \theta_M$, which solving for the number of firms that merge, $k$, yield

$$k \geq \frac{2n+3-\sqrt{4n+5}}{2} \equiv \tilde{k}$$

as in our analysis of mergers between several firms (where the merger does not produce synergies). If, instead, $x > 0$, and only two firms merge, $k = 2$, condition $\theta \geq \theta_M$ becomes

$$\theta \geq \frac{[(\sqrt{2} - 1) n - 1]}{n^2 - 1} \equiv \theta_M(2, n).$$

Cutoff $\theta_M(2, n)$ lies below 0.05 for most values of $n$. For instance, if $n = 3$, cutoff $\theta_M(2, 3) = 0.03$, indicating that, for the merger to be profitable, the cost-reduction effect (relative to the market size) must be larger than 3%.