EconS 594 - Industrial Organization
Homework #1 - Answer key

1. **Cournot with convex and asymmetric costs.** Consider a duopoly market with inverse demand curve $p(Q) = 85 - \frac{Q}{20}$, where $Q = q_1 + q_2$ denotes aggregate output. Firm 1 faces cost function $c(q_1) = 3,000 + 9q_1 + \frac{q_1^2}{200}$, while firm 2's cost function is $c(q_2) = 3,500 + 8q_2 + \frac{q_2^2}{200}$. Firms compete a la Cournot.

   (a) Find each firm's best response function. Identify the output levels making each firm shut down. Plot them in a figure.

   (b) Find the Cournot equilibrium output pair, equilibrium price, consumer surplus, and social welfare.

   - See answer key at the end of this handout.

2. **Salop circle with quadratic transportation costs.** Consider the Salop circle we presented in class, but assume that transportation costs are now τd^2, where d denotes the distance that the consumer travels to his selected shop.

   (a) Find the equilibrium price in this setting. How does it differ from that under linear transportation costs?

   - Assume that there are n firms. Consider firm i's choice of price p_i given that the other firms charge p. A consumer located at distance $x < \frac{1}{n}$ from firm i is indifferent between firm i and the nearest competitor (firm j) if

   $$p_i + \tau x^2 = p_j + \tau \left(\frac{1}{n} - x \right)^2.$$

   Solving for x, we obtain

   $$x = \frac{n(p_j - p_i) + \frac{\tau}{n}}{2\tau}$$

   which implies that firm i's demand is twice this amount (consumers to the left and right of the indifferent consumer), that is,

   $$Q_i(p_i, p) = 2x = \frac{1}{n} - \frac{n(p_i - p_j)}{\tau}.$$

   The firm then solves the following profit-maximization problem

   $$\max_{p_i} \; (p_i - c) Q_i(p_i, p).$$

   Differentiating with respect to p_i, yields

   $$\frac{1}{n} - \frac{n(2p_i - p_j - c)}{\tau} = 0$$

   Invokying symmetry, $p_i = p_j = p$, and solving for p, we obtain the equilibrium price

   $$p^* = c + \frac{\tau}{n^2}.$$

(b) Find the equilibrium number of firms entering the industry, n^e, when entry cost is $e > 0$.

- Firm profits in equilibrium are

$$\begin{aligned}\pi^*(e) &= (p^* - c)Q_i(p^*, p^*) - e \\ &= \left(c + \frac{\tau}{n^2} - c\right)\left(\frac{1}{n} - \frac{n(p_i - p_j)}{\tau}\right) - e \\ &= \frac{\tau}{n^3} - e.\end{aligned}$$

Using the zero-profit condition, $\pi^*(e) = 0$, and solving for e, we obtain the equilibrium number of firms entering the industry, as follows

$$n^e = \left(\frac{\tau}{e}\right)^{1/3}$$

which entails an equilibrium price of

$$p^* = c + \frac{\tau}{\left[\left(\frac{\tau}{e}\right)^{1/3}\right]^2} = c + \tau^{1/3}e^{2/3}.$$

(c) Find the socially optimal number of firms entering the industry, n^{SO}, when entry cost is $e > 0$.

- To find the socially optimal number of firms entering the industry, recall that social welfare in this context coincides with total entry costs plus total transportation costs. This is because consumers purchase one unit of the good and all the market is covered, entailing that loss in consumer surplus coincides with the increase in firm revenues.
Therefore, we only need to minimize the sum of total entry costs and total transportation costs, as follows

$$\min_n \; ne + 2n\tau \int_0^{\frac{1}{2n}} x^2 dx = \min_n \; ne + \frac{\tau}{12n^2}$$

Differentiating with respect to n, yields

$$e - \frac{\tau}{6n^3} = 0$$

which, solving for n, yields the socially optimal number of firms entering the industry

$$n^{SO} = \left(\frac{\tau}{6e}\right)^{1/3} = \frac{1}{6^{1/3}}n^e \simeq 0.55n^e.$$

(d) Compare your results in parts (b) and (c). Interpret.

- The socially optimal number of firms is lower than the equilibrium number of firms, $n^{SO} < n^e$, since every firm does not internalize the business-stealing effect that its entry imposes on other firms. The regulator internalizes this external effect, seeking less entry.

(e) Compare your result in part (d) against the case in which transportation costs are linear (check your class notes). Interpret.

- Equilibrium entry with linear transportation costs is $n^e = \left(\frac{\tau}{e}\right)^{1/2}$, thus being larger than under quadratic transportation costs. We can also measure the excessive entry in each context as follows. Under linear transportation costs, excessive entry is given by

$$\text{Excessive Entry} = \left(\frac{\tau}{e}\right)^{1/2} - \frac{1}{2}\left(\frac{\tau}{e}\right)^{1/2} = 0.5\left(\frac{\tau}{e}\right)^{1/2}$$

while under quadratic transportation costs, we have

$$\text{Excessive Entry} = \left(\frac{\tau}{e}\right)^{1/3} - \left(\frac{\tau}{6e}\right)^{1/3} \simeq 0.45\left(\frac{\tau}{e}\right)^{1/3}.$$

3. **An investment game.** Consider a duopoly market with a continuum of homogeneous consumers of mass 1. Consumers derive utility $v_i \in \{v^H, v^L\}$ for product i depending on whether the product is of high or low quality. Firms play the following 2-stage game:

1st At stage 1, firms simultaneously invest in quality. The more a firm invests the higher is its probability λ_i of obtaining a high-quality product. The associated investment cost is denoted by $I(\lambda_i)$ and satisfies standard properties that ensure an interior solution: $I(\lambda_i)$ is continuous for $\lambda_i \in [0,1)$, $I'(\lambda_i) > 0$ and $I''(\lambda_i) > 0$ for $\lambda_i \in (0,1)$, and the Inada conditions $\lim_{\lambda_i \to 0^+} I'(\lambda_i)$ and $\lim_{\lambda_i \to 1^-} I'(\lambda_i) = +\infty$. Before the beginning of stage 2 qualities become publicly observable— i.e., all uncertainty is resolved.

2nd At stage 2, firms simultaneously and independently set prices.

a) For any given (λ_1, λ_2), what are the expected equilibrium profits? In case of multiple equilibria select the (from the view point of the firms) Pareto-dominant equilibrium.

b) Are investments strategic complements or substitutes? Explain your finding.

c) Provide the equilibrium condition at the investment stage.

d) How do equilibrium investments change as valuation v^H increases? How do they change when v^L increases?

- See answer key at the end of this handout.

4. **Vertical product differentiation and cost of quality.** A consumer with income m who consumes a product of quality s_i and pays p_i obtains the utility $\frac{s_i m}{6} - p_i$. If instead the consumer decides not buy the good, the resulting utility is zero. Consumer income m is uniformly distributed on the interval $[2, 8]$, so its density is $f(m) = \frac{1}{8-2} = \frac{1}{6}$. The total mass of consumers is equal to 1.

There are two firms in the market, 1 and 2, offering qualities s_1 and s_2, respectively. Assume that $s_1, s_2 \in [1, 2]$. Label firms such that $s_1 \leq s_2$. Suppose that firm i has constant marginal cost equal to $c \times s_i$ where $c < 2$.

(a) Derive the demand of firms 1 and 2, and calculate the best-response functions of the two firms presuming that first-order conditions hold with equality. Distinguish between full and partial market coverage.

(b) Calculate the Nash Equilibrium in prices and find the equilibrium profits as a function of s_1 and s_2. Distinguish between full and partial market coverage.

(c) What are the equilibrium quality choices of the two firms? (Again distinguish between the full and market coverage cases using first-oder conditions.)

(d) Which firm is more profitable? Consider the two cases mentioned above.

(e) How does in the partial coverage equilibrium an increase in cost c affect the profits of the two firms?

- See answer key at the end of this handout.

HOMEWORK #1, ECONS 594

EXERCISE #1

In a duopoly market, let the equation of the inverse demand curve be

$$p(q_1+q_2) = 85 - \frac{1}{20}(q_1+q_2),$$

and let the equations of the cost functions of the two firms be

$$c_1(q_1) = 3000 + 9q_1 + \frac{1}{200}q_1^2$$

and

$$c_2(q_2) = 3500 + 8q_2 + \frac{1}{200}q_2^2$$

respectively.
(a) Find the equations of the best response functions, identifying the levels of rival firm output at which each firm would shut down because its maximum profit would be negative if it produced positive output. Graph the best response functions on the same diagram; indicate the Cournot equilibrium point.

Firm 1's profit is

$$\pi_1 = \left[85 - \frac{1}{20}(q_1+q_2)\right] q_1 - 3000 - 9q_1 - \frac{1}{200}q_1^2$$

$$= \left(76 - \frac{11}{200}q_1 - \frac{1}{20}q_2\right) q_1 - 3000$$

The first-order condition to maximize π_1 (the equation of firm 1's best response function) is

$$76 - \frac{11}{200}q_1 - \frac{1}{20}q_2 - \frac{11}{200}q_1 = 0.$$

Hence along the best response function

$$76 - \frac{11}{200}q_1 - \frac{1}{20}q_2 = \frac{11}{200}q_1$$

and firm 1's payoff is

$$\pi_1 = \frac{11}{200}q_1^2 - 3000.$$

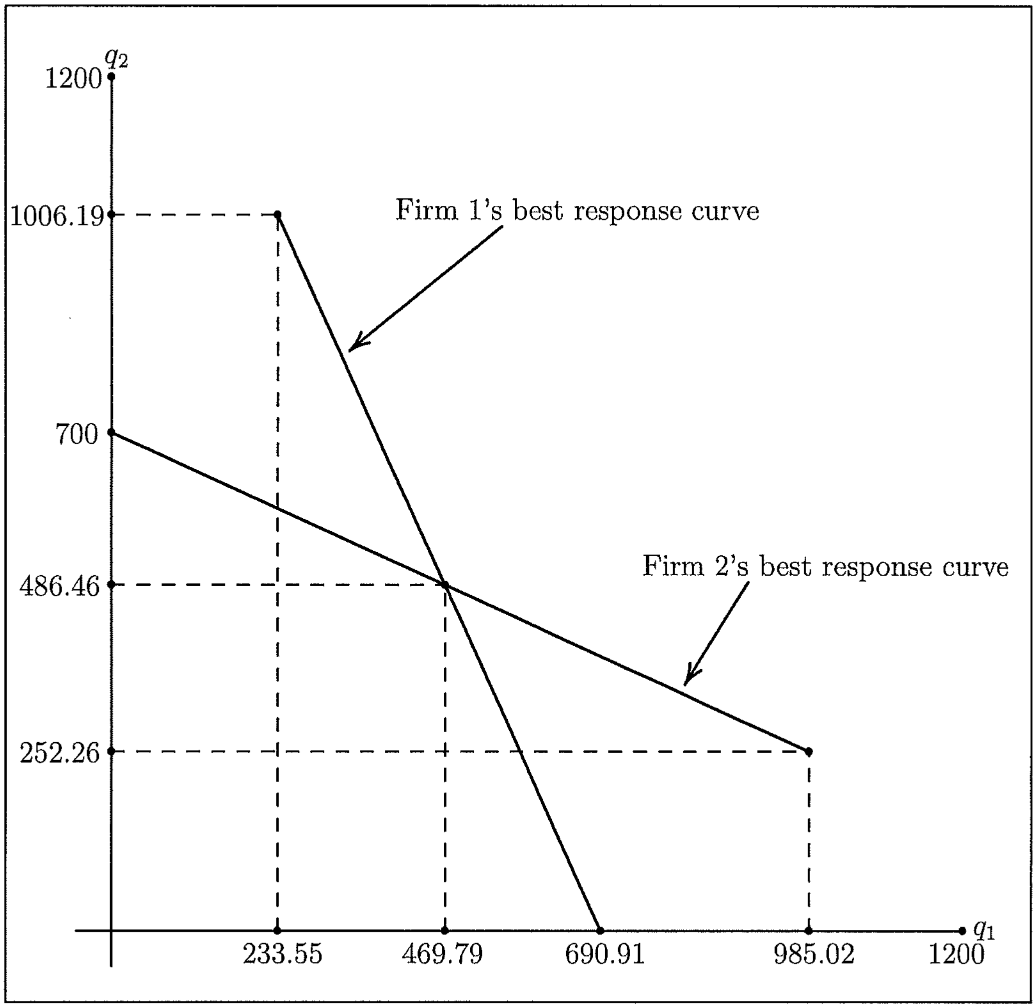

Figure 2.1: Best response curves, Stigler duopoly example

Firm 1 will produce only if its payoff is nonnegative. Hence firm 1 will shut down if

$$q_1 < \sqrt{\frac{200}{11}(3000)} = 233.55.$$

The equation of firm 1's best response function can also be written

$$11q_1 + 5q_2 = 7600,$$

and this is valid for $q_1 \geq 233.55$.

$$\pi_2 = \left[85 - \frac{1}{20}(q_1 + q_2)\right] q_2 - 3500 - 8q_2 - \frac{1}{200}q_2^2$$

$$= \left(77 - \frac{1}{20}q_1 - \frac{11}{200}q_2\right) q_1 - 3500.$$

Firm 2's payoff along its best response function is

$$\pi_2 = \frac{11}{200}q_2^2 - 3500;$$

firm 2 will shut down for

$$q_2 < \sqrt{\frac{200}{11}(3500)} = 252.26.$$

The equation of firm 2's best response function is

$$5q_1 + 11q_2 = 7700,$$

and this is valid for $q_2 \geq 252.26$.

The best response curves are shown in Figure 2.1.

(b) Find the Cournot equilibrium outputs by solving the equations of the best response functions. What is Cournot equilibrium price, consumer surplus, and net social welfare?

Write the system of equations of the best response functions as

$$\begin{pmatrix} 11 & 5 \\ 5 & 11 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 100 \begin{pmatrix} 76 \\ 77 \end{pmatrix}$$

$$96\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 100\begin{pmatrix} 11 & -5 \\ -5 & 11 \end{pmatrix}\begin{pmatrix} 76 \\ 77 \end{pmatrix} = 100\begin{pmatrix} 451 \\ 467 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{25}{24}\begin{pmatrix} 451 \\ 467 \end{pmatrix} = \begin{pmatrix} 469.79 \\ 486.46 \end{pmatrix}.$$

Equilibrium payoffs are

$$\pi_1 = \frac{11}{200}(469.79)^2 - 3000 = 9138.6$$

$$\pi_2 = \frac{11}{200}(486.46)^2 - 3500 = 9515.4.$$

Cournot equilibrium output and price are

$$Q = 469.79 + 486.46 = 956.25$$

$$p = 85 - \frac{1}{20}(956.25) = 37.19.$$

Consumers' surplus, the area of the triangle below the inverse demand curve and above the line $p = 37.19$, is

$$\frac{1}{2}(85 - 37.19)(956.25) = 22859.16.$$

Net social welfare is the sum of firm profits and consumers' surplus:

$$9138.6 + 9515.4 + 22859.16 = 45513.2.$$

EXERCISE #3

1. Bertrand competition: If $v_1 = v_2$, $\pi_1^* = \pi_2^* = 0$. If $v_i > v_j$, $\pi_i^* = \Delta$ and $\pi_j = 0$, $i \neq j$. The expected equilibrium profit is $E\pi_i^* = \lambda_i(1-\lambda_j)\Delta$.

2. At stage 1, each firm solves $\max_{\lambda_i} \lambda_i(1-\lambda_j)\Delta - I(\lambda_i)$. First-order condition of profit maximization is

$$(1-\lambda_j)\Delta = I'(\lambda_i)$$

Since $I(\lambda_i)$ is strictly convex $I'(\lambda_i)$ is monotone and thus invertible.

$$\lambda_i = (I')^{-1}[(1-\lambda_j)\Delta]$$

Since $I'(\lambda_i)$ is increasing the best response of firm i is decreasing in λ_j and investment decisions are strategic substitutes.

3. The equilibrium investment decisions $\lambda_1^* = \lambda_2^* \equiv \lambda^*$ are characterized by

$$(1-\lambda^*)\Delta = I'(\lambda^*).$$

There is a unique solution to this equation.

4. Rewriting the above equation as

$$\Delta = \frac{I'(\lambda^*)}{1-\lambda^*}$$

we see that the numerator on the right-hand side is increasing in λ^* while the denominator on that side is decreasing in λ^*. Thus, an increase in Δ implies a larger λ^*. Due to the nature of Bertrand competition only the quality difference but not the absolute levels of qualities affect investment incentives.

Exercise 10 *Hotelling model*

Reconsider the simple Hotelling model in which consumers are uniformly distributed on the unit interval and firms are located at the extremes of this interval. Now take consumers' participation constraint explicitly into account. Derive the equilibrium depending on the parameter τ. [Be careful to distinguish between different regimes with respect to competition between firms!]

Exercise 11 *Price and quantity competition*

Reconsider the duopoly model with linear individual demand and differentiated products. Show that profits under quantity competition are higher than under price competition if products are substitutes and that the reverse holds if products are complements.

Exercise 12 *Asymmetric duopoly [included in the 2nd edition of the book]*

EXERCISE #4

Assumptions:

- $v_i = \frac{s_i m}{6} - p_i$ (indirect utility) and $v_0 = 0$ if no good is consumed
- Income m is uniformy distributed on $[2;8]$. Thus $f(m) = \frac{1}{8-2} = \frac{1}{6}$
- 2 firms with qualities such that $s_1 \leq s_2, s_i \in [1;2]$
- $MC_i = cs_i$
- There may be full or partial market coverage.

1. Find consumer $\hat{m}$ who is indifferent between low and high quality.

$$\frac{s_1 \hat{m}}{6} - p_1 = \frac{s_2 \hat{m}}{6} - p_2$$

Thus,

$$\hat{m} = \frac{6(p_2 - p_1)}{s_2 - s_1}$$

Under full market coverage (condition for all consumers to buy: even for $\hat{m} = 2$, $v_1 \geq 0$):

$$\frac{2s_1}{6} - p_1 \geq 0 \quad \Leftrightarrow \quad p_1 \leq \frac{s_1}{3}$$

Find demand for given s_1, s_2.

Case 1: Full coverage with $p_1 \leq \frac{s_1}{3}$.

$$Q_1(p_1, p_2) = \int_2^{\hat{m}} \frac{1}{6} dm = \frac{\hat{m}}{6} - \frac{1}{3} = \frac{p_2 - p_1}{s_2 - s_1} - \frac{1}{3}$$

$$Q_2(p_1, p_2) = \int_{\hat{m}}^8 \frac{1}{6} dm = \frac{4}{3} - \frac{p_2 - p_1}{s_2 - s_1}$$

Case 2: Partial coverage with $p_1 > \frac{s_1}{3}$. Marginal consumer $\widetilde{m}$ such that $v_1(p_1) = 0$

$$\frac{s_1 \widetilde{m}}{6} - p_1 = 0 \quad \Leftrightarrow \quad \widetilde{m} = \frac{6p_1}{s_1}$$

$$\text{thus,} Q_1(p_1, p_2)\Big|_{p_1 > \frac{s_1}{3}} = \int_{\widetilde{m}}^{\hat{m}} \frac{1}{6} dm = \frac{\hat{m}}{6} - \frac{\widetilde{m}}{6} = \frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1}$$

Find best reply for both cases:

$$\max_{p_i} \quad \pi_i = (p_i - cs_i) Q_i(p_i, p_j)$$

$$
\begin{aligned}
p_1^{br}(p_2) &= \begin{cases} \frac{p_2 + cs_1}{2} - \frac{1}{6}(s_2 - s_1) & \text{if foc with full coverage holds} \\ \frac{p_2 \frac{s_1}{s_2} + cs_1}{2} & \text{if foc with partial coverage holds} \end{cases} \\
p_2^{br}(p_1) &= \frac{p_1 + cs_2}{2} + \frac{2}{3}(s_2 - s_1)
\end{aligned}
$$

Firm 1 exhibits a kinked demand curve. Best-response function is reported only when foc's holds with equality. When the price at the kink is profit maximizing, we simple have as the best response $p_1 = s_1/3$; this case is ignored below.

2. Case 1 (using best response with full coverage)

   Equilibrium at the second stage: Solve $p_1^{br}(p_2^{br}(p_1))$ for p_1.

$$
p_1^* = \frac{1}{3}c(s_2 + 2s_1) + \frac{2}{9}(s_2 - s_1) \qquad Q_1^* = \frac{1}{3}c + \frac{2}{9}
$$
$$
p_2^* = \frac{1}{3}c(s_1 + 2s_2) + \frac{7}{9}(s_2 - s_1) \qquad Q_2^* = -\frac{1}{3}c + \frac{7}{9}
$$

$$
\pi_1^*(s_1, s_2) = (\frac{1}{3}c + \frac{2}{9})^2(s_2 - s_1)
$$
$$
\pi_2^*(s_1, s_2) = (\frac{1}{3}c - \frac{7}{9})^2(s_2 - s_1)
$$

   Case 2 (using best response with full coverage)

   Equilibrium at the second stage:

$$
\begin{aligned}
p_1^* &= \frac{s_1((4+9c)s_2 - 4s_1)}{3(4s_2 - s_1)} \\
p_2^* &= \frac{s_2((8+6c)s_2 - (8-3c)s_1))}{3(4s_2 - s_1)} \\
Q_1^* &= \frac{s_2(4-3c)}{3(4s_2 - s_1)} \\
Q_2^* &= \frac{2s_2(4-3c)}{3(4s_2 - s_1)} \\
\widetilde{\pi}_1^*(s_1, s_2) &= \frac{(4-3c)^2 s_1 s_2 (s_2 - s_1)}{9(4s_2 - s_1)^2} \\
\widetilde{\pi}_2^*(s_1, s_2) &= \frac{4(4-3c)^2 s_2^2 (s_2 - s_1)}{9(4s_2 - s_1)^2}
\end{aligned}
$$

   Note: This is not a full equilibrium characterization, as firm 1 has kinked demand and may set price at the kink.

3. Case 1: At the first stage, determine s_1^*, s_2^*.

$\frac{\partial \pi_1^*}{\partial s_1} < 0$ and $\frac{\partial \pi_2^*}{\partial s_2} > 0$
Firms will choose s_1, s_2 to maximally differentiate $s_2^* = 2$ and $s_1^* = 1$.

$$\pi_1^{**}(c) = \pi_1^*(1,2) = (\frac{1}{3}c + \frac{2}{9})^2$$
$$\pi_2^{**}(c) = \pi_2^*(1,2) = (\frac{1}{3}c - \frac{7}{9})^2$$

Case 2: At the first stage, determine s_1^*, s_2^*.

First-order conditions w.r.t. s_i yields best replies for s_i

$s_1^*(s_2) = \frac{4}{7}s_2$.

$s_2^*(s_1)$ no interior solution $\longrightarrow$ use Kuhn-Tucker $\rightarrow s_2^* = 2$.

In equilibrium, $s_2^* = 2$ and $s_1^* = \frac{8}{7}$

$$\widetilde{\pi}_1^{**}(c) = \frac{(4-3c)^2}{216} \text{ and } \widetilde{\pi}_2^{**}(c) = \frac{7(4-3c)^2}{216}$$

4. Show that $\widetilde{\pi}_1 < \widetilde{\pi}_2$ in case 1:

$$\begin{aligned}
(\frac{1}{3}c + \frac{2}{9})^2 &< (\frac{1}{3}c - \frac{7}{9})^2 \\
\frac{c^2}{9} + \frac{4}{27}c + \frac{4}{81} &< \frac{c^2}{9} - \frac{14}{27}c + \frac{49}{81} \\
\frac{6}{9}c &< \frac{45}{81} \\
c &< \frac{5}{6}
\end{aligned}$$

and thus always holds when case 1 applies. Case 2: $\widetilde{\pi}_2 > \widetilde{\pi}_1$ always satisfied.

5.

$$\begin{aligned}
\frac{d\widetilde{\pi}_1^{**}}{dc} &= -\frac{(4-3c)}{36} < 0 \text{ if } c < \frac{4}{3} \\
\frac{d\widetilde{\pi}_2^{**}}{dc} &= -\frac{7(4-3c)}{36} < 0 \text{ if } c < \frac{4}{3}
\end{aligned}$$

$\widetilde{\pi}_2$ falls more strongly than $\widetilde{\pi}_1$. Differentiation becomes more costly for firm 2 ($s_2 = 2$) as c increases. Price p_2 increases more strongly than p_1 and firm 2 will loose consumers to firm 1. As c increases, firm 1 experiences higher demand because consumers switch from 2 to 1.

Exercise 8 *The quality-quantity trade-off under vertical differentiation*[2] *[included in 2nd edition of the book]*

[2]This exercise draws from McCannon, B.C.. (2008). The Quality-Quantity Trade-off, *Eastern Economic Journal* 34, 95-100.