EconS 503 - Microeconomic Theory II
Final Exam - Answer Key

1. **Excess demand in General Equilibrium.** Consider a two-commodity exchange economy, with two agents $i = \{A, B\}$ whose utility functions are

$$\begin{aligned} u^A(x^A) &= \log(x_1^A) + 2\log(x_2^A) \\ u^B(x^B) &= 2\log(x_1^B) + \log(x_2^B) \end{aligned}$$

Initial endowments are $\omega^A = (9, 3)$ and $\omega^B = (12, 6)$.

(a) Find the excess demand function for each good. Verify that Walras' Law holds.

- Since we need the demand functions for each individual, we first solve the UMP. The Lagrangian for consumer A is

$$\mathcal{L}^A = \log(x_1^A) + 2\log(x_2^A) + \lambda_A \left[9p_1 + 3p_2 - p_1 x_1^A - p_2 x_2^A\right]$$

Taking first-order conditions,

$$\begin{aligned} \frac{1}{x_1^A} - \lambda_A p_1 &= 0 \\ \frac{2}{x_2^A} - \lambda_A p_2 &= 0 \\ 9p_1 + 3p_2 - p_1 x_1^A - p_2 x_2^A &= 0 \end{aligned}$$

Define $p = \frac{p_1}{p_2}$, so we normalize $p_2 = 1$. This normalization helps us more compactly express the first-order conditions as

$$\begin{aligned} \frac{1}{\lambda_A} &= p x_1^A \\ \frac{2}{\lambda_A} &= x_2^A \\ 9p + 3 &= p x_1^A + x_2^A \end{aligned}$$

Subtracting the first two equations from the third , we find that $\lambda_A^* = \frac{1}{1+3p}$. Substituting back $\lambda_A^* = \frac{1}{1+3p}$ into the first two equations above, we obtain Walrasian demands

$$x_1^{A,*} = 3 + \frac{1}{p} \qquad \text{and} \qquad x_2^{A,*} = 6p + 2$$

- Following similar steps for consumer B we would find Walrasian demands

$$x_1^{B,*} = 8 + \frac{4}{p} \qquad \text{and} \qquad x_2^{B,*} = 4\rho + 2$$

- *Excess Demands.* Using the definition of excess demand:

$$z_k(p) = x_k^{A,*} + x_k^{B,*} - \omega_k^A - \omega_k^B$$

for every good $k = \{1, 2\}$. Doing this we obtain

$$\begin{aligned} z_1(p) &= \overbrace{3+\frac{1}{p}}^{x_1^{A,*}} + \overbrace{8+\frac{4}{p}}^{x_1^{B,*}} - \overbrace{9}^{\omega_1^A} - \overbrace{12}^{\omega_1^B} = \frac{5}{p} - 10 \\ z_2(p) &= \underbrace{6p+2}_{x_2^{A,*}} + \underbrace{4p+2}_{x_2^{B,*}} - \underbrace{3}_{\omega_2^A} - \underbrace{6}_{\omega_2^B} = 10p - 5 \end{aligned}$$

- *Walras' Law.* We can now check if Walras' law holds, i.e., $\sum_{k=1}^N p_k z_k(p) = 0$. Since in this case $p_2 = 1$ and $\frac{p_1}{p_2} \equiv p$, Walras' law becomes

$$p \cdot z_1(p) + z_2(p) = 0.$$

which holds since

$$\underbrace{p\left(\frac{5}{p} - 10\right)}_{z_1(p)} + \underbrace{(10p-5)}_{z_2(p)} = (5 - 10p) + (10p - 5) = 0$$

(b) Find the equilibrium price ratio.

- Solving for $z_1(p) = \frac{5}{p} - 10 = 0$ in part (a), we find

$$p^* = \frac{1}{2}$$

for the (relative) equilibrium prices.

(c) What is the WEA?

- Substituting the equilibrium price ratio $p^* = \frac{1}{2}$ into the Walrasian demands found in part (a) yields

$$x_1^{A,*} = 3 + \frac{1}{p^*} = 5 \qquad \text{and} \qquad x_2^{A,*} = 6p^* + 2 = 5$$

for consumer A, and

$$x_1^{B,*} = 8 + \frac{4}{p^*} = 16 \qquad \text{and} \qquad x_2^{B,*} = 4p^* + 2 = 4$$

for consumer B. Hence, the WEA is

$$(x_1^A, x_2^A; x_1^B, x_2^B; p) = \left(5, 5; 16, 4; \frac{1}{2}\right).$$

2. **Grobes-Loeb mechanism.** Consider a divisible public good y with cost function $c(y) = y$. Every agent $i \in N$ enjoys a benefit $b_i(y, \theta) = \theta_i \sqrt{y}$ from y units of the public good, and $\theta_i > 0$ denotes agent i's valuation.

(a) Find the socially optimal amount of the public good, y^{SO}. [*Hint*: For compactness, let $\theta \equiv \sum_{i=1}^{N} \theta_i$.]

- The social planner chooses y to solve the following welfare maximization problem:

$$\max_{y \geq 0} \; W(y, \theta) = \sum_{i=1}^{N} b_i(y, \theta) - c(y)$$
$$= \left(\sum_{i=1}^{N} \theta_i \right) \sqrt{y} - y$$

For compactness, let $\theta \equiv \sum_{i=1}^{N} \theta_i$. Differentiating with respect to y and assuming interior solutions, we obtain

$$\frac{\theta}{2\sqrt{y}} = 1$$

Solving for y in the above expression, yields

$$y^{SO} = \frac{\theta^2}{4}.$$

(b) Consider a direct revelation mechanism (DRM) where every agent reports her valuation of the public good θ_i, and then outcome $(y^{SO}, c_1, c_2, ..., c_N)$ is implemented, where $c_i = \frac{1}{4}\theta_i \sum_{i=1}^{N} \theta_i$ represents agent i's cost share. Show that this DRM is not strategy-proof.

- Let the social planner implements $y = \frac{1}{4}\left(\sum_{i=1}^{N} \hat{\theta}_i \right)^2$, where $\hat{\theta}_i$ is the reported valuation by each agent i; then letting $\theta_{-i} \equiv \sum_{j \neq i} \theta_j$ be the actual valuations of all other agents $j \neq i$, agent i solves the following utility maximization problem:

$$\max_{\hat{\theta}_i \geq 0} u_i\left(\hat{\theta}_i, \theta\right) = b_i\left(\hat{\theta}_i, \theta_{-i}\right) - c_i\left(\hat{\theta}_i, \theta_{-i}\right)$$
$$= \frac{\theta_i}{4}\left(\hat{\theta}_i + \theta_{-i}\right) - \frac{1}{4}\hat{\theta}_i\left(\hat{\theta}_i + \theta_{-i}\right)$$
$$= \frac{2\theta_i - \hat{\theta}_i}{2}\left(\hat{\theta}_i + \theta_{-i}\right)$$

Differentiating with respect to agent i's report $\hat{\theta}_i$, and assuming interior solutions, we find

$$-\hat{\theta}_i - \theta_{-i} + 2\theta_i - \hat{\theta}_i = 0$$

Rearranging the above expression, yields

$$\hat{\theta}_i = \theta_i - \frac{\theta_{-i}}{2}$$

such that agent i has an incentive to *underreport* his valuation $\hat{\theta}_i$ below his true valuation θ_i, and the same applies to all other agents $j \neq i$. As a result, this DRM is not strategyproof.

(c) Let us now consider an alternative DRM with the following transfer function

$$t_i = \frac{1}{4}\theta_i^2 + \frac{1}{2(N-2)} \sum_{j,k \neq i,\, j<k} \theta_j \theta_k$$

implying that individual i's utility function becomes $u_i(y, \theta) = \theta_i \sqrt{y} - t_i$. Assume that the public project is budget balanced, $\sum_{i=1}^{N} t_i \geq c(y)$. Show that the above mechanism is dominant strategy incentive compatible (DSIC).

- Agent i chooses a report, $\hat{\theta}_i$, that solves

$$\max_{\hat{\theta}_i \geq 0} \; u_i\left(\hat{\theta}_i, \theta\right) = b_i\left(\hat{\theta}_i, \theta_{-i}\right) - t_i\left(\hat{\theta}_i, \theta_{-i}\right)$$

$$= \underbrace{\frac{\theta_i}{2}\left(\hat{\theta}_i + \theta_{-i}\right)}_{b_i\left(\hat{\theta}_i, \theta_{-i}\right)} - \underbrace{\left(\frac{1}{4}\hat{\theta}_i^2 + \frac{1}{2(N-2)} \sum_{j,k \neq i,\, j<k} \theta_j \theta_k\right)}_{t_i\left(\hat{\theta}_i, \theta_{-i}\right)}$$

Differentiating with respect to agent i's report, $\hat{\theta}_i$, and assuming interior solutions, we obtain

$$\frac{\theta_i - \hat{\theta}_i}{2} = 0$$

Solving for agent i's report $\hat{\theta}_i$, yields

$$\hat{\theta}_i = \theta_i$$

so that every agent i has incentives to truthfully report his type θ_i. This result holds regardless of the report that all other agents submit, $\hat{\theta}_{-i}$, which could coincide with their true profile of types θ_{-i} or not. Therefore, player i finds truthfully reporting his type to be a strictly dominant strategy, as required for DSIC.

3. **Checking properties on a social welfare function - Kaneko and Nakamura (1979)**[1] Consider the following social welfare function

$$SW(x) = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_N^{\alpha_N} = \prod_{i=1}^{N} x_i^{\alpha_i}$$

where $x = (x_1, x_2, \ldots x_N)$ denotes an alternative, where $x \in \mathbb{R}^N$, and $\alpha_i > 0$ represents the weight that the social planner assigns to agent i. Alternatively, this social welfare

[1] Kaneko M. and Nakamura K. (1979). The Nash Social Welfare Function. *Econometrica,* 47(2), pp. 423-35.

function can be represented in its linear form by applying logs, as follows,

$$\alpha_1 \ln x_1 + \alpha_2 \ln x_2 \cdot ... \cdot \alpha_N \ln x_N = \sum_{i=1}^{N} \alpha_i \ln x_i$$

In this exercise, we show that ordinal preferences among the alternatives can be represented by the above (cardinal) social welfare function that satisfies:

- Paretian (P),
- Anonymity (A),
- Neutrality (N), and
- Independence of Irrelevant Alternatives (IIA).

Show that the above social welfare function $SW(x)$ satisfies these five properties.

- *Paretian.* Consider two alternatives, x and y, where $x \geq y$ for all agents, then

$$\begin{aligned} SW(x) &= \prod_{i=1}^{N} x_i^{\alpha_i} \\ &\geq \prod_{i=1}^{N} y_i^{\alpha_i} = SW(y) \end{aligned}$$

  so that, when all agents prefer alternatives x to y, the social planner also prefers alternative x to y; thereby satisfying the Pareto Optimality (P) condition.

- *Anonymity.* Consider a permutation of agents, where agent i is assigned a new identity, $\pi(i) \neq i$.

$$\begin{aligned} SW(\pi(x)) &= \prod_{\pi_i=1}^{N} x_{\pi_i}^{\alpha_{\pi_i}} \\ &= \prod_{i=1}^{N} x_i^{\alpha_i} = SW(x) \end{aligned}$$

  because when the agents are assigned a new identity, both their endowments x and weights α are permuted, so that the social welfare generated from the allocation $\pi(x)$ coincides with that emerges from the original allocation x; thereby satisfying the Anonymity (A) condition.

- *Neutrality.* Consider a permutation of alternatives, where $x_i = \beta_i y_i$, for which $\beta_i > 0$ and $\beta_i \neq \beta_j$ in general (that is, we consider alternative x to be different from alternative y by a factor of β_i that is generally not the same for each agent

i), then

$$\begin{aligned} SW\left(x\right) &= \prod_{i=1}^{N} x_i^{\alpha_i} \\ &= \prod_{i=1}^{N} \left(\beta_i y_i\right)^{\alpha_i} \\ &= \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right)\left(\prod_{i=1}^{N} y_i^{\alpha_i}\right) \\ &= \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) SW\left(y\right) \end{aligned}$$

so that when the alternatives are assigned a new identity, social welfare is also scaled by a factor of $\prod_{i=1}^{N} \beta_i^{\alpha_i}$ that preserves the ranking of alternatives, where

$$\left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) SW\left(y\right) = \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) SW\left(\frac{x}{\beta}\right) = \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{x_i}{\beta_i}\right)^{\alpha_i} = SW\left(x\right);$$

thereby satisfying the Neutrality (N) condition.

- *Independence of Irrelevant Alternatives.* Consider an alternative z. Since $SW\left(x\right)$ only considers alternative x and $SW\left(y\right)$ only considers alternative y, the ranking of x and y does not depend on z; thereby satisfying the Independence of Irrelevant Alternatives (IIA) condition.