EconS 503 - Microeconomic Theory II
Midterm Exam #2 - Answer key

1. **All-pay auction.** Consider an independent private value all-pay auction (APA), where every bidder must pay his bids, regardless whether he wins the object or not. As in other auctions, let v_i denote bidder i's privately observed valuation, and b_i his bid.

   (a) Find the equilibrium bidding function allowing for valuations to be distributed according to a general cdf $F(v_i)$ with associated density $f(v_i) > 0$ in all v_i. Assume that N bidders participate in this auction.

   - The expected payoff of a bidder i with valuation v_i is

$$\begin{aligned} EU_i(v) &= prob_i(win)\left[v_i - b_i(v_i)\right] + \left[1 - prob_i(win)\right](-b_i(v_i)) \\ &= prob_i(win)v_i - b_i \end{aligned}$$

   where, if winning, bidder i obtains a net payoff $v_i - b_i(v_i)$; but, if losing, which occurs with probability $1 - prob_i(win)$, he must still pay the bid he submitted.
   We know that bidder i wins the auction if his bid, b_i, exceeds that of the highest competing bidder, $\beta(Y_i)$, where Y_1 represents the highest valuation among the $N-1$ remaining bidders (the first order statistic). Hence, the probability of winning is given by:

$$prob\,(win) = prob\,(\beta(Y_1) \leq b_i) = prob\left(Y_1 \leq \beta^{-1}(b_i)\right) = G\left(\beta^{-1}(b_i)\right)^{N-1}$$

   Thus, the expected payoff function for a bidder with valuation v_i is given by:

$$EU_i(v_i) = G\left(\beta^{-1}(b_i)\right)^{N-1}(v_i - b_i) + \left[1 - q\left(\beta^{-1}(b_i)\right)^{N-1}\right](-b_i)$$

   or, rearranging,

$$EU_i(v_i) = G\left(\beta^{-1}(b_i)\right)^{N-1} v_i - b_i$$

   We can now take first-order conditions with respect to the optimal bid of this player, b_i, we obtain

$$\frac{dEU_i(v_i)}{db_i} = \frac{g\left(\beta^{-1}(b_i)\right)}{\beta'\left(\beta^{-1}(b_i)\right)} v_i - 1 = 0$$

   Multiplying both sides by $\beta'\left(\beta^{-1}(b_i)\right)$, and rearranging,

$$g\left(\beta^{-1}(b_i)\right) v_i - \beta'\left(\beta^{-1}(b_i)\right) = 0$$

   Given that in equilibrium we have that $\beta(v_i) = b_i$, we can invert this bidding function to obtain $v_i = \beta^{-1}(b_i)$, Therefore, we can rewrite the above expression as:

$$g(v_i)\, v_i - \beta'(v_i) = 0$$

Given that this equality holds for any value of v, we can integrate on both sides of the equality, to obtain $\beta(v) = \int_0^v g(y)\, y dy + C$, where C is the integration constant and it is zero given that $\beta(0) = 0$. Thus, the optimal bidding function in an APA is:

$$\beta(v) = \int_0^v (g(y)\, y)\, dy$$

(b) *Uniform distribution.* For the rest of the exercise assume that valuations are uniformly distributed in $[0,1]$. Show that, in the case of two bidders, the equilibrium bidding function you found in part (a) can be represented as a *convex* function of his private valuation v_i for the object.

- Given that in this case we know that the valuations are drawn from a uniform distribution $F(x) = x$ with density $f(x) = 1$ and support $[0,1]$, $F(x)^{N-1} = x^{N-1}$ and its derivative is $dF(x)^{N-1} = (N-1)\, x^{N-2}$, implying that the optimal bidding function becomes

$$\begin{aligned} \widehat{b}(v_i) &= \int_0^v x(N-1)\, x^{N-2} dx = (N-1)\int_0^v x^{N-1} dx \\ &= (N-1)\left[\frac{x^N}{N}\right]_0^v = \frac{N-1}{N} v^N \end{aligned}$$

Therefore, in this case of $N = 2$ bidders, the optimal bidding function for the APA will be $\widehat{b}(v_i) = \frac{1}{2} v_i^2$, which is increasing and convex in bidder i's valuation.

(c) *Comparative statics.* Still assuming $v_i \sim U[0,1]$, analyze how the optimal bidding function varies in the number of bidders, N.

- When increasing the number of bidders, N, we obtain a more convex function. In particular, the optimal bidding function in the APA with two bidders is $\frac{v^2}{2}$, with three bidders becomes $\frac{2}{3} v^3$, and with ten bidders is $\frac{9}{10} v^{10}$. Intuitively, as more bidders compete for the object, bid shading becomes substantial when bidder i has a relatively low valuation, but induces him to bid more aggressively when his valuation is likely the highest among all other players, i.e., when $v_i \to 1$.

(d) *Comparison with FPA.* Still assuming $v_i \sim U[0,1]$, how do players' bids compare to those in the first-price auction? What is the intuition behind this difference in bids? [Recall that, in the first-price auction with uniformly distributed valuations, we found that the equilibrium bidding function was $b^{FPA}(v_i) = \frac{N-1}{N} v_i$.]

- The optimal bidding function in the FPA, $b^{FPA}(v) = \frac{N-1}{N} v$, lies above that in the APA, $b^{APA}(v) = \frac{N-1}{N} v^N$, since the difference

$$b^{FPA}(v) - b^{APA}(v) = \frac{(N-1)\left(v - v^N\right)}{N}$$

is positive, since $N \geq 2$ and $v > v^N$ (graphically, the 45^0-line lies above any convex function such as v^N). Hence, every bidder i in the APA bids less aggressively than in the FPA since he has to pay the bid he submits.

2. **Signaling and Limit pricing.** Consider a market with inverse demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Let us analyze an entry game with an incumbent monopolist (Firm 1) and an entrant (Firm 2) who analyzes whether or not to join the market. The incumbent's marginal costs are either high H or low L, i.e., $c_1^H = \frac{1}{2} > c_1^L = \frac{1}{3}$, while it is common knowlegde that the entrant's marginal costs are high, i.e., $c_2 = c_1^H = \frac{1}{2}$. To make the entry decision interesting, assume that when the incumbent's costs are low, entry is unprofitable; whereas when the incumbent's costs are high, entry is profitable. (Otherwise, the entrant would enter regardless of the incumbent's cost, or stay out regardless of the incumbent's cost.) For simplicity, assume no discounting of future payoffs throughout all the exercise.

   (a) *Complete information.* Let us first examine the case in which entrant and incumbent are informed about each others' marginal costs. Consider a two-stage game where, in the first stage, the incumbent has monopoly power and selects an output level. In the second stage, a potential entrant decides whether or not to enter. If entry occurs, agents compete as Cournot duopolists, simultaneously and independently selecting production levels. If entry does not occur, the incumbent maintains its monopoly power during both periods. Find the subgame perfect equilibrium (SPNE) of this complete information game.

   - We next apply backward induction, starting from the second-period game.
   - *Second period.* When no entry occurs, the incumbent solves

   $$\max_{x_1} (1 - x_1)x_1 - c_1^K x_1$$

   thus selecting monopoly output $x_1^{K,m} = \frac{1-c_1^K}{2}$ for every incumbent type $K = \{H, L\}$. If entry occurs, every firm $i = \{1, 2\}$ solves

   $$\max_{x_i} (1 - x_i - x_j)x_i - c_i^K x_i$$

   which, after finding best response functions and simultaneously solving for incumbent and entrant's outputs, yields equilibrium output $x_1^{K,d} = \frac{1+c_2-2c_1^K}{3}$ for the incumbent and $x_2^{K,d} = \frac{1-2c_2+c_1^K}{3}$ for the entrant.
   - *First period.* Regardless of the entrant's entry decision during the second period, the incumbent selects the standard monopoly output $q^{K,Info} = \frac{1-c_1^K}{2}$ in the first period. This is because the incumbent's output choice in this complete information setting does not affect the entrant's entry decision.

   (b) *Incomplete information.* In this section we investigate the case where the incumbent is privately informed about its marginal costs, while the entrant only observes the incumbent's first-period output which the entrant uses as a signal to infer the incumbent's cost. The time structure of this signaling game is as follows:

   1. Nature decides the realization of the incumbent's marginal costs, either high or low, with probabilities $p \in (0, 1)$ and $1 - p$, respectively. The incumbent privately observes this realization but the entrant does not.

2. The incumbent chooses its first-period output level, q.
3. Observing the incumbent's output decision, the entrant forms beliefs about the incumbent's initial marginal costs. Let $\mu(c_1^H|q)$ denote the entrant's posterior belief about the initial costs being high after observing a particular first-period output from the incumbent q.
4. Given the above beliefs, the entrant decides whether or not to enter the industry.
5. If entry does not occur, the incumbent maintains its monopoly power; whereas if entry occurs, both agents compete as Cournot duopolists and the entrant observes the incumbent's type.

Write down the incentive compatibility conditions that must hold for a separating Perfect Bayesian Equilibrium (PBE) to be sustained. Then find the set of separating PBEs.

- In a separating equilibrium in which the high-cost firm selects q^H while the low-cost firm chooses q^L information about the incumbent's type is conveyed to the potential entrant, who responds entering after observing the incumbent producing q^H, and does not enter after observing q^L. For simplicity, we assume that all other output levels $q \neq q^H \neq q^L$ (i.e., off-the-equilibrium outputs) also lead the entrant to enter the industry; as depicted in figure 1.

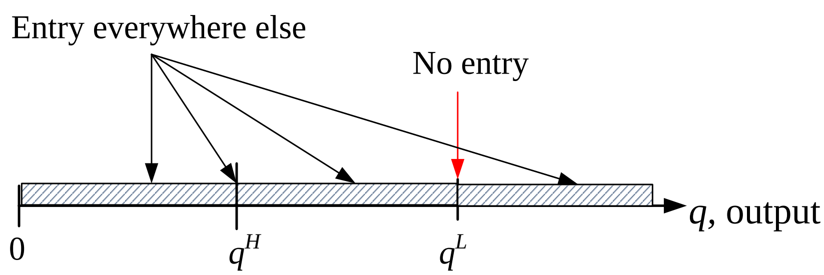

Fig 1. Output choices and entry in the separating PBE.

Let us next separately analyze each type of incumbent.

- *High-cost incumbent.* Since, by selecting q^H this type of incumbent attract entry, this firm selects the output that maximizes its first-period (monopoly) profits, that is, q^H coincides with its output under complete information $q^{H,Info} = \frac{1-c_1^H}{2}$. If, instead, the incumbent deviates towards the low-cost incumbent's output q^L, it conceals its type from the entrant and deters entry. Hence, the high-cost incumbent selects its equilibrium output q^H rather than deviating if $M_1^H(q^{H,Info}) + \delta D_1^H \geq M_1^H(q^L) + \delta \overline{M}_1^H$, where

$$M_1^H(q) = (1-q)q - c^H q \quad \text{for every output } q$$

denotes the incumbent's first-period monopoly profits, D_1^H represents second-period duopoly profits when the incumbent's costs are high, and $\overline{M}_1^H$ indicates the second-period monopoly profits for the incumbent (in the case of no entry) when its costs are high. We can now rewrite the above incentive compatibility condition as follows

$$M_1^H(q^{H,Info}) - M_1^H(q^L) \geq \delta \left[\overline{M}_1^H - D_1^H\right] \qquad (IC_H)$$

(where we grouped first-period profits on the left-hand side, and discounted second-period profits on the right-hand side). For our parameter values, we obtain profits of $M_1^H(q^{H,Info}) = \overline{M}_1^H = \frac{1}{16}$ since $c_1^H = 1/2$, and $D_1^H = \frac{1}{36}$ given that $c_1^H = c_2 = 1/2$. Hence, condition IC_H reduces to

$$\frac{1}{16} - \left[(1-q^L)q^L - \frac{1}{2}q^L\right] \geq \delta \left[\frac{1}{16} - \frac{1}{36}\right]$$

Figure 2 depicts IC_H. Specifically, the curve depicting the difference in first-period profits, $M_1^H(q^{H,Info}) - M_1^H(q^L)$, becomes zero at $q^L = q^{H,Info}$ since at that point $M_1^H(q^{H,Info}) = M_1^H(q^L)$, but otherwise is positive since $M_1^H(q^{H,Info}) > M_1^H(q^L)$ for all $q^L \neq q^{H,Info}$. In contrast, the difference in discounted second-period profits, $\delta \left[\overline{M}_1^H - D_1^H\right]$, is constant in first-period output q^L. Hence, IC_H holds if output q^L lies in the range depicted in the horizontal axis of figure 2.

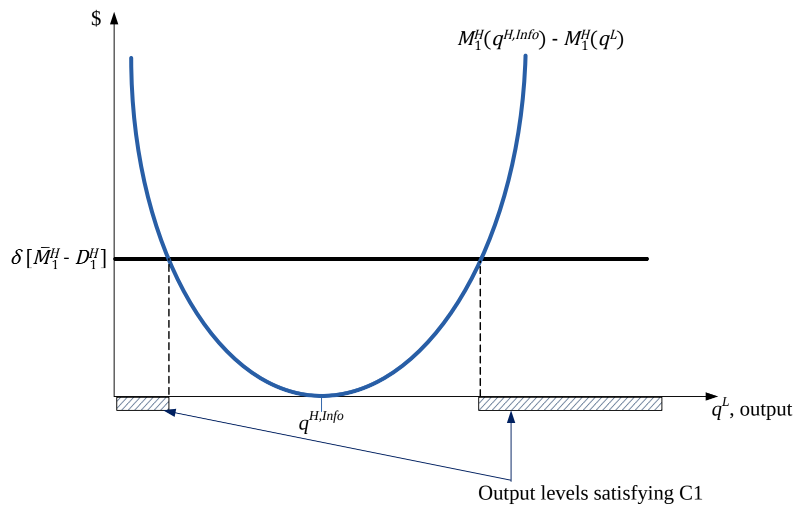

Fig 2. Incentive compatibility condition IC_H.

- *Low-cost incumbent.* If the low-cost incumbent chooses the equilibrium output q^L, it deters entry. If instead the incumbent deviates towards the high-cost incumbent's output, q^H, it attracts entry. Conditional on attracting entry, the low-cost incumbent would select output $q^{L,Info}$, since such output maximizes its first-period profits, yielding $M_1^L(q^{L,Info}) + \delta D_1^L$. Thus, the low-cost incumbent selects its equilibrium output of q^L if $M_1^L(q^{L,Info}) + \delta D_1^L \leq M_1^L(q^L) + \delta \overline{M}_1^L$, or equivalently,

$$M_1^L(q^{L,Info}) - M_1^L(q^L) \leq \delta \left[\overline{M}_1^L - D_1^L\right] \qquad (IC_L)$$

which, for our parameter values, yields $M_1^L(q^{L,Info}) = \overline{M}_1^L = 1/9$ and $D_1^L = \frac{25}{324}$ given that $c_1^L = 1/3$ and $c_2 = 1/2$. Hence, condition IC_L reduces to

$$\frac{1}{9} - \left[(1-q^L)q^L - \frac{1}{3}q^L\right] \leq \delta\left[\frac{1}{9} - \frac{25}{324}\right]$$

A similar argument as for IC_H applies to the graphical representation of IC_L. As figure 3 illustrates, the curve depicting the difference in first-period profits, $M_1^L(q^{L,Info}) - M_1^L(q^L)$, becomes zero at $q^L = q^{L,Info}$ since at that point $M_1^L(q^{L,Info}) = M_1^L(q^L)$, but otherwise is positive since $M_1^L(q^{L,Info}) > M_1^L(q^L)$ for all $q^L \neq q^{L,Info}$. In contrast, the difference in discounted second-period profits, $\delta\left[\overline{M}_1^L - D_1^L\right]$, is constant in first-period output q^L. Hence, IC_L holds if output q^L lies in the range depicted in the horizontal axis of figure 3.

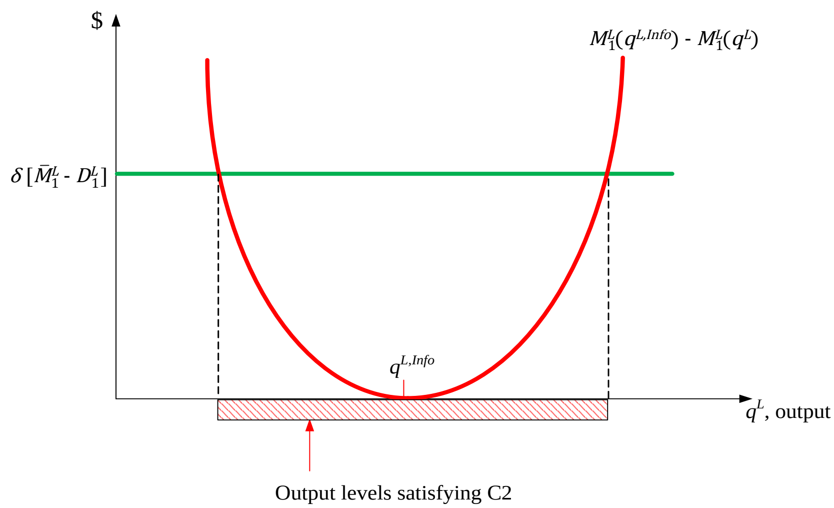

Fig 3. Incentive compatibility condition IC_L.

- *Combining both ICs.* Superimposing figures 2 and 3, we can examine the set of output levels that simultaneously satisfy condition IC_H and IC_L, as depicted in figure 4. In particular, the overlap between the range of outputs identified in figures 2 and 3 provides us with the set of output levels that constitute a separating PBE of the signaling game. Intuitively, the high-cost incumbent does not have incentives to mimic the output level chosen by the low-cost firm, i.e., $q^L \in [q^A, q^B]$. The low-cost firm, by contrast, has incentives to choose an output level in the interval $q^L \in [q^A, q^B]$, which is above its first-period output under complete information, $q^{L,Info} = \frac{1-c_1^L}{2}$. Thus, the low-cost incumbent increases its first-period output in order to communicate its efficient costs to the potential entrant, deterring entry as a result.

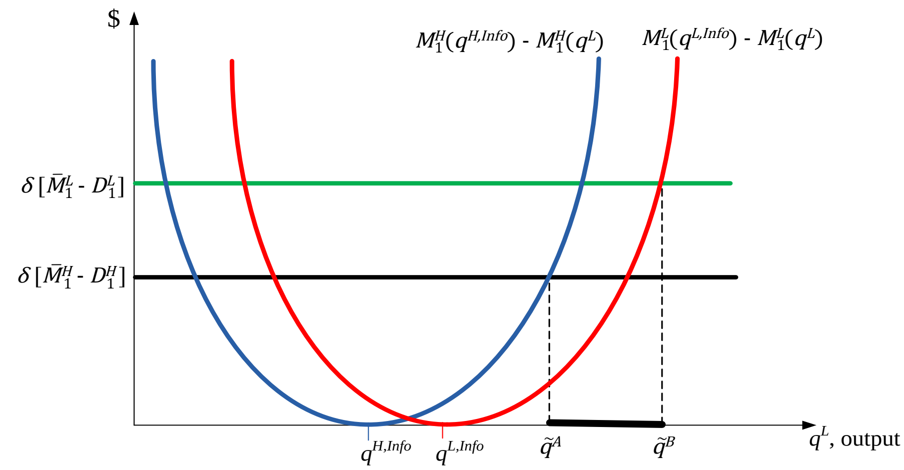

Fig 4. Separating equilibria in the limit pricing model.

In particular, the lower-bound output q^A solves condition IC_H with equality, and the upper-bound output q^B solves IC_L with equality. Rearranging condition IC_H, and assuming that there is no discounting, $\delta = 1$, we obtain

$$1 - 1 + \frac{1}{36} = (1 - q^L)q^L - \frac{1}{2}q^L$$

or

$$36\left(q^L\right)^2 - 18q^L + 1 = 0$$

and solving for output q^L yields two roots for the lower bound q^A, $q^A = 0.06$ and $q^A = 0.43$. Similarly operating with condition IC_L in order to obtain the upper bound q^B, and since there is no discounting, $\delta = 1$, IC_L simplifies to

$$324\left(q^L\right)^2 - 216q^L + 25 = 0.$$

Solving for output q^L yields two roots for the upper bound q^B, $q^B = 0.14$ and $q^B = 0.51$. Hence, the set of separating output levels for the low-cost firm must lie on the interval $q^L \in [0.43, 0.51]$.

(c) Which separating PBEs of those you found in part (b) survive the Cho and Kreps' Intuitive Criterion?

- Starting from the separating PBE in which the low-cost incumbent chooses the highest output level $q^L = q^B$, a deviation toward any output level in $q^L \in [q^A, q^B)$ can only be profitable for the low-cost incumbent (but not for the high-cost firm). Formally, deviating towards $q^L \in [q^A, q^B)$ is "equilibrium dominated" for the high-cost incumbent alone. Hence, the potential entrant would update its beliefs accordingly, making such a deviation profitable for the low-cost firm. A similar argument applies to all other separating PBEs in the interval $q^L \in (q^A, q^B)$ but not for $q^L = q^A$, the least-costly separating PBE (also known as the "Riley outcome").

- Summarizing, the low-cost incumbent raises its first-period output from $q^{L,Info} = \frac{1-c_1^L}{2} = \frac{1-\frac{1}{3}}{2} = 0.33$, under complete information, to $q_1^A = 0.43$, under the separating equilibrium. Hence, the "separating effort" that this firm must exert in order to reveal its type to the potential entrant (and thus deter entry) is measured by the distance $q^A - q^{L,Info} = 0.43 - 0.33 = 0.10$.
- *Remark:* Importantly, the low-cost incumbent chooses his output level q^A by considering the incentive compatibility condition of the high-cost incumbent (condition IC_H). This is not a mistake! The low-cost firm uses IC_H to identify its optimal output q^A because this is the lowest output level that would make the high-cost firm indifferent between mimicking its output decision q^A and produce $q^{H,Info}$. Output levels above q^A would be strictly unprofitable to imitate by the high-cost firm, while output levels strictly lower than q^A would be profitably mimicked by the high-cost incumbent. The low-cost firm then increases its output from $q^{L,Info}$ to q^A to successfully make mimicking unprofitable for the high-cost firm (often referred as that the low-cost firm "separates" from the high-cost firm), thus conveying its type to the potential entrant, who is deterred from entering the market.

3. **Rent seeking and moral hazard.**[1] Consider a firm delegating a supervisor to monitor the performance of an agent. The agent can invest in two levels of effort, e_1 and e_2. Effort e_1 generates productive output for the firm, $y = e_1 + \varepsilon$ where $\varepsilon \sim N(0, \sigma^2)$, and effort e_2 has no productive value to the firm but allows the agent to impress his supervisor in writing a favorable report to the firm. In particular, assume that the supervisor writes a report $x = y + e_2$ for the agent, which sums up the agent's productive output y and the impression that the supervisor has on the agent e_2; and in return, the agent receives a wage of $W = F + bx$, where F is the fixed wage and b is the bonus rate set by the firm. Let us assume that the agent has zero reservation utility and a negative exponential utility in the form of

$$U(e_1, e_2) = -\exp\left(-\eta\left[W - \frac{c_1}{2}e_1^2 - \frac{c_2}{2}e_2^2\right]\right)$$

where η stands for the absolute risk aversion of the agent.

(a) Find the expected utility of the agent.

- As in prevoius exercises, the expected utility is characterized by the certainty equivalent payment,

$$\begin{aligned} EU(e_1, e_2) &= E\left[-\exp\left(-\eta\left[F + b(y + e_2) - \frac{c_1}{2}e_1^2 - \frac{c_2}{2}e_2^2\right]\right)\right] \\ &= F + be_1 + be_2 - \frac{c_1}{2}e_1^2 - \frac{c_2}{2}e_2^2 - \frac{\eta b^2\sigma^2}{2}, \end{aligned}$$

that induces the agent to accept the contract offered by the firm.

(b) What are the optimal levels of effort e_1 and e_2 exerted by the agent?

[1] Milgrom P. (1988). Employment Contracts, Influence Activities, and Efficient Organization Design. *Journal of Political Economy*, 96(1), 42-60.

- The agent chooses the pair of effort e_1 and e_2 that maximizes his expected utility.

$$\frac{\partial EU(e_1, e_2)}{\partial e_1} = b - c_1 e_1 = 0$$
$$\frac{\partial EU(e_1, e_2)}{\partial e_2} = b - c_2 e_2 = 0$$

- Assuming interior solutions, where $e_1 > 0$ and $e_2 > 0$, we have

$$e_1(b) = \frac{b}{c_1}$$
$$e_2(b) = \frac{b}{c_2}$$

Intuitively, the higher the bonus rate b, the higher will be the agent's effort levels; but the higher the costs of effort, the lower will be the respective effort levels.

(c) What is the firm's optimal contract? For the remainder of the exercise, please consider $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{3}$, $\eta = \frac{1}{4}$, and $\sigma^2 = \frac{1}{9}$.

- Based on the supervisor's report x, the firm solves

$$\max_{F,b} E[y - W] = E[y - F - bx]$$
$$\text{subject to } \; EU(e_1, e_2) \geq 0 \qquad \text{(PC)}$$

which can be rewritten as

$$\begin{aligned} \max_{F,b} E[y - W] &= E[y - F - bx] \\ &= E[y - F - b(y + e_2(b))] \\ &= E[(1-b)y - F - be_2(b)] \\ &= (1-b)e_1(b) - be_2(b) - F \end{aligned}$$
$$\text{subject to } \; EU(e_1, e_2) \geq 0 \qquad \text{(PC)}$$

where the last line emerges from the fact that the firm is risk neutral, that is $E(y) = E(e_1 + \varepsilon) = e_1 + \underbrace{E(\varepsilon)}_{=0} = e_1$.

- Since the agent's reservation utility is zero, the participation constraint holds with equality

$$EU(e_1, e_2) = F + be_1 + be_2 - \frac{c_1}{2}e_1^2 - \frac{c_2}{2}e_2^2 - \frac{\eta b^2 \sigma^2}{2} = 0$$

which, after solving for F, yields

$$F = \frac{c_1}{2}[e_1(b)]^2 + \frac{c_2}{2}[e_2(b)]^2 + \frac{\eta b^2 \sigma^2}{2} - be_1(b) - be_2(b)$$

which means that the agent's fixed wage must compensate for the difference between his variable wage (i.e., expected bonus) and the sum of costs of effort and risk premium.

- Operating by backwards induction, we substitute the agent's best response functions, $e_1(b) = \frac{b}{c_1}$ and $e_2(b) = \frac{b}{c_2}$ from part (b), and the binding PC condition, into the firm's objective function,

$$\max_b \left[(1-b)\, e_1(b) - b e_2(b) - F\right]$$

$$= (1-b)\, e_1(b) - b e_2(b) + \underbrace{b e_1(b) + b e_2(b) - \frac{c_1}{2}\left[e_1(b)\right]^2 - \frac{c_2}{2}\left[e_2(b)\right]^2 - \frac{\eta b^2 \sigma^2}{2}}_{-F}$$

$$= \frac{b}{c_1} - \frac{c_1}{2}\frac{b^2}{c_1^2} - \frac{c_2}{2}\frac{b^2}{c_2^2} - \frac{\eta b^2 \sigma^2}{2}$$

$$= \frac{1}{c_1}\left[b - \frac{b^2}{2}\left(1 + \frac{c_1}{c_2} + \eta c_1 \sigma^2\right)\right]$$

Differentiating the above expected profit with respect to bonus b, we obtain

$$1 - b\left(1 + \frac{c_1}{c_2} + \eta c_1 \sigma^2\right) = 0$$

Assuming interior solutions, we fomd

$$b^* = \frac{c_2}{c_1 + c_2 + \eta c_1 c_2 \sigma^2}$$

Substituting the optimal bonus b^* into the fixed wage F, yields

$$F^* = \frac{c_1}{2}\left[\frac{b}{c_1}\right]^2 + \frac{c_2}{2}\left[\frac{b}{c_2}\right]^2 + \frac{\eta b^2 \sigma^2}{2} - \frac{b^2}{c_1} - \frac{b^2}{c_2}$$

$$= -\frac{b^2}{2c_1} - \frac{b^2}{2c_2} + \frac{\eta b^2 \sigma^2}{2}$$

$$= -\frac{1}{2}\left(\frac{1}{c_1} + \frac{1}{c_2} - \eta \sigma^2\right)\left(\frac{1}{1 + \frac{c_1}{c_2} + \eta c_1 \sigma^2}\right)^2$$

$$= -\frac{c_1 + c_2 - c_1 c_2 \eta \sigma^2}{2 c_1 c_2}\left(\frac{c_2}{c_1 + c_2 + \eta c_1 c_2 \sigma^2}\right)^2$$

$$= \frac{c_2\left(c_1 c_2 \eta \sigma^2 - c_1 - c_2\right)}{2 c_1\left(c_1 + c_2 + \eta c_1 c_2 \sigma^2\right)^2}$$

such that when the agent is sufficiently risk averse, in particular, $\eta \geq \frac{c_1 + c_2}{c_1 c_2 \sigma^2}$, the firm needs to offer a fixed wage $F^* \geq 0$ in order to induce the agent to accept the contract. Otherwise, the firm can charge an entry fee, where $F^* < 0$ indicates a transfer from the agent to the firm (e.g., negative fixed wage, such as a training fee), that makes the agent indifferent between accepting the contract or not.[2]

[2]If the agent is protected from limited liability, $F \geq 0$, he earns a positive utility when he is not too risk averse, that is, $\eta < \frac{c_1 + c_2}{c_1 c_2 \sigma^2}$. In this case, the firm sets $F^* = 0$, meaning that it neither offers a fixed wage nor charges a fee to the agent.

- Summarizing, the firm's optimal contact is given by

$$(F^*, b^*) = \left(\frac{c_2 \left(c_1 c_2 \eta \sigma^2 - c_1 - c_2\right)}{2c_1 \left(c_1 + c_2 + \eta c_1 c_2 \sigma^2\right)^2}, \frac{c_2}{c_1 + c_2 + \eta c_1 c_2 \sigma^2} \right)$$

(d) What is the agent's optimal levels of efforts? How about his expected utility?

- Substituting the optimal bonus, b^*, into the agent's best response functions, yields

$$e_1^* = \frac{b^*}{c_1} = \frac{c_2}{c_1 \left(c_1 + c_2 + \eta c_1 c_2 \sigma^2\right)}$$
$$e_2^* = \frac{b^*}{c_2} = \frac{1}{c_1 + c_2 + \eta c_1 c_2 \sigma^2}$$

- Since the firm sets the fixed wage w^* that binds the agent's participation constraint, the agent's expected utility is zero, that is, $EU\left(e_1^*, e_2^*\right) = 0$.
- Substituting the above parameter values into the agent's optimal levels of effort,

$$e_1^* = \frac{\frac{1}{3}}{\frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\frac{1}{2}\frac{1}{3}\frac{1}{9}\right)} = \frac{144}{181} \approx 0.780$$
$$e_2^* = \frac{1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\frac{1}{2}\frac{1}{3}\frac{1}{9}} = \frac{216}{181} \approx 1.193$$

- Substituting the above parameter values into the firm's optimal contract,

$$F^* = \frac{\frac{1}{3}\left(\frac{1}{2}\frac{1}{3}\frac{1}{4}\frac{1}{9} - \frac{1}{2} - \frac{1}{3}\right)}{2\frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\frac{1}{2}\frac{1}{3}\frac{1}{9}\right)^2} = -\frac{12888}{32761} \approx -0.393$$
$$b^* = \frac{\frac{1}{3}}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\frac{1}{2}\frac{1}{3}\frac{1}{9}} = \frac{72}{181} \approx 0.398$$