EconS 503 - Microeconomic Theory II
Homework #8 - Answer key

1. **Cheap talk vs. Delegation.** Consider a cheap talk game where an expert privately observes the realization of parameter $\theta \sim U[0,1]$, sends a message $m \in [0,1]$ to the uninformed politician who responds with policy $p \in [0,1]$. The politician's utility function is $u_P(p,\theta) = -(p-\theta)^2$, while the expert's is $u_E(p,\theta) = -(p-(\theta+\delta))^2$, where parameter $\delta > 0$ represents the expert's upward bias.

   (a) *Delegation.* Consider an alternative communication setting where the politician delegates the decision of policy p to the expert. Which policy the expert chooses under delegation? Which are the expected utilities for expert and politician?

   - After observing the realization of θ, the expert chooses the policy p that maximizes his utility, that is,

     $$\max_{p \geq 0} \; -(p-(\theta+\delta))^2$$

     Differentiating with respect to p, and solving for p, we obtain $p^{Del} = \theta + \delta$, where the superscript Del denotes "delegation."

   - Since the expert chooses policy $p^{Del} = \theta + \delta$ under delegation, his utility becomes

     $$u_E^{Del} = -(p^{Del} - (\theta+\delta))^2 = -(\underbrace{\theta+\delta}_{p^{Del}} - (\theta+\delta))^2 = 0$$

     Intuitively, he chooses his ideal policy $\theta + \delta$, so his utility is maximized. However, the politician's expected utility from p^{Del} becomes

     $$u_P^{Del} = -E\left[(p^{Del} - \theta)^2\right] = -E[(\underbrace{\theta+\delta}_{p^{Del}} - \theta)^2] = -\delta^2.$$

   (b) *Do-it-yourself.* Consider now that the receiver (politician) ignores the sender's messages, which is often known as if the politician took a "do-it-yourself" approach. Find the policy he responds with, and his expected utility in this setting.

   - Since the politician ignores the messages from the expert in this setting, he chooses the policy that maximizes his expected utility, as follows

     $$\max_{p \geq 0} \; -E\left[(p-\theta)^2\right]$$

     $$\text{subject to } \; \theta \sim U[0,1]$$

Then, the politician solves the following expected utility maximization problem:

$$\max_{p\geq 0} \quad -E\left[(p-\theta)^2\right] = -\int_0^1 (p-\theta)^2 f(\theta)\, d\theta$$
$$= \frac{1}{3}\left[(p-\theta)^3\right]_0^1$$
$$= \frac{1}{3}\left[(p-1)^3 - p^3\right]$$
$$= -\frac{1}{3}\left(3p^2 - 3p + 1\right)$$

Differentiating the politician's expected utility with respect to policy p, yields

$$\frac{dEU[p]}{dp} = 1 - 2p$$

Solving for $\frac{dEU[p]}{dp} = 0$, the politician chooses a policy $p^{DIY} = \frac{1}{2}$ that maximizes his expected utility, where the superscript DIY denotes the "do-it-yourself" approach. (Note that the above expected utility is concave in policy p since $\frac{d^2EU[p]}{dp^2} = -2 < 0$; confirming that $p^{DIY} = \frac{1}{2}$ is a maximum.)

- The politician's expected utility in the "do-it-yourself" approach then becomes

$$u_P^{DIY} = -E\left[\left(\frac{1}{2} - \theta\right)^2\right] = -Var(\theta) = -\frac{1}{12}.$$

where the second equality follows because the expected value of θ is $\frac{1}{2}$ given that $\theta \sim U[0,1]$.

(c) *Comparison.* Under which values of δ the politician prefers the "do-it-yourself" approach from part (b) than the delegation approach from part (a)?

- Then, the politician prefers the "do-it-yourself" approach over delegation if $u_P^{DIY} \geq u_P^{Del}$, entailing

$$-\frac{1}{12} \geq -\delta^2$$

which simplifies to $\delta \geq \frac{1}{2\sqrt{3}} \simeq 0.28$. Intuitively, when the expert's preference bias δ is sufficiently strong, the politician prefers to ignore the expert's messages and choose a policy that coincides with the expected value of random parameter θ. Otherwise, the politician prefers to delegate the policy decision to the expert since his preferences and the expert's are relatively similar.

(d) Recall that, in a cheap talk game, an N-partition equilibrium can be sustained if $\delta \leq \frac{1}{2N(N-1)}$. Show that the politician's equilibrium utility is $u_P = -\frac{1}{12N^2} - \frac{\delta^2(N^2-1)}{3}$, while that of the expert is $u_E = u_P^{CT} - \delta^2$. Then show that the expert prefers delegation than the cheap talk game under all parameter conditions. Then show that the politician prefers delegation only if δ is sufficiently small.

- *Politician's utility.* After receiving message m in the interval $[\theta_{i-1}, \theta_i]$, the politician responds with a policy p that coincides with the midpoint of that interval, that is, $p = \frac{\theta_{i-1}+\theta_i}{2}$. Therefore, since the politician's utility function is $u_P(p,\theta) = -(p-\theta)^2$, his expected utility in a cheap talk game with N partitions is

$$\begin{aligned}
u_P^{CheapTalk} &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(\frac{x_{i-1}+x_i}{2} - \theta \right)^2 d\theta \\
&= \sum_{i=1}^{N} -\frac{1}{24} \left(x_{i-1} + x_i - 2\theta \right)^3 \Big|_{x_{i-1}}^{x_i} \\
&= \sum_{i=1}^{N} -\frac{1}{24} \left[\left(x_{i-1} - x_i \right)^3 - \left(x_i - x_{i-1} \right)^3 \right] \\
&= \sum_{i=1}^{N} -\frac{1}{24} \left[2 \left(x_{i-1} - x_i \right)^3 \right] \\
&= -\frac{1}{12} \sum_{i=1}^{N} \left(x_i - x_{i-1} \right)^3 \\
&= -\frac{1}{12N^2} - \frac{\delta^2 (N^2-1)}{3}
\end{aligned}$$

- *Expert's utility.* Following a similar approach, we find that the expert's equilibrium utility in a cheap talk game with N partitions is

$$\begin{aligned}
u_E^{CheapTalk} &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(\frac{x_{i-1}+x_i}{2} - (\theta+\delta) \right)^2 d\theta \\
&= \underbrace{\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(\frac{x_{i-1}+x_i}{2} - \theta \right)^2 d\theta}_{u_P^{CheapTalk}} - \delta^2 \\
&= u_P^{CheapTalk} - \delta^2 \\
&= \left(-\frac{1}{12N^2} - \frac{\delta^2(N^2-1)}{3} \right) - \delta^2 \\
&= -\frac{1 + 4\delta^2 N^2 (2+N^2)}{12N^2}.
\end{aligned}$$

- *Expert.* The expert's expected utility from delegation is $u_E^{Del} = 0$, while that from playing the cheap talk game is negative since $u_E < 0$. Therefore, he prefers delegation regardless of the number of partitions, N, and independently on his preference bias, δ.
- *Politician.* The politician's expected utility from delegation is $u_P^{Del} = -\delta^2$,

which exceeds that from playing the cheap talk game, u_P^{CT}, if

$$-\delta^2 \geq -\frac{1}{12N^2} - \frac{\delta^2(N^2-1)}{3}$$

After rearranging, we obtain

$$\frac{1}{12N^2} \geq \delta^2 \left(\frac{4-N^2}{3} \right)$$

and solving for preference divergence parameter, δ, we find

$$\delta \leq \frac{1}{2N\sqrt{4-N^2}}$$

We can now evaluate condition $\delta \leq \frac{1}{2N\sqrt{4-N^2}}$ at different numbers of partitions:

  - When $N = 1$ (meaning that the pooling equilibrium arises, which entails uninformative messages for the politician), we obtain that $\delta \leq \frac{1}{2\sqrt{3}} \simeq 0.28$. In this case, the politician prefers delegation to receiving a cheap-talk message from the expert if the bias parameter is sufficiently small, that is, $\delta \leq 0.28$.
  - In contrast, for any larger number of partitions, $N \geq 2$, we find that $\delta \leq +\infty$. In words, delegation is optimal for the politician when there is an informative PBE with two or more partitions in the cheap-talk game for all values of the divergence parameter δ.

- Therefore, delegation is preferred over cheap talk for the politician under relatively large parameter values because the welfare loss caused by self-interested communication in the cheap talk game is larger than the cost that the politician experiences from letting the expert choose a relatively biased policy under delegation.

2. **Cheap talk when the expert receives imprecise signals.** Consider the following cheap talk model between an expert (E), such a special interest group, and a decision maker (DM), such as a politician. For simplicity, assume that the state of the world is discrete, either $\theta = 1$ or $\theta = 0$ with prior probability $p \in (0, 1)$ and $1 - p$, respectively. The expert privately observes an informative but noisy signal s, which also takes two discrete values $s \in \{0, 1\}$. The precision of the signal is given by the conditional probability

$$\text{prob}(s = k|\theta = k) = q,$$

where $k = \{0, 1\}$, and $q > \frac{1}{2}$. In words, the probability that the signal s coincides with the true state of the world θ is q (precise signal), while the probability of an imprecise signal where $s \neq \theta$ is $1 - q$. The time structure of the game is as follows:

1) Nature chooses θ according to the prior p.
2) Expert observes signal s and reports a message $m \in \{0, 1\}$
3) Decision maker observes m and responds with $x \in \{0, 1\}$

4) θ is observed and payoffs are realized

The payoff function for the decision maker is

$$u(x,\theta) = \left(\theta - \frac{1}{2}\right) x$$

while that of the expert is

$$v(m,\theta) = \begin{cases} 1, & \theta = m \\ 0, & \theta \neq m \end{cases}$$

which, in words, indicates that the expert's payoff is 1 when the message she sends coincides with the true realization of the state of the world, but becomes zero otherwise. Importantly, her payoff is unaffected by the signal, which she only uses to infer the actual realization of parameter θ. Intuitively, $v(m,\theta)$ is often understood as a "reputation function" since it provides the expert with a payoff of 1 only when his message was an accurate representation of the true state of the world (which in this model he does not precisely observe).

(a) Is there a Perfect Bayesian equilibrium (PBE) in which the expert reports his signal truthfully?

- *Updated beliefs.* In a strategy profile where the expert sends a message that coincides with the signal she receives (that is, sending message $m = 1$ after receiving signal $s = 1$, but sending message $m = 0$ after receiving signal $s = 0$)[1], the decision maker and the expert sustain the same beliefs about θ since $m = s$. Specifically, after receiving a signal of $s = 1$ (a message of $m = 1$), both expert and decision maker use Bayes' rule to update their beliefs yielding

$$\mu_1 = \frac{pq}{pq + (1-p)(1-q)}$$

while after receiving a signal of $s = 0$ (a message of $m = 0$), both expert and receiver updated their beliefs as follows

$$\mu_0 = \frac{p(1-q)}{p(1-q) + (1-p)q}$$

- *Decision maker's response.* Given the above beliefs, after receiving a message $m = 1$ from the expert, the decision maker responds with $x = 1$ if

$$\mu_1 \left(1 - \frac{1}{2}\right) 1 + (1-\mu_1)\left(0 - \frac{1}{2}\right) 1 \geq \mu_1 \left(1 - \frac{1}{2}\right) 0 + (1-\mu_1)\left(0 - \frac{1}{2}\right) 0$$

or

$$\mu_1 \frac{1}{2} + (1-\mu_1)\left(-\frac{1}{2}\right) \geq 0$$

[1] This truthful reporting of signals can be described more compactly by saying that the expert's strategy is a message $m(s) = s$ for every signal $s \in \{0, 1\}$.

or, simplifying, $\mu_1 \geq \frac{1}{2}$. From the above expression of posterior belief μ_1, this condition holds if

$$\frac{pq}{pq + (1-p)(1-q)} \geq \frac{1}{2}$$

or, after rearranging, $\mu \geq 1 - q$, which holds by assumption. That is, the decision maker responds with $x = 1$ after receiving message $m = 1$ for all admissible parameter values. Similarly, after receiving a message $m = 0$, the decision maker responds with $x = 1$ if

$$\mu_0 \left(1 - \frac{1}{2}\right) 1 + (1 - \mu_0) \left(0 - \frac{1}{2}\right) 1 \geq \mu_0 \left(1 - \frac{1}{2}\right) 0 + (1 - \mu_0) \left(0 - \frac{1}{2}\right) 0$$

or, after simplying, $\mu_0 \geq \frac{1}{2}$. From the above expression of posterior belief μ_0, this condition holds if

$$\frac{p(1-q)}{p(1-q) + (1-p)q} \geq \frac{1}{2}$$

or $p \geq q$. In words, the decision maker responds with $x = 1$ after observing message $m = 0$ when the probability of $\theta = 1$, p, is higher than the probability of the expert receiving precise signals, q. Otherwise (when $p < q$), the decision maker responds with $x = 0$ after observing message $m = 0$. Therefore, when $p < q$ we can say that the decision maker responds with an action $x(m) = m$ to every message $m \in \{0, 1\}$ he receives from the sender.

- *Expert's messages - After receiving signal $s = 1$.* If the expert reports her signal truthfully (sending message $m = 1$), her expected payoff is

$$\mu_1 v(1, 1) + (1 - \mu_1) v(1, 0) = \mu_1$$

Intuitively, the above expression says that the expert sends a message $m = 1$ but does not know if the state of the world is $\theta = 1$, which yields a payoff of 1 since $\theta = m$; or if the state of the world is $\theta = 0$, which yields a payoff of zero for her since $\theta \neq m$. If, instead, she misreports her signal (sending message $m = 0$), her expected payoff becomes

$$\mu_1 v(0, 1) + (1 - \mu_1) v(0, 0) = 1 - \mu_1$$

Therefore, the expert truthfully reports her signal if $\mu_1 \geq 1 - \mu_1$, or $\mu_1 \geq \frac{1}{2}$. Using the expression of posterior belief μ_1, we obtain that

$$\frac{pq}{pq + (1-p)(1-q)} \geq \frac{1}{2}$$

which collapses to $p \geq 1 - q$. In words, after receiving a signal of $s = 1$, the expert truthfully conveys her signal if the probability of receiving such a signal is higher than the probability of an imprecise signal, $1 - q$.

- *Expert's messages - After receiving signal $s = 0$.* If the expert reports his signal truthfully (that is, sending message $m = 0$), her expected payoff is

$$\mu_0 v(0, 1) + (1 - \mu_0) v(0, 0) = \mu_0 0 + (1 - \mu_0) 1 = 1 - \mu_0$$

Intuitively, the above expression says that the expert sends a message $m = 0$ but does not know if the state of the world is $\theta = 1$, which yields a payoff of zero for her since $\theta \neq m$; or if the state of the world is $\theta = 0$, which yields a payoff of 1 since $\theta = m$. If, instead, the expert sends message $m = 1$ (lying about her message), her expected payoff becomes

$$\mu_0 v(1,1) + (1-\mu_0)v(1,0) = \mu_0 v(1,1) + (1-\mu_0)v(1,0) = \mu_0$$

Therefore, the expert truthfully reports her signal if $1 - \mu_0 \geq \mu_0$, or $\frac{1}{2} \geq \mu_0$. Examining the expression of posterior belief μ_0, we find that

$$\frac{p(1-q)}{p(1-q)+(1-p)q} \leq \frac{1}{2}$$

simplifies to $p \leq q$. In words, after receiving a signal of $s = 0$, the expert truthfully conveys her signal if the probability of an accurate signal, q, is higher than the probability of receiving a signal of $s = 1$. Combining the above conditions $p \geq 1 - q$ and $p \leq q$, we obtain $1 - q \leq p \leq q$.

- *Summary*:
  - When $p \geq q$, a PBE where the expert truthfully reports her signal can be sustained if $1 - q \leq p \leq q$ (from the expert) and $p \geq q$ (from the decision maker), which are only compatible when $p = q$. In words, the prior probability of the state of the world being $\theta = 1$, p, must coincide with the probability with which the expert receiving precise signals, q. While the expert truthfully reports her signals to the decision maker, the decision maker does not follow the expert's advise when observing a message of $m = 0$.
  - When $p < q$, a PBE where the expert truthfully reportsher signal can be sustained if $1 - q \leq p \leq q$ (from the expert) and $p < q$ (from the decision maker), where the expert sends a message $m(s) = s$ for every signal $s \in \{0, 1\}$ she received, while the decision maker responds with an action $x(m) = m$ for every message $m \in \{0, 1\}$ he receives. In this PBE, the expert truthfully reports her signals to the decision maker, and the decision maker follows the expert's advise after every message.

3. **Policy announcements as signals.** Consider Downs' (1957) model of voting with a continuum of voters with policy ideals in the interval $[0, 1]$, distributed according to cummulative distribution function $F(x)$ with positive and continuous density in $[0, 1]$. The median voter $x = m$ satisfies $F(m) = \frac{1}{2}$, and is either low (L) or high (H), where $L < H$, with equal probabilities. The time structure of the game is the following:

   1) Political candidate 1 privately observes the position of the median voter (that is, $m = L$ or $m = H$), and announces a policy position p_1.
   2) Candidate 2 observes p_1, and updates its beliefs about the position of the median voter. Candidate 2 then responds announcing his own policy p_2.
   3) After observing policies p_1 and p_2, voters vote for the candidate who is closest to their ideal policy. In case of a tie, you can assume that candidates evenly share votes.

Candidates only care about winning the election and assign a payoff of 1 to winning, $\frac{1}{2}$ to a tie, and 0 to losing.

(a) Find at least one separating Perfect Bayesian Equilibria (PBEs).

- In a separating strategy profile, candidate 1's policies are different after observing a low and a high median voter, that is, $p_1(L) \neq p_1(H)$.
- *Candidate 2's beliefs.* Candidate 2's beliefs must be concentrated, that is, after observing policy $p_1(L)$, he beliefs htat $\mu(L|p_1(L)) = 1$; while after observing policy $p_1(H)$ he beliefs that $\mu(H|p_1(H)) = 1$. All other policies $p_1 \neq p_1(L) \neq p_1(H)$ are regarded as off-the-equilibrium messages, and Bayes' rule yields an undefined belief $\frac{0}{0}$, entailing that we need to leave off-the-equilibrium beliefs unrestricted, that is, $\mu(H|p_1) \in [0,1]$.
- *Candidate 2's response.* Given the above updated beliefs, candidate 2 responds with policy $p_2 = L$ after $p_1(L)$, and with policy $p_2 = H$ after $p_1(H)$ since that maximizes his chances of winning the election (if player 1 announces $p_1(L) \neq L$ and $p_1(H) \neq H$ respectively) or his chances of tying the election (if player 1 announces $p_1(L) = L$ and $p_1(H) = H$ respectively).

  After any off-the-equilibrium policy announcement from candidate 1, $p_1 \neq p_1(L) \neq p_1(H)$, candidate 2 exhibits off-the-equilibrium belief $\mu \equiv \mu(H|p_1) \in [0,1]$, and thus responds with the *expected* median policy

$$\mu H + (1-\mu)L$$

  where the first term represents the probability that the median voter is high type, while the second term indicates the probability he is low type.
- *Candidate 1's messages.* Candidate 1, when choosing his policy announcement p_1 in the first stage, anticipates candidate 2's optimal responses; as discussed in the previous bullet point.
  - When he observes that the median voter is $m = L$, he chooses a policy $p_1(L) = L$, because otherwise candidate 2 could win for sure by responding with $p_2 = L$ when player 1 chooses any other policy $p_1(L) \neq L$.
  - Similarly, after observing that the median voter is $m = H$, candidate 1 chooses a policy $p_1(H) = H$; as otherwise candidate 2 could win for sure by responding with $p_2 = H$ after $p_1(H) \neq H$.
- Therefore, in the separating equilibrium, after observing that the median voter is $m = k$, where $k = \{H, L\}$:
  - candidate 1 responds with a policy that coincides with the median voter's ideal, $p_1(k) = k$, and
  - candidate 2 responds with the same policy $p_2(p_1(k)) = k$.

  As a result, there is a tie in the election, and each player receives an expected payoff of $\frac{1}{2}$.

(b) Find at least one pooling Perfect Bayesian Equilibrium (PBE).

- In a pooling strategy profile, candidate 1 announces the same policy after observing a low and a high median voter, that is, $p_1(L) = p_1(H)$

- *Candidate 2's beliefs.* Candidate 2 posterior beliefs after observing the pooling policy $p_1(L) = p_1(L) = \overline{p}_1$ cannot be updated with Bayes' rule, so they coincide with his priors, $\frac{1}{2}$, that is,

$$\mu(L|\overline{p}_1) = \mu(H|\overline{p}_1) = \frac{1}{2}.$$

(Recall that both types of median voters are equally likely.) Like in part (a) of the exercise, all policies different than the pooling policy $\overline{p}_1$, $p_1 \neq \overline{p}_1$, are regarded as off-the-equilibrium messages, and off-the-equilibrium beliefs are left unrestricted, that is,

$$\mu(H|p_1) \in [0, 1].$$

- *Candidate 2's response.* Given the above updated beliefs, after extremely low policies from candidate 1, $\overline{p}_1 < L$, candidate 2 responds with policy $p_2 = L$, since by doing so he win the election for sure. Intuitively, even if the median voter was $m = L$, candidate 1's policy is so radicalized that responding with $p_2 = L$ is enough to win the election, both if the median voter is L and H.
Likewise, if candidate 1 announces extremely high policies, $\overline{p}_1 > H$, candidate 2 responds with $p_2 = H$, since that also lets candidate 2 to win the election with certainty. A similar argument as above explains this optimal response from candidate 2.
Finally, when candidate 1 announces intermediate policies, $H \geq \overline{p}_1 \geq L$, candidate 2 can optimally respond by mimicking candidate 1's announcement, $\overline{p}_1$. Doing so leads to a tie, with an expected payoff of $\frac{1}{2}$; while choosing a different response in that interval, $p_2 \in [L, H]$, leads to candidate 1 winning the election with probability 50% (which yields a payoff of zero to candidate 2) or a tie (with expected payoff $\frac{1}{2}$), but never wins the election.
Summarizing, player 2 responds mimicking candidate 1's announcement, $\overline{p}_1$.
- *Candidate 1's messages.* When candidate 1 observes that the median voter is $m = L$, he chooses a pooling policy $\overline{p}_1$, where $\overline{p}_1 \in [L, H]$. This leads candidate 2 to respond mimicking that announcement, $p_2 = \overline{p}_1$, ultimately producing a tie in the election and an expected payoff of $\frac{1}{2}$.
If, instead, candidate 1 deviates to another policy $\overline{p}_1 < L$, player 2 responds with $p_2 = L$ and wins the election for sure. Similarly, if candidate 1 deviates to $\overline{p}_1 > H$, player 2 responds with $p_2 = H$ and wins the election for sure.
A similar argument applies when candidate 1 observes that the median voter is $m = H$, where he chooses a pooling policy $\overline{p}_1 \in [L, H]$, which is also mimicked by candidate 2, $p_2 = \overline{p}_1$, leading to a tie as well.
- Therefore, in the pooling equilibrium, after observing that the median voter is $m = k$, where $k = \{H, L\}$:
  - candidate 1 responds with a pooling policy $\overline{p}_1 \in [L, H]$, and
  - candidate 2 responds with the same policy $p_2 = \overline{p}_1$.

  As a result, there is a tie in the election, and each player receives an expected payoff of $\frac{1}{2}$.

4. **Exercise 4, Chapter 10, MIT Press book.** **[Moral Hazard Application - Credit Rationing]** Consider a setting with a bank (lender) and N firms seeking to

obtain a loan for a project. All agents are risk neutral. Upon receiving the loan, a firm can use it to invest \$$I$ in either of two projects, S (safe) or R (risky), where the return from the risky project is higher, $X_R > X_S$, but its expected return is lower, $p_S X_S > p_R X_R$ because $p_S > p_R$. Both projects are, however, profitable in expectation, i.e., $p_i X_i > I$ for all project $i = S, R$. In addition, if the investment fails (which occurs with probability $1 - p_i$ for all $i = \{R, S\}$), the borrowing firm makes no money, and defaults on both its principal and interest to the bank. Assuming that the interest rate that the bank charges is $r \in (0, 1)$ so its total revenue is $R \equiv (1 + r)I$, the bank's profit from project i is

$$\pi(R) = p_i R - I$$

while the profit that a firm obtains from investing in project i is

$$u(R) = p_i(X_i - R)$$

In addition, assume that the bank cannot offer loans to all firms, i.e., $N \cdot I > L$, but the loan amount is enough to at least offer loans to one firm, $L > I$.

(a) *Symmetric Information:* Assume that the bank could observe the project in which the firm invests the loan. Show that there is no credit rationing under this setting.

- In this context, the bank would solve the following problem,

$$\max_{i=\{S,R\},R} \quad p_i R - I$$
$$\text{subject to} \quad p_i(X_i - R) \geq 0$$

  Intuitively, the bank maximizes its profits from a given project $i = \{S, R\}$ subject to the firm's voluntary participation (participation constraint, PC). The bank can increase the interest rate enough to extract as much surplus from the firm and still guarantee participation, implying that PC holds with equality. Therefore, $p_i(X_i - R) = 0$, or $X_i = R$, which allows us to rewrite the above program as the following unconstrained problem

$$\max_{i=\{S,R\}} \quad p_i X_i - I$$

  In this context, since the bank's choice variable is binary (i.e., fund the safe or risky project), the above program is equivalent to a simple profit comparison i.e., $p_S X_S \lesseqgtr p_R X_R$. Given that $p_S X_S > p_R X_R$ by assumption, the bank chooses the safe project, yielding an expected profit of $p_S X_S - I$, and charging an interest $R = p_S X_S$ (which leaves no surplus to the firm). In this case there is no credit rationing, since all firms planning to invest in a safe project obtain funds from the bank.

(b) *Asymmetric information:* When the bank cannot observe the use of the funds by the firms that receive a loan, the above model describes a moral hazard setting similar to that existing between principal and agents where the former cannot observe the latter's effort. In this context, the banks needs to first anticipate the firm's subsequent use of the loan, and incorporate that expected behavior into its profit maximization problem. Show that in this asymmetric information setting, credit rationing can occur.

- *Second stage*. For a given interest R the firm solves

$$\max \quad \{p_S(X_S - R),\ p_R(X_R - R)\}$$

That is, it chooses the safe project if and only if

$$p_S(X_S - R) \geq p_R(X_R - R)$$

Solving for R, yields,

$$R \leq \frac{p_S X_S - p_R X_R}{p_S - p_R} \equiv \bar{R}$$

As figure 10.1 depicts, if the interest rate is sufficiently low, $R \leq \bar{R}$, the firm chooses the safe project, but otherwise it chooses the risky project. (Needless to say, if $R > X_R$, even the risky project becomes unprofitable since the interest rate exceeds its return, leading the firm to not invest in any project).

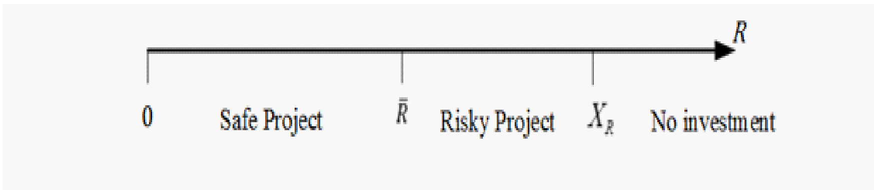

Figure 10.1. Firm's investment as a function of R.

First stage. Anticipating the cutoff rule that the firm will use in the second stage of the game, the bank's profits are,

$$\pi(R) = \begin{cases} p_S R - I, & \text{if } 0 \leq R \leq \bar{R} \\ p_R R - I, & \text{if } \bar{R} < R \leq X_R \end{cases}$$

Since $p_S > p_R$, we can depict the bank's profits as a function of R with two lines, as in figure 10.2. Intuitively, when $R \leq \bar{R}$ the bank anticipates that the firm invests in the safe project yielding profits of $p_S R - I$ for the bank. However, when $R > \bar{R}$, the firm invests in the risky project yielding profits of $p_R R - I$. (The case of $R > X_R$ is uninteresting as the firm does not invest in either project, and thus bank profits are 0).

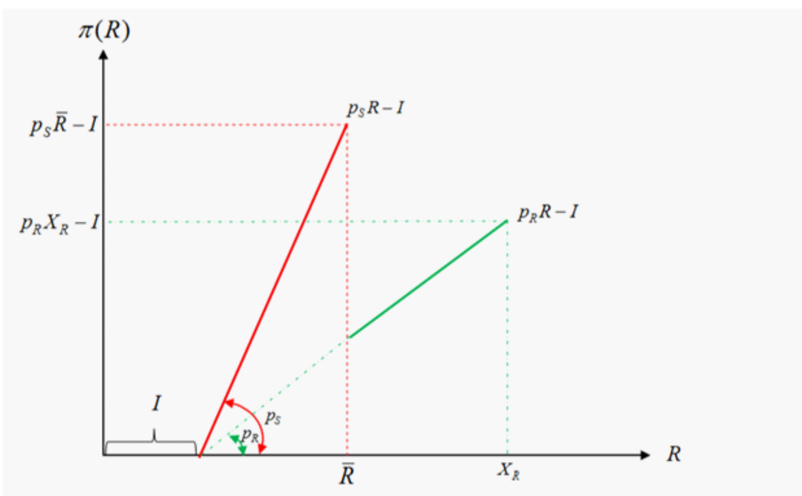

Figure 10.2. Bank's profits as a function of R.

Importantly, if X_R is sufficiently high, the line $p_R R - I$ could keep growing (see right-hand side of the figure), ultimately reaching a height above $p_S \bar{R} - I$ (the highest height of the line $p_S \bar{R} - I$). That is,

– If

$$p_R X_R - I > p_S \bar{R} - I, \quad \text{or} \quad p_R X_R > p_S \bar{R},$$

the bank obtains a higher expected profit inducing the risky project by setting the highest possible interest rate, i.e., $R = X_R$, which yields no profit for the firm since $p_R(X_R - R) = 0$, than inducing the safe project.

– If, instead, X_R is not very high (as depicted in the figure) the height of profit line $p_R X_R - I$ lies below that of the safe project at $\bar{R}, p_S \bar{R} - I$, implying that the bank's expected profit is higher with the safe project. This occurs when

$$p_R X_R - I < p_S \bar{R} - I, \quad \text{or} \quad p_R X_R < p_S \bar{R}$$

and the bank sets the interest rate $R = \bar{R}$ that induces firms to choose the safe project. In this case, the firm's expected profit is

$$u(\bar{R}) = p_S(X_s - \bar{R}) > 0$$

which induces all firms to ask for a loan in order to implement the safe project. In addition, note that $p_S(X_s - \bar{R})$ is positive since

$$X_s - \bar{R} = X_s - \frac{p_S X_S - p_R X_R}{p_S - p_R} = \frac{p_R\,(X_R - X_S)}{p_S - p_R} > 0$$

given that $X_R > X_S$ and $p_S > p_R$ by definition, i.e., the risky project yields a higher return than the safe project, but is less likely to be successful. However, since $N \cdot I > L$ by definition, the bank cannot offer loans to all firms, ultimately giving rise to credit rationing.

- *Numerical Example:* Consider $N = 10$ firms each seeking to invest $I = 0.2$ in either a safe project yielding a return of $X_S = 2$ or a risky project with return $X_R = 3$, with associated probabilities $p_s = 0.8$ and $p_R = 0.3$ (so the initial condition $p_S Xs > p_R X_R$ holds). Assume that the bank only has a loan amount of $L = 1$. In this setting, cutoff $\bar{R} = 1.4$. Therefore, by setting the interest rate at $R = \bar{R} = 1.4$, the bank's profit is $p_S \bar{R} - I = 0.92$, whereas by setting the interest rate at $R = X_R$ the bank's profit is only $p_R X_R - I = 0.7$. Hence, the bank prefers to set the interest rate at $R = 1.4$ in order to induce all firms to select the safe project. However, since $N * I = 10 * 0.2 = 2 > 1 = L$, credit rationing emerges. (Finally, note that firms voluntary participate since $p_S \left(X_S - \bar{R}\right) = 0.8(2 - 1.4) = 0.48 > 0$.)

5. **Exercise 8, Chapter 10, MIT Press book. [Moral Hazard - Using a Signal of Effort as a Bonus for the Agent]** Consider the same model as in Exercise 7. Let us allow, however, the principal to use not only the observed output q, but also a signal

s (correlated with ε) to design the agent's compensation. For example, if the agent is selling smartphones, signal s could be the sales of other agents. Let the distribution of ε and s be given by

$$\begin{pmatrix} \varepsilon \\ s \end{pmatrix} \backsim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon s} \\ \sigma_{\varepsilon s} & \sigma_s^2 \end{bmatrix}\right)$$

Suppose the principal uses the linear wage function:

$$w(q, s) = F + b(q + \alpha s)$$

where the agent receives a fixed payment F, but his bonus now depends on both his own sales, q, and that of the salesman that the principal uses as a signal, s. Finally parameter α indicates the sensitivity of the bonus to the signal. For instance, when $\alpha = 0$ the principal does not use the signal to determine the agent's compensation (as in the standard model analyzed in the previous exercise) while if $\alpha = 1$ the principal considers that the signal is as informative about the agent's unobserved effort as the agent's own sales.

(a) Using the same approach as in the previous exercise, solve for the optimal value of parameter α. Does the optimal value of α increase or decrease when ε and s become more correlated? Interpret.

- The principal's problem in this context is

$$\max_{F,b,\alpha,e} \quad e - (F + be)$$

$$\text{subject to} \quad F + be - \frac{1}{2}\text{Var}(w) - c(e) \geq 0 \qquad (PC)$$

$$e \in \arg\max_{e' \geq 0} \quad F + be' - \frac{1}{2}\text{Var}(w) - c(e') \qquad (IC)$$

where now the principal must choose not only the optimal value of F, b and e (as in the previous exercise), but also the optimal value of α. Using the binding PC constraint, we can find the expression for F:

$$F = c(e) + \frac{\text{Var}(w)}{2} - be = c(e) + \frac{b^2\text{Var}(\varepsilon + \alpha s)}{2} - be$$

since $\text{Var}(w) = b^2\text{Var}(\varepsilon + \alpha s)$. Substituting F in the principal's objective function yields

$$e - \left(c(e) + \frac{1}{2}b^2\text{Var}(\varepsilon + \alpha s) - be + be\right) = e - c(e) - \frac{1}{2}b^2\text{Var}(\varepsilon + \alpha s)$$

Hence, the principal problem simplifies, as follows

$$\max_{F,b,\alpha,e} \quad e - \frac{1}{2}b^2\text{Var}(\varepsilon + \alpha s) - c(e)$$

$$\text{subject to} \quad e \in \arg\max_{e' \geq 0} \quad F + be' - \frac{1}{2}b^2\text{Var}(\varepsilon + \alpha s) - c(e') \qquad (IC)$$

We can now further simplify the maximization problem by noticing that : (1) $\text{Var}(\varepsilon + \alpha s) = \sigma_\varepsilon^2 + \alpha^2\sigma_s^2 + 2\alpha\sigma_{\varepsilon s}$ given the distribution of ε and s provided at the beginning of the exercise[2]; and (2) the agent's maximization problem in the IC constraint has only two terms that are dependent on his choice of effort, e, implying that the arg max of $F + be' - \frac{1}{2}b^2\text{Var}(\varepsilon + \alpha s) - c(e')$ coincides with that of $be' - c(e')$. Hence, the maximization problem becomes,

$$\max_{b,\alpha,e} \quad e - \frac{1}{2}b^2[\sigma_\varepsilon^2 + \alpha^2\sigma_s^2 + 2\alpha\sigma_{\varepsilon s}] - c(e)$$

$$\text{subject to } e \in \arg\max_{e' \geq 0} \quad be' - c(e') \qquad (IC)$$

Using the first-order approach, we can now differentiate with respect to effort e' in the agent's IC constraint, to obtain $b = c'(e)$, which we can substitute in the principal's objective function, yielding the following unconstrained problem

$$\max_{\alpha,e} \quad e - \frac{1}{2}[c'(e)]^2[\sigma_\varepsilon^2 + \alpha^2\sigma_s^2 + 2\alpha\sigma_{\varepsilon s}] - c(e)$$

Taking FOC with respect to e and α, we obtain

$$1 - c'(e)c''(e)[\sigma_\varepsilon^2 + \alpha^2\sigma_s^2 + 2\alpha\sigma_{\varepsilon s}] - c'(e) = 0 \qquad (\text{FOC}_e)$$

$$-2\alpha\sigma_s^2 - 2\sigma_{\varepsilon s} = 0 \qquad (\text{FOC}_\alpha)$$

Solving for α in FOC_α yields,

$$\alpha^* = -\frac{\sigma_{\varepsilon s}}{\sigma_s^2}$$

Intuitively, if ε and s are:

- Uncorrelated, $\sigma_{\varepsilon s} = 0$, then parameter α^* becomes $\alpha^* = 0$, thus indicating that the principal does not use signal s to design the agent's compensation, i.e., the agent's wage simplifies to $w = F + bq$.
- Positively correlated $\sigma_{\varepsilon s} > 0$, then $\alpha^* < 0$, implying that a high signal (e.g., many sales by the other agent) means that it is easy to sell output (e.g., perhaps the economic conditions are improving in the area), and the agent should receive a smaller bonus.
- The opposite argument applies when ε and s are negatively correlated $\sigma_{\varepsilon s} < 0$, leading to $\alpha^* > 0$. In words, sales by the other agent mean now that the agent's output is affected by smaller shocks, and thus should be provided a larger bonus for every unit sold, as each unit is more indicative of a higher effort.

In addition, from the above objective function in the principal's problem, it is easy to see that parameter α only affects the second term, $-\frac{1}{2}b^2\text{Var}(\varepsilon + \alpha s)$, which enters negatively. Hence, the principal chooses α to minimize the variance $\text{Var}(\varepsilon + \alpha s)$.

[2] Recall that, for two random variables x and y, the variance of the sum $ax + by$ is given by $\text{Var}(ax + by) = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}$.

- Finally, we can solve for $c'(e)$ in FOC_e, and use the first-order approach $b = c'(e)$, to obtain that the optimal bonus

$$b^* = \frac{1}{1 + c''(e)[\sigma_\varepsilon^2 + \alpha^2\sigma_s^2 + 2\alpha\sigma_{\varepsilon s}]}$$

And inserting our above result, $\alpha^* = -\frac{\sigma_{\varepsilon s}}{\sigma_s^2}$, yields

$$\begin{aligned} b^* &= \frac{1}{1 + c''(e)\left[\sigma_\varepsilon^2 + \frac{\sigma_{\varepsilon s}^2}{\sigma_s^4}\sigma_s^2 - 2\frac{\sigma_{\varepsilon s}}{\sigma_s^2}\sigma_{\varepsilon s}\right]} \\ &= \frac{1}{1 + c''(e)\left[\sigma_\varepsilon^2 - \frac{\sigma_{\varepsilon s}^2}{\sigma_s^2}\right]} \end{aligned}$$

Note that when $\sigma_{\varepsilon s} = 0$, we obtain the same bonus as in the previous exercise, i.e., $b^* = \frac{1}{1+c''(e)\sigma_\varepsilon^2}$. However, under noise correlation, $\sigma_{\varepsilon s} \neq 0$ (which allows for it to positive or negative), bonus b^* found above is larger than when $\sigma_{\varepsilon s} = 0$ since

$$\frac{1}{1 + c''(e)\left[\sigma_\varepsilon^2 - \frac{\sigma_{\varepsilon s}^2}{\sigma_s^2}\right]} > \frac{1}{1 + c''(e)\sigma_\varepsilon^2}.$$

Intuitively, the principal should pay more to the risk-averse agent if he faces a higher risk undertainty in the realization of effort to output.

(b) Assuming the same parametric expressions as in part (b) of the previous exercise, i.e., cost function $c(e) = \theta e^2$, where $\theta > 0$, find the optimal values e^*, b^*, F^* and α^*.

- Using FOC_e, we obtain

$$1 - 4\theta^2 e^*[\sigma_\varepsilon^2 + (\alpha^*)^2\sigma_s^2 + 2\alpha^*\sigma_{\varepsilon s}] - 2\theta e^* = 0 \qquad (\text{FOC}_e)$$

which, solving for effort e^*, yields

$$\begin{aligned} e^* &= \frac{1}{2\theta\left[1 + 2\theta\left[\sigma_\varepsilon^2 + \left(\frac{\sigma_{\varepsilon s}}{\sigma_s^2}\right)^2\sigma_s^2 - 2\left(\frac{\sigma_{\varepsilon s}}{\sigma_s^2}\right)\sigma_{\varepsilon s}\right]\right]} \\ &= \frac{\sigma_s^2}{2\theta\left[\sigma_s^2 + 2\theta\left(\sigma_\varepsilon^2\sigma_s^2 - \sigma_{\varepsilon s}^2\right)\right]} \end{aligned}$$

Now, using the expression of bonus, b^*, found in part (a) we get

$$\begin{aligned} b^* &= \frac{1}{1 + 2\theta\left[\sigma_\varepsilon^2 + \left(\frac{\sigma_{\varepsilon s}}{\sigma_s^2}\right)^2\sigma_s^2 - 2\left(\frac{\sigma_{\varepsilon s}}{\sigma_s^2}\right)\sigma_{\varepsilon s}\right]} \\ &= \frac{1}{1 + 2\theta\left[\sigma_\varepsilon^2 - \frac{\sigma_{\varepsilon s}^2}{\sigma_s^2}\right]} \end{aligned}$$

Last, from the PC constraint we know that the fixed payment satisfies $F = c(e) + \frac{b^2 \text{Var}(\varepsilon+\alpha s)}{2} - be$. Since

$$\begin{aligned}\text{Var}(\varepsilon + \alpha s) &= \sigma_\varepsilon^2 + \alpha^2 \sigma_s^2 + 2\alpha\sigma_{\varepsilon s} \\ &= \sigma_\varepsilon^2 + \frac{\sigma_{\varepsilon s}^2}{\sigma_s^4}\sigma_s^2 + 2\frac{\sigma_{\varepsilon s}}{\sigma_s^2}\sigma_{\varepsilon s} \\ &= \sigma_\varepsilon^2 - \left(\frac{\sigma_{\varepsilon s}}{\sigma_s}\right)^2\end{aligned}$$

The fixed payment F becomes $F = c(e) + \frac{b^2\left(\sigma_\varepsilon^2 - \left(\frac{\sigma_{\varepsilon s}}{\sigma_s}\right)^2\right)}{2} - be$. Evaluating F at e^* and b^*, yields

$$\begin{aligned}F^* &= \theta e^2 + \frac{b^2\left(\sigma_\varepsilon^2 - \left(\frac{\sigma_{\varepsilon s}}{\sigma_s}\right)^2\right)}{2} - be \\ &= \theta e^2 + \frac{b^2}{2\sigma_s}\left(\sigma_\varepsilon^2\sigma_s^2 - \sigma_{\varepsilon s}^2\right) - be \\ &= \left(\frac{\sigma_\varepsilon^2\sigma_s^2 - \sigma_{\varepsilon s}^2}{2\sigma_s^2} - \frac{1}{4\theta}\right)\left(\frac{\sigma_s^2}{\sigma_s^2 + 2\theta\left(\sigma_\varepsilon^2\sigma_s^2 - \sigma_{\varepsilon s}^2\right)}\right)^2 \\ &= \frac{\sigma_s^4\left(2\theta\sigma_\varepsilon^2 - 1\right) - 2\theta\sigma_\varepsilon^2\sigma_{\varepsilon s}^2}{4\theta\left[\sigma_s^2 + 2\theta\left(\sigma_\varepsilon^2\sigma_s^2 - \sigma_{\varepsilon s}^2\right)\right]}\end{aligned}$$

Finally, parameter α satisfies $\alpha^* = -\frac{\sigma_{\varepsilon s}}{\sigma_s^2}$.

(c) Evaluate your results in part (b) at

$$\begin{bmatrix}\sigma_\varepsilon^2 & \sigma_{\varepsilon s} \\ \sigma_{\varepsilon s} & \sigma_s^2\end{bmatrix} = \begin{bmatrix}1 & 1/2 \\ 1/2 & 1\end{bmatrix}$$

Compare the optimal values of e^*, b^*, F^* and α^* against those in part (b) of the previous exercise.

- Considering this variance-covariance matrix, we obtain that $\alpha^* = -\frac{1/2}{1} = -\frac{1}{2}$, and that

$$\begin{aligned}\text{Var}(\varepsilon + \alpha s) &= \sigma_\varepsilon^2 - \left(\frac{\sigma_{\varepsilon s}}{\sigma_s}\right)^2 \\ &= 1^2 - \left(\frac{1/2}{1}\right)^2 \\ &= \frac{3}{4}\end{aligned}$$

Hence, effort and bonus are

$$\begin{aligned}e^* &= \frac{1^2}{2\theta\left[1^2 + 2\theta\left(1 - (1/2)^2\right)\right]} \\ &= \frac{1}{\theta(2+3\theta)}\end{aligned}$$

and

$$\begin{aligned} b^* &= \frac{1}{1+2\theta\left[1^2-\frac{(1/2)^2}{1^2}\right]} \\ &= \frac{2}{2+3\theta} \end{aligned}$$

and the fixed payment becomes

$$\begin{aligned} F^* &= \frac{1^4\,(2\theta-1)-2\theta}{4\theta\,[1+2\theta\,(1-(1/2)^2)]} \\ &= \frac{3\theta-2}{2\theta(2+3\theta)^2} \end{aligned}$$

Hence, for this parametric example, effort e^* becomes higher than when the principal only uses the sales of the agent to design his compensation, i.e., $\frac{1}{\theta(2+3\theta)} > \frac{1}{2\theta(1+2\theta)}$ for all θ; the bonus b^* is also higher, i.e., $\frac{2}{2+3\theta} > \frac{1}{1+2\theta}$ for all θ; and the fixed payment F is higher, i.e., $\frac{3\theta-2}{2\theta(2+3\theta)^2} > -\frac{16\theta^2(1+\theta)+2\theta+1}{4\theta(1+2\theta)^2}$ for all θ since the agent receives a fixed payment when the principal uses two instruments (sales and a signal) to determine the agent's compensation, but must pay (negative F) when the principal only uses sales to determine the agent's compensation.