# ECONS 424 – STRATEGY AND GAME THEORY

# HOMEWORK #7 – *ANSWER KEY*

**<u>Exercise 5-Chapter 28-Watson (*Signaling between a judge and a defendant*)</u>**

***a. This game has a unique PBE. Find and report it.***

After $E^1$, the judge chooses $\bar{y}$ such that:

$$\underset{\bar{y}}{Max} -(\bar{y}-1)^2$$

Taking FOCs with respect to $\bar{y}$, we obtain:

$$-2(\bar{y}-1)=0$$
$$\rightarrow \quad \bar{y}=1$$

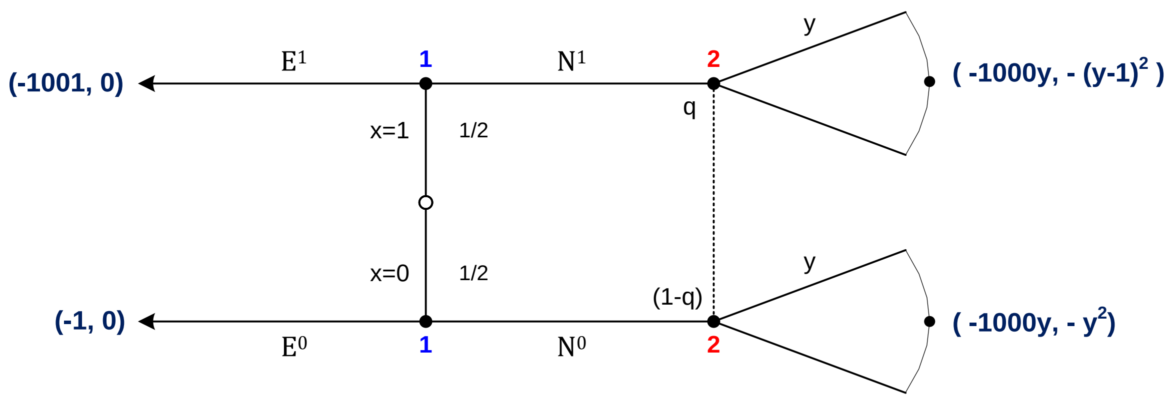
Similarly, after $E^0$, the judge chooses $\underline{y}$ such that:

$$\underset{\underline{y}}{Max} -\underline{y}^2$$

Taking FOCs with respect to $\underline{y}$, we obtain:

$$-2\underline{y} \leq 0 \quad \rightarrow \quad \underline{y}=0$$

Hence, the game becomes:




- ***Let us first check for the existence of a separating PBE where $E^0$ and $N^1$:***

1. *Belief*: q=1 since N only comes from x=1

2. *Judge (second mover)*: After observing N, the judge selects y assigning full probability to being in the open node of his information set (see figure below)

Hence,

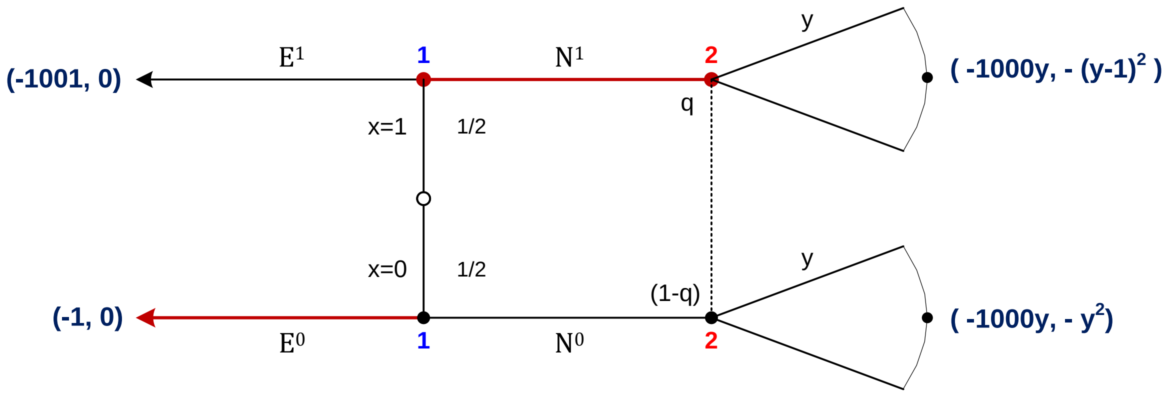
$$\underset{y}{Max} -(y-1)^2$$

Taking FOCs with respect to $y$, we obtain:

$$-2(y-1)=0, \text{ which implies } y=1$$

3. *Defendant (first mover)*:
   - If $x=1$, the defendant compares:
     - -1001 if he chooses $E^1$
     - -1000 if he chooses $N^1$
     - So, $N^1$ is better.

- If $x = 0$, the defendant compares:
  - -1 if he chooses $E^0$
  - -1000 if he chooses $N^0$
  - So, $E^0$ is better.




Hence, this separating PBE can be supported.

- ***Let us now check the separating $N^0E^1$***

1. *Beliefs*: $q = 0$ since $N$ only comes from $x = 0$
2. *Judge*: After observing $N$, the judge assigns full probability to lower node of his information set. Then, he selects $y$ such that:

$$\underset{y}{Max} -y^2$$

Taking FOCs with respect to $y$, we obtain:

$$-2y \leq 0, \text{ which implies } y = 0$$

3. *Defendant*:
   - If $x = 1$, the defendant compares:
     - -1001 if he chooses $E^1$
     - 0 if he chooses $N^1$
     - So, $N^1$ is better $\leftarrow$ Deviation from the prescribed separating $N^0E^1$.
   - If $x = 0$, the defendant compares:
     - -1 if he chooses $E^0$
     - 0 if he chooses $N^0$
     - So, $N^0$ is better. Hence, the separating $N^0E^1$ cannot be supported as PBE.

- ***Let us now check if a pooling PBE where $N^0N^1$ can be sustained***

1. *Beliefs*:

$$q = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times 1} = \frac{1}{2}$$

2. *Judge*: After observing N, given his beliefs q=1/2, he must choose y in order to maximize his expected utility:

$$\underset{y}{Max} \ \frac{1}{2}[-(y-1)^2] + \frac{1}{2}[-y^2]$$

Taking FOCs with respect to $y$, we obtain:

$$-\frac{1}{2}\times 2(y-1)-\frac{1}{2}\times 2y=0, \text{ which implies } y=\frac{1}{2}$$

3. *Defendant*:

- If $x=1$, the defendant compares:
  $-1001$ if he chooses $E^1$
  $-1000\times\frac{1}{2}=-500$ if he chooses $N^1$
  So, $N^1$ is better
- If $x=0$, the defendant compares:
  $-1$ if he chooses $E^0$
  $-1000\times\frac{1}{2}=-500$ if he chooses $N^0$
  So, $N^0$ is better → Deviation from the prescribed pooling. Hence, the pooling $N^0N^1$ cannot be sustained.

- ***Let us now check if a pooling PBE where $E^0E^1$ can be sustained.***

1. *Beliefs*: $q\in[0,1]$ since N is only observed off-the-equilibrium
2. *Judge*: From his beliefs, he chooses y in order to maximize his expected utility:

$$\underset{y}{Max}\ \ q[-(y-1)^2]+(1-q)[-y^2]$$

Taking FOCs with respect to $y$, we obtain:

$$-q\times 2(y-1)-(1-q)\times 2y=0, \text{ which implies } y=q$$

3. *Defendant*:

- If $x=1$, the defendant compares:
  $-1001$ if he chooses $E^1$
  $-1000q$ if he chooses $N^1$
  So, $N^1$ is better for any $q<1$ → Deviation from the prescribed pooling.
- If $x=0$, the defendant compares:
  $-1$ if he chooses $E^0$
  $-1000q$ if he chooses $N^0$
  So, $E^0$ is better for any $q>1/1000$ → the pooling $E^0E^1$ cannot be supported as PBE either.

***b. Explain why the result of part (a) is interesting from an economic standpoint?***

The only equilibrium that we can support in this game is the separating equilibrium in which the innocent defendant provides evidence of his innocence, whereas the guilty defendant does not provide such evidence. This is something desirable, since the judge can perfectly infer the true innocence of a defendant by simply observing whether he/she presented evidences.

***c. When $x\in[0,K]$ with each value equally likely, compute the PBE.***

We are going to test the equilibrium where all types of x={0,…,K-1} present evidence (E), but the last type x=K presents no evidence (N).

1) <u>Beliefs</u>

After observing the evidence presented by the defendant, the judge can perfectly observe his type 0,1,2,…,K-1. In these cases we don't need to specify beliefs. When no evidence(N) is presented, the judge's beliefs are:

$$\mu(t_j|N) = 0 \ \forall \ j = \{0, \dots, K-1\}$$

$$\mu(t_K|N) = 1$$

Which implies that after receiving no evidence, the judge assigns full probability to the K-type, and therefore no probability to any of the 0,1,2,…,K-1 types.

2) Judge's Best Response:

Given N:

$$\max_y -(y-K)^2 \quad \rightarrow \quad y^N = K$$

Given E(where there is no information set and the judge knows what type has played E):

$$\max_y [-(y-x)^2]$$

where x is the specific type of the defendant that presented evidence (a type that is observed by the judge thanks to the presentation of evidence). Taking FOCs with respect to y, we obtain

$$-2y + 2x = 0$$

$$\rightarrow y^E = x$$

3) Defendant's Best Response:

**For types 0,…,K-1** : if he provides evidence, E, then they get $y^E$ from the judge, providing:

$$-1000y^E - 1$$

which must exceed his payoff from not presenting evidence: $-1000y^N = -1000K$

(in this case the judge interprets that the defendant is a K-type and chooses a sentence $y^N = K$)

- Note, type x=0 prefers the payoff he obtains by presenting evidence, $-1000y^E - 1 = -1$, than his payoff from not presenting evidence, $-1000K$ (since K > 2 given that there are more than two types of defendants).
- Similarly for type x=1, where $-1000y^E - 1 = -1001 > -1000 * K$; and for all other types x=2,3,…
- The defendant who obtains the lowest equilibrium payoff from providing evidence is x=$K$-1, who obtains $-1000y^E - 1 = -1000(K-1) - 1$. Let us check if his equilibrium payoff from providing evidence is larger than from deviating, that is:

$$-1000K + 1000 - 1 > -1000K$$

$$-1000K + 999 > -1000K$$

$$999 > 0$$

This obviously holds, so the defendant behaves as prescribed when his type is x=0,…,K-1

**For type K**: if he doesn't provide evidence, N, (as initially prescribed) then he gets a sentence $y^N = K$ from the judge, providing a payoff of:

$$-1000K$$

This must exceed his alternative payoff from providing evidence (E):

$$-1000K - 1$$

The condition reduces to:

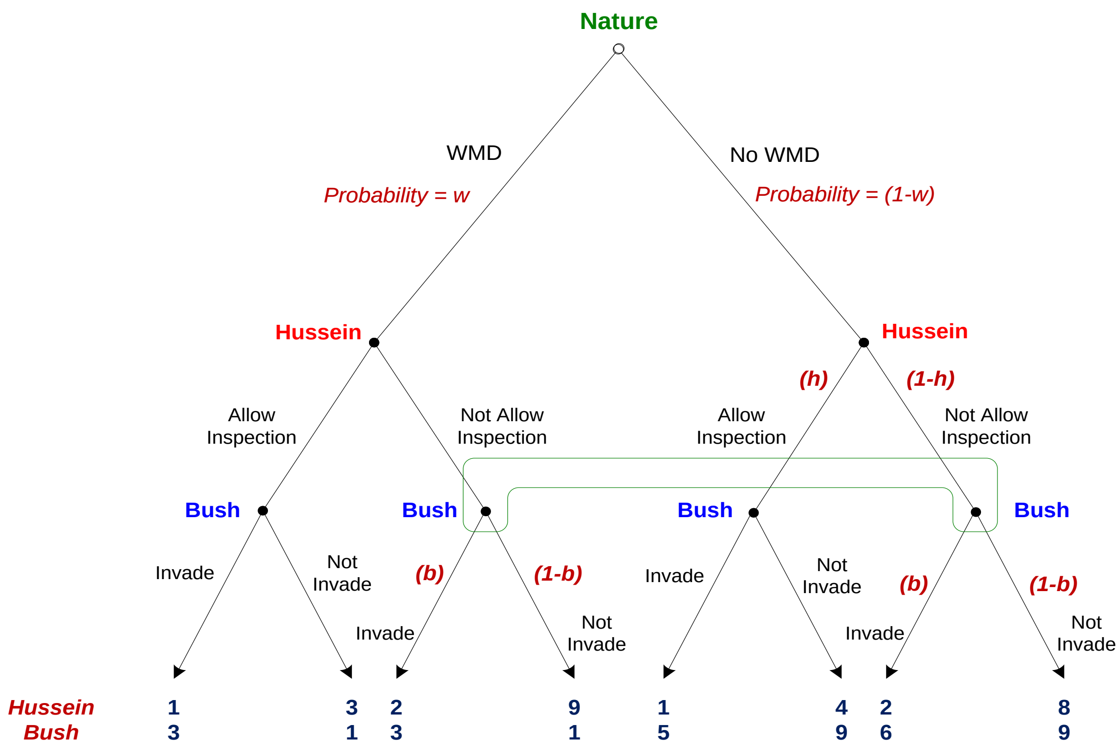
$$-1000K > -1000K - 1$$

$$0 > -1$$

This as well holds, showing the initially stated strategy, where types x=0,1,2,…,K-1 present evidence but type x=K does not, can be sustained as a PBE.

**Exercise 4-Chapter 11-Harrington**

The extensive form of the WMD game:




1. Nature moves first determining a presence of WMD:

- with probability $w$ Hussein has WMD
- with probability $(1-w)$ he does not, where $0 < w < 1/3$

2. After observing his own type, Hussein's strategies are the following:

- when he <u>has</u> WMD then he does not allow inspections with probability 1.
- when he <u>does not have</u> WMD then he can choose either to allow inspection with probability h or do not allow – with probability $(1-h)$.

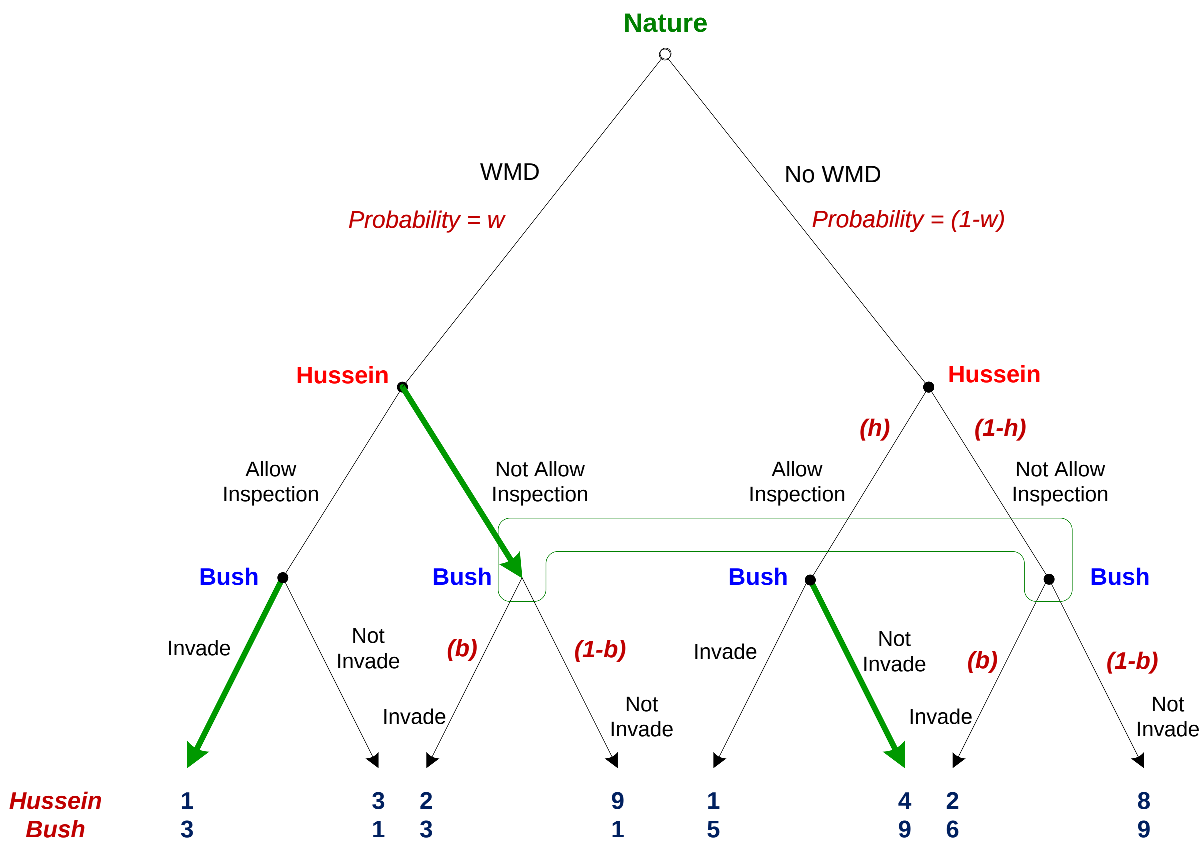
3. Assumptions:

- If Hussein has WMD, then Bush found out it and then Bush wants to invade;
- If Hussein does not have WMD, then Bush does not found it and he prefers not to invade.

After observing Hussein's decision about inspection, Bush strategies are:

- if Hussein allows inspections and WMD are found, then invade with probability 1.
- if Hussein allows inspections and WMD are not found, then do not invade with probability 1.
- if Hussein does not allow inspections, then Bush can guess that with some probability Hussein has WMD, so that Bush invade with probability $b$.

See graph 1 below:




**Steps**:

- **Step 1: Bush's beliefs**

  If Hussein does not allow inspections, then the probability of Hussein's having WMD is given by Bayes's rule:

$$P(WMD|Not\ Allow) = \frac{P(WMD) \times P(Not\ Allow, WMD)}{P(Not\ Allow)} = \frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}$$

  where Sadam has WMD with probability $w$, and in that event, he does not allow inspections with probability 1; and while with probability $(1-w)$, Sadam has WMD and, in that event, he does not allow inspections with probability $(1-h)$.

- **Step 2: Bush's optimal strategy given his beliefs**

Its optimality is clear when there are inspections, whether WMD are found or not.

- When inspections <u>are not allowed</u>, Bush is content to randomize (that is, $0 < b < 1$) if and only if:

$$E^{Bush}[INV|WMD\ or\ No\ WMD] = E^{Bush}[No\ INV|WMD\ or\ No\ WMD\ and\ NA]$$

$$3\left[\frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}\right] + 6\left[\frac{w \times (1-h)}{w \times 1 + (1-w) \times (1-h)}\right] = \left[\frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}\right] + 9\left[\frac{w \times (1-h)}{w \times 1 + (1-w) \times (1-h)}\right]$$

The left-hand expression is the expected payoff from invading, and the right-hand expression is the expected payoff from not invading. Solving this equation for h yields:

$$h = \frac{3-5w}{3(1-w)}$$

Note that:

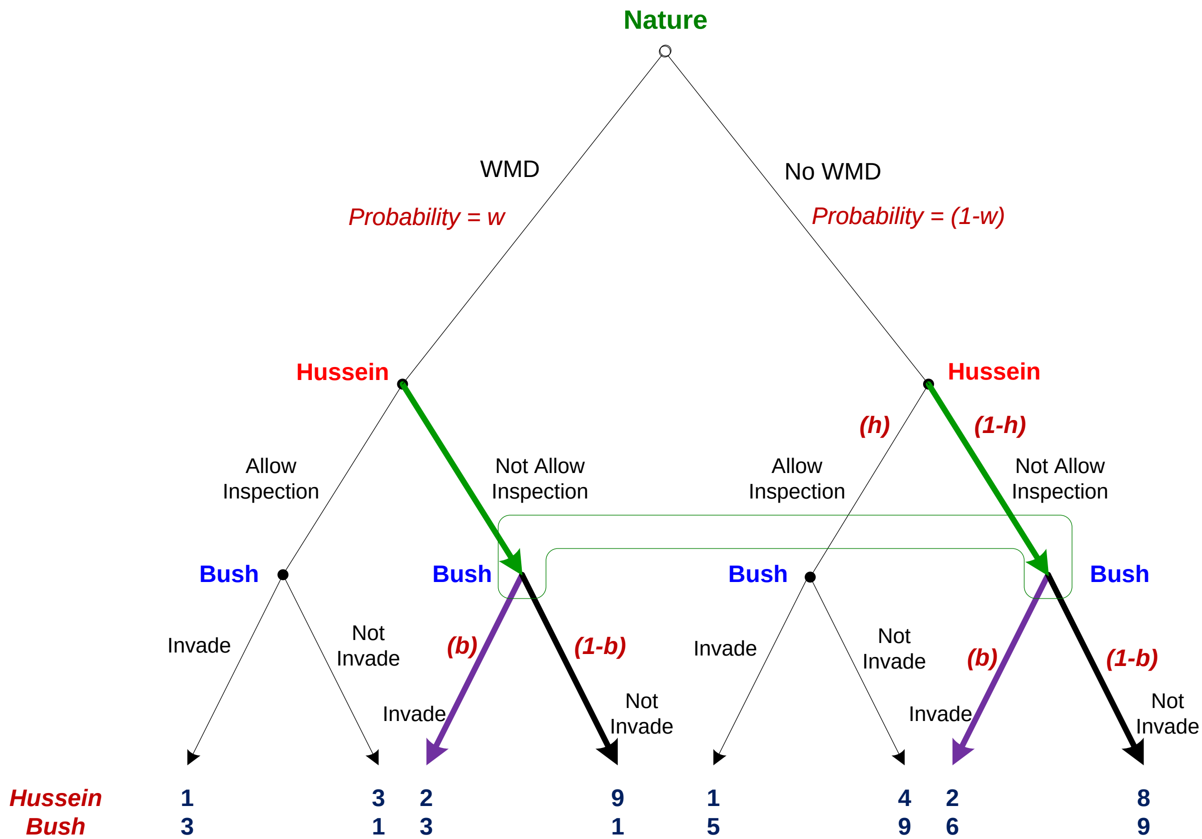
$$0 < \frac{3-5w}{3(1-w)} < 1 \quad when\ \ 0 < w < \frac{3}{5}$$

The latter condition was assumed. See graph 2 below.

- When Sadam <u>has WMD</u>, it is clearly optimal for him to not allow inspections. When he does not have WMD, it is optimal to randomize if and only if:

$$2b + 8(1-b) = 4$$

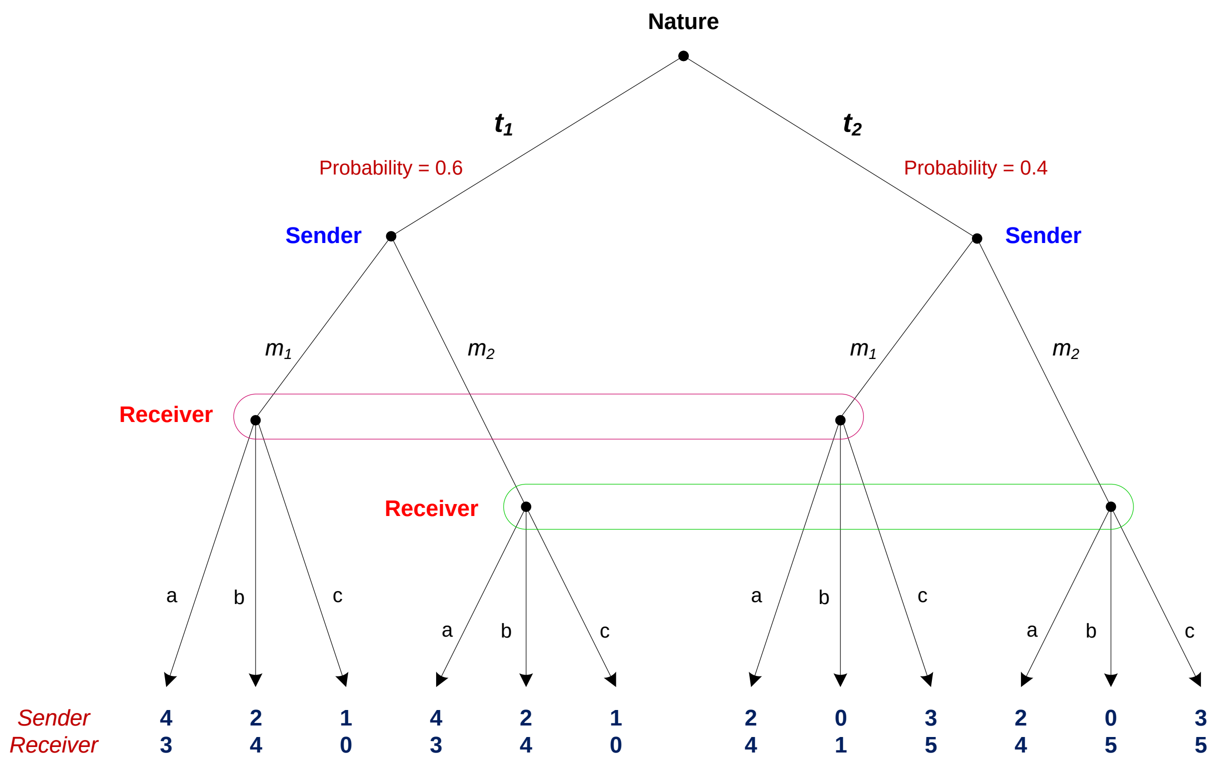
where he earns a payoff of 4 by allowing inspections – in which case there is no invasion-- and gets an expected payoff of $2b + 8(1-b) = 4$ from not allowing inspections (where there is an invasion with probability b). Solving this equation, we can get $b = 2/3$.




$$E^{Bush}[INV|WMD\ or\ No\ WMD] = P(NA|WMD) \times U(INV) + P(NA|No\ WMD) \times U(INV)$$
$$= 3 \times P(NA|WMD) + 6 \times P(NA|No\ WMD)$$
$$E^{Bush}[NINV|WMD\ or\ No\ WMD\ and\ NA] = 1 \times P(NA|WMD) + 9 \times P(NA|No\ WMD)$$

## Exercise 5-Chapter 12-Harrington

Consider the cheap talk game:




***a. Find a separating PBNE.***

With a separating equilibrium, the sender chooses distinct messages, so let us presume that the sender chooses $m_1$ when his type is $t_1$ and chooses $m_2$ when his type is $t_2$. (We could instead have supposed that the sender's strategy is to choose $m_2$ when his type is $t_1$, and $m_1$ when his type is $t_2$.)

Receiver's beliefs

After observing message $m_1$,

$$\mu(t_1|m_1) = 1$$
$$\mu(t_2|m_1) = 0$$

And after observing message $m_2$,

$$\mu(t_1|m_2) = 0$$
$$\mu(t_2|m_2) = 1$$

Receiver's optimal response

- After observing $m_1$, the receiver believes that such a message can only originate from a $t_1$-type of sender. Hence, his optimal response is $b$ given that it yields a payoff of 4 (higher than what he gets from $a$, 3, and $c$, 0.)
- After observing message $m_2$, the receiver believes that such a message can only originate from a $t_2$-type of sender. Hence, his optimal response is either $b$ or $c$, since both yield a payoff of 5, rather than $a$, which only provides a payoff of 4. For simplicity, we choose $c$.

Sender's optimal messages

- If his type is $t_1$, by sending $m_1$ he obtains a payoff of 2 (since $m_1$ is responded with $b$), but a lower payoff of 1 if he deviates towards message $m_2$ (since such message is responded with $c$). Hence, the sender doesn't want to deviate from $m_1$. [Note that if $m_2$ were responded with $b$, then the sender would be indifferent between $m_1$ and $m_2$ (both would yield a payoff of 2). Strictly speaking, he wouldn't have incentives to deviate from message $m_1$].
- If his type is $t_2$, he obtains a payoff of 3 by sending message $m_2$ (which is responded with $c$) and a payoff of 0 if he deviates to message $m_1$ (which is responded with $b$). Hence, he doesn't have incentives to deviate from $m_2$. [Similarly as above, note that if message $m_2$ were responded with $b$ the sender would obtain the same payoff sending $m_2$ and $m_1$, 0. Nonetheless, $t_2$-sender wouldn't have incentives to deviate from his initially prescribed message of $m_2$].

Hence, the initially prescribed separating strategy profile can be supported as a PBE.
Also note that there is another separating equilibrium in which $m_1$ and $m_2$ are exchanged.

***b. Find a pooling PBNE.***

With a pooling PBNE, the sender chooses the same message regardless of his type. Let this message be $m_1$.

Receiver's beliefs

After observing message $m_1$ (in problem),

$$\mu(t_1|m_1) = \frac{0.6 * 1}{0.6 * 1 + 0.4 * 1} = 0.6$$

$$\mu(t_2|m_1) = \frac{0.4 * 1}{0.6 * 1 + 0.4 * 1} = 0.4$$

After receiving message $m_2$ (off-the-equilibrium),

$$\mu(t_1|m_1) = \frac{0.6 * 0}{0.6 * 0 + 0.4 * 0} = \frac{0}{0}$$

and hence beliefs must be arbitrarily specified, i.e. $\mu \in [0,1]$.

<u>Receiver's optimal response</u>

- After receiving a message $m_1$, the receiver's expected utility from responding with actions $a$, $b$, and $c$ are

$$\text{Action } a\text{:} \quad 0.6 \times 3 + 0.4 \times 4 = 3.4$$
$$\text{Action } b\text{:} \quad 0.6 \times 4 + 0.4 \times 1 = 2.8$$
$$\text{Action } c\text{:} \quad 0.6 \times 0 + 0.4 \times 5 = 2.0$$

Hence, the receiver's optimal strategy is to choose action $a$ in response to message $m_1$.

- After receiving message $m_2$ (off-the-equilibrium), the receiver's EUs from each of his three possible responses are

$$EU_{Receiver}(a|m_2) = \mu * 3 + (1-\mu) * 4 = 4 - \mu$$
$$EU_{Receiver}(b|m_2) = \mu * 4 + (1-\mu) * 5 = 5 - \mu$$
$$EU_{Receiver}(c|m_2) = \mu * 0 + (1-\mu) * 5 = 5 - 5\mu$$

  Where it's clear that the EU from responding with $b$, $5 - \mu$, is the highest EU payoff the responder can obtain given that $\mu \in [0,1]$.

<u>Sender's optimal message</u>

- If his type is $t_1$, the sender obtains a payoff of 4 from sending $m_1$ (since it is responded with $a$), but a payoff of only 2 when deviating towards $m_2$ (since it is responded with $b$). Hence, he doesn't have incentives to deviate from $m_1$.
- If his type is $t_2$ , the sender obtains a payoff of 2 by sending $m_1$ (since it is responded with $a$), but a payoff of only 0 by deviating towards $m_2$ (since it is responded with $b$). Hence, he doesn't have incentives to deviate from $m_1$.

Therefore, the initially prescribed pooling strategy profile where both types of sender select $m_1$can be sustained as a PBE of the game.

There are other babbling equilibria that differ in terms of the message sent by the sender and the receiver's beliefs in response to a message that the sender never sends (according to his strategy). For any babbling equilibrium, it must be the case that the receiver ends up choosing action $a$.

***c. Suppose the probability that the sender is type $t_1$ is p and the probability that the sender is type $t_2$ is $(\mathbf{1}-\boldsymbol{p})$. Find the values for p such that there is a pooling PBNE in which the receiver chooses action b.***

For any pooling equilibrium, the sender's strategy has him choose the same message—let it be $m_1$ – for any type and, in response to observing that message; the receiver's beliefs are her prior beliefs.

Receiver's beliefs

After observing message $m_1$,

$$\mu(t_1|m_1) = \frac{p*1}{p*1+(1-p)*1} = p$$

$$\mu(t_2|m_1) = \frac{(1-p)*1}{p*1+(1-p)*1} = 1-p$$

After receiving message $m_2$ (off-the-equilibrium),

$$\mu(t_1|m_1) = \frac{p*0}{p*0+(1-p)*0} = \frac{0}{0}$$

And hence beliefs must be arbitrarily specified, i.e. $\mu \in [0,1]$.

Receiver's optimal response

- After receiving a message $m_1$ (in equilibrium), the receiver's expected utility from responding with actions $a$, $b$, and $c$ are

$$\text{Action } a\text{:}\ \ p \times 3 + (1-p)*4 = 4-p$$

$$\text{Action } b\text{:}\ \ p \times 4 + (1-p)*1 = 1+3p$$

$$\text{Action } c\text{:}\ \ p \times 0 + (1-p)*5 = 5-5p$$

For it to be optimal to choose action b, it must be the case that

$$1+3p \geq 4-p \quad \rightarrow \quad p \geq \frac{3}{4} \quad and \quad 1+3p \geq 5-5p \quad \rightarrow \quad p \geq \frac{1}{2}$$

Thus if $p < 3/4$, then the receiver does not choose action $b$ at a pooling equilibrium, as she would prefer action $a$. If $p \geq \frac{3}{4}$, then it is the receiver's optimal strategy to choose $b$ in response to message $m_1$.

- After receiving message $m_2$ (off-the-equilibrium), the receiver's EUs from each of his three possible responses are

$$EU_{Receiver}(a|m_2) = \mu*3+(1-\mu)*4 = 4-\mu$$

$$EU_{Receiver}(b|m_2) = \mu*4+(1-\mu)*5 = 5-\mu$$

$$EU_{Receiver}(c|m_2) = \mu * 0 + (1-\mu) * 5 = 5 - 5\mu$$

Where it's clear that the EU from responding with $b$, $5-\mu$, is the highest EU payoff the responder can obtain given that $\mu \in [0,1]$.

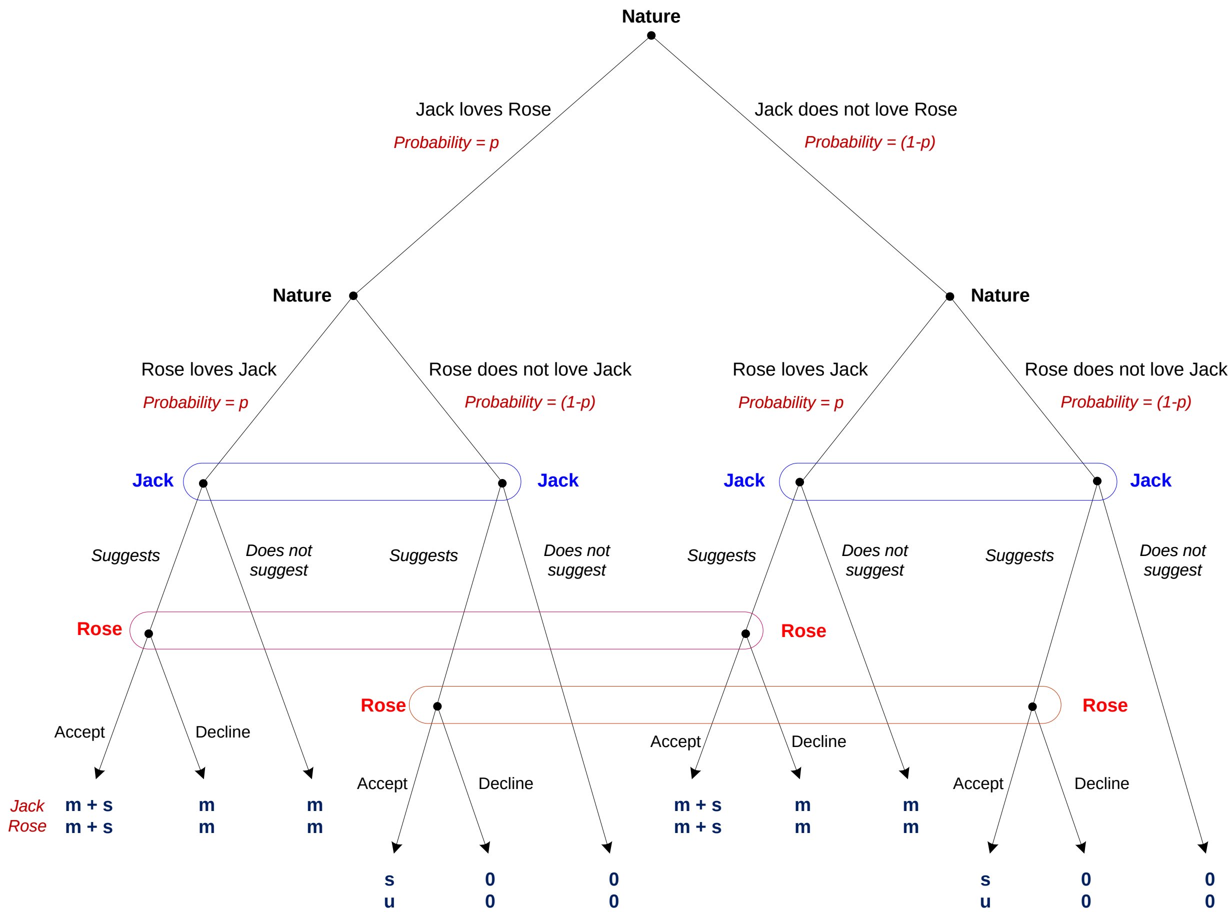
<u>Sender's optimal message</u>

- If his type is $t_1$, the sender obtains a payoff of 2 from sending $m_1$ (since it is responded with $b$), but a payoff of only 2 when deviating towards $m_2$ (since it is responded with $b$). Hence, he doesn't have incentives to deviate from $m_1$.
- If his type is $t_2$ , the sender obtains a payoff of 0 by sending $m_1$ (since it is responded with $b$), but a payoff of only 0 by deviating towards $m_2$ (since it is responded with $b$). Hence, he doesn't have incentives to deviate from $m_1$.

Therefore, the initially prescribed pooling strategy profile where both types of sender select $m_1$can be sustained as a PBE of the game when $p \geq \frac{3}{4}$.

**<u>Exercise 7-Chapter 12-Harrington</u>**

Consider the Courtship Game with Cheap Talk




***Show that there is no PBNE in which premarital sex occurs.***

Consider a strategy profile in which Rose accepts Jack's proposal when it is made. To begin, it is clear that she would never accept having sex with someone she doesn't love. Doing so results in a payoff of $u < 0$ (as she knows she isn't going to marry Jack), while not having sex results in a payoff of zero. Thus, if there is an equilibrium with premarital sex, it would only involve Rose having sex with Jack if she loves him. Suppose Rose does act in that manner, accepting if she loves Jack but declining if she does not.

***Jack.*** *What is an optimal response for Jack?* Regardless whether or not he loves Rose, his payoff is higher by having sex. Thus, he'll ask for sex; he has nothing to lose. If Rose doesn't love him, then she'll decline and his payoff is zero. If she does love him, then his payoff is higher by $s$. More specifically, if he loves Rose, then his expected payoff from asking for sex is:

$$EU_{Jack}(sex|loves\ Rose) = p \times (m+s) + (1-p) \times 0 = p(m+s)$$

and from not asking is $EU_{Jack}(not\ sex|loves\ Rose) = p * m + (1-p) * 0 = p * m$. Thus, Jack asks for sex.

***Rose.*** Is Rose's strategy of accepting if she loves Jack optimal given Jack asks regardless whether he loves her? Her payoff from accepting his proposition is:

$$EU_{Rose}(accept|loves\ Jack) = p \times (m+s) + (1-p) \times u$$

Since both Jack types ask, Rose doesn't learn anything about whether he wants to marry her from the fact that he wants to have sex with her. Recall that we are evaluating this in the case when she loves Jack. If she doesn't have sex, then her expected payoff is:

$$EU_{Rose}(reject|loves\ Jack) = p \times m + (1-p) \times 0 = pm$$

Thus, it is indeed optimal for Rose to accept if and only if:

$$p \times (m+s) + (1-p) \times u > p * m$$

which is equivalent to $ps + (1-p)u > 0$.

If this condition does not hold, then Rose would prefer to decline even if she loves Jack. In that case, there is no PBE in which premarital sex occurs.