# ECONS 424 – STRATEGY AND GAME THEORY
# MIDTERM EXAM #2 – Answer key

**Exercise #1. Hawk-Dove game.** Consider the following payoff matrix representing the Hawk-Dove game. Intuitively, Players 1 and 2 compete for a resource, each of them choosing to display an aggressive posture (hawk) or a passive attitude (dove). Assume that payoff $V > 0$ denotes the value that both players assign to the resource, and $C > 0$ is the cost of fighting, which only occurs if they are both aggressive by playing hawk in the top left-hand cell of the matrix.

| | | Player 2 | |
|---|---|---|---|
| | | *Hawk* | *Dove* |
| Player 1 | *Hawk* | $\frac{V-C}{2}, \frac{V-C}{2}$ | $V, 0$ |
| | *Dove* | $0, V$ | $\frac{V}{2}, \frac{V}{2}$ |

a) Show that if $C < V$, the game is strategically equivalent to a Prisoner's Dilemma game.

b) The Hawk-Dove game commonly assumes that the value of the resource is less than the cost of a fight, i.e., $C > V > 0$. Find the set of pure strategy Nash equilibria.

***Answer:***

**Part (a)**

- When Player 2 (in columns) chooses *Hawk* (in the left-hand column), Player 1 (in rows) receives a positive payoff of $\frac{V-C}{2}$ by paying *Hawk*, which is higher than his payoff of zero from playing *Dove*. Therefore, for Player 1 *Hawk* is a best response to Player 2 playing *Hawk*. Similarly, when Player 2 chooses *Dove* (in the right-hand column), Player 1 receives a payoff of $V$ by playing *Hawk*, which is higher than his payoff from choosing *Dove*, $\frac{V}{2}$; entailing that *Hawk* is Player 1's best response to Player 2 choosing *Dove*. Therefore, Player 1 chooses *Hawk* as his best response to all of Player 2's strategies, implying that *Hawk* is a strictly dominant strategy for Player 1.

- Symmetrically, Player 2 chooses *Hawk* in response to Player 1 playing *Hawk* (in the top row) and to

Player 1 playing *Dove* (in the bottom row), entailing that playing *Hawk* is a strictly dominant strategy for Player 2 as well.

- Since $C \leq V$ ,by assumption, the unique pure strategy Nash equilibrium is {*Hawk*, *Hawk*}, although {*Dove*, *Dove*} Pareto dominates the NE strategy as both players can improve their payoffs from $\frac{V-C}{2}$ to $\frac{V}{2}$. Since every player is choosing a strictly dominant strategy in the pure-strategy Nash equilibrium of the game, despite being Pareto dominated by another strategy profile, this game is strategically equivalent to a Prisoner's Dilemma game.

**Part (b)**

- *Finding best responses.* When Player 2 plays *Hawk* (in the left-hand column), Player 1 receives a negative payoff of $\frac{V-C}{2}$ since $C > V > 0$. Player 1 then prefers to choose *Dove* (with a payoff of zero) in response to Player 2 playing *Hawk*. However, when Player 2 plays *Dove* (in the right-hand column), Player 1 receives a payoff of $V$ by responding with *Hawk*, and half of this payoff, $\frac{V}{2}$, when responding with *Dove*. Then, Player 1 plays *Hawk* in response to Player 2 playing *Dove*.
- Symmetrically, Player 2 plays *Dove* (*Hawk*) in response to Player 1 playing *Hawk* (*Dove*) to maximize his payoff.
- *Finding pure strategy Nash equilibria.* For illustration purposes, we underline the best response payoffs in the matrix below.

| | | Player 2 | |
|---|---|---|---|
| | | *Hawk* | *Dove* |
| Player 1 | *Hawk* | $\frac{V-C}{2}, \frac{V-C}{2}$ | $\underline{V}, \underline{0}$ |
| | *Dove* | $\underline{0}, \underline{V}$ | $\frac{V}{2}, \frac{V}{2}$ |

We can therefore identify two cells where the payoffs of both players were underlined as best response payoffs: (*Hawk, Dove*) and (*Dove, Hawk*). These two strategy profiles are the two pure strategies Nash equilibria.

Our results then resemble those in the Chicken game, since every player seeks to miscoordinate by choosing the opposite strategy of his opponent.

**Exercise #2. Collusion in donations – Infinitely repeated public good game.** Consider a public good game between two players, A and B. The utility function of every player $i=\{A,B\}$ is

$$u_i(g_i) = (w - g_i)^{1/2}\left[m\left(g_i + g_j\right)\right]^{1/2}$$

where the first term, $w - g_i$, represents the utility that the player receives from money (i.e., the amount of his wealth, $w$, not contributed to the public good). The second term indicates the utility he obtains from aggregate contributions to the public good, $g_i + g_j$, where $m \geq 0$ denotes the return from aggregate contributions. For simplicity, assume that both players have equal wealth levels $w$.

a) Find every player $i$'s best response function, $g_i\left(g_j\right)$. Interpret.
b) Find the Nash Equilibrium of this game, $g^*$.
c) Find the socially optimal contribution level, $g^{SO}$, that is, the donation that each player should contribute to maximize their joint utilities. Compare it with the Nash equilibrium of the game. Which of them yields the highest utility for every player $i$?
d) Consider the infinitely repeated version of the game. Show under which values of players' discount factor $\delta$ you can sustain cooperation, where cooperation is understood as contributing the socially optimal amount (the one that maximizes joint utilities) you found in part (c). For simplicity, you can consider a "Grim-Triger strategy" where every player starts cooperating (that is, donating $g^{SO}$ to the public good), and continues to do so if both players cooperated in all previous periods. However, if one or both players deviate from $g^{SO}$, he reverts to the Nash equilibrium of the unrepeated game, $g^*$, thereafter.

***Answer:***

**Part (a).** Every individual $i$ chooses his contribution to the public good, to solve the following utility maximization problem:

$$\max_{g_i > 0} \; u_i(g_i) = (w - g_i)^{1/2}\left[m\left(g_i + g_j\right)\right]^{1/2}$$

Differentiating $u_i(g_i)$ with respect to $g_i$ gives us

$$\frac{\partial u_i(g_i)}{\partial g_i} = \frac{m\sqrt{w - g_i}}{2 \times \sqrt{m\left(g_i + g_j\right)}} - \frac{\sqrt{m\left(g_i + g_j\right)}}{2 \times \sqrt{w - g_i}} = 0$$

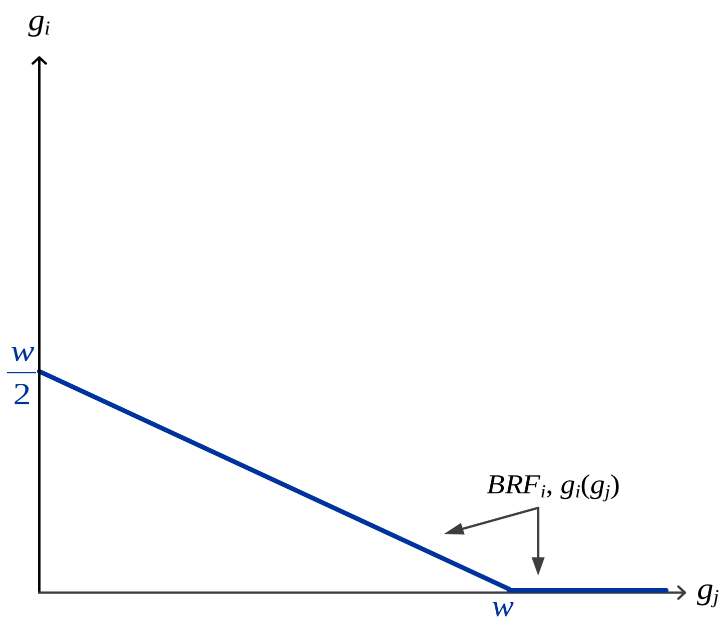
Rearranging, we obtain

$$\frac{m(w - g_i)}{2} = \frac{m(g_i + g_j)}{2}$$

Solving this equation yields a best response function of

$$g_i(g_j) = \frac{w}{2} - \frac{1}{2} g_j \qquad (BRF_i)$$

The figure below depicts this best response function. It originates at $\frac{w}{2}$, indicating that player *i* contributes half of his wealth towards the public good when his opponent does not contribute anything, $g_j = 0$; see vertical intercept on the left-hand side of the figure. However, when player *j* makes a positive donation, player *i* reduces his own. Specifically, for every additional dollar that player *j* contributes, player *i* decreases his donation by half a dollar. Finally, when player *j* donates $g_j = w$, player *i*'s contribution collapses to zero. Graphically, $g_j = w$ is the horizontal intercept of the best response function for player *i*. Intuitively, when player *j* contributes all his wealth to the public good, player *i* does not need to donate anything.

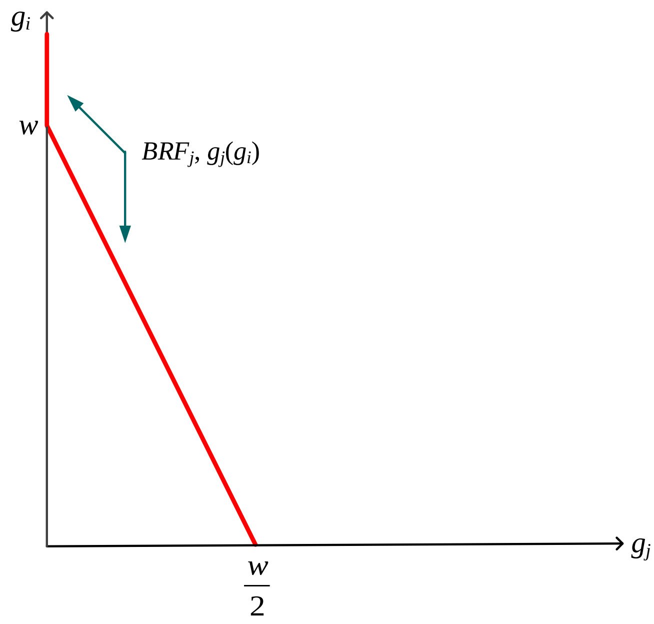



Players are symmetric since they have the same wealth level and face the same utility function. Therefore, player *j*'s best response function is analogous to that of player *i*, and can be expressed as follows (note that we only switch the subscripts relative to $BRF_i$)

$$g_j(g_i) = \frac{w}{2} - \frac{1}{2} g_i \qquad (BRF_j)$$

where we only switch the subscripts relative to $BRF_i$. A similar intuition as that above applies to this best response function. The following figure depicts this best response function, using the same axis as for

*BRFᵢ*, which will help us superimpose both best response functions on the same figure in the next part of the exercise.



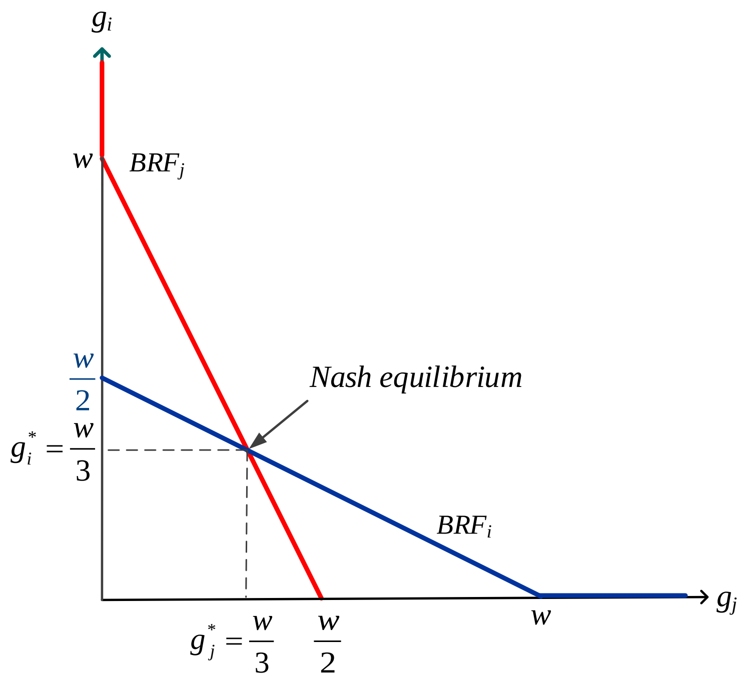

**Part (b).** In a symmetric equilibrium, both players contribute the same amount to the public good, which entails that $g_i = g_j = g$. Substituting this result in the best response function of either player yields

$$g = \frac{w}{2} - \frac{1}{2} g$$

Solving this for $g$ gives us the equilibrium contribution, as follows,

$$g^* = \frac{w}{3}$$

which represents the donation that every player *i* and *j* submit in the Nash equilibrium of the game. The next figure superimposes both players' best response functions, illustrating that the Nash equilibrium occurs at the point where both best response functions cross each other (that is, representing a mutual best response).




**Part (c).** The socially optimal donation maximizes players' joint utility, as follows

$$\max_{g_i, g_j > 0} \; u_i(g_i, g_j) + \; u_j(g_i, g_j) = (w - g_i)^{1/2}\left[m(g_i + g_j)\right]^{1/2} + (w - g_j)^{1/2}\left[m(g_i + g_j)\right]^{1/2}$$

Differentiating with respect to $g_i$, we get,

$$\frac{\partial u_i(g_i, g_j)}{\partial g_i} = -\frac{\sqrt{m(g_i + g_j)}}{2 \times \sqrt{w - g_i}} + \frac{m\sqrt{w - g_i}}{2 \times \sqrt{m(g_i + g_j)}} + \frac{m\sqrt{w - g_j}}{2 \times \sqrt{m(g_i + g_j)}} = 0$$

Invoking symmetry, $g_i = g_j = g$, the above expression simplifies to

$$\frac{m\left(\sqrt{w - g} + \sqrt{w - g}\right)}{\sqrt{2mg}} = \frac{\sqrt{2mg}}{\sqrt{w - g}}$$

Rearranging, we find

$$m(w - g) = mg$$

Finally, solving for $g$, yields a socially optimal donation of

$$g_i^{SO} = g_j^{SO} = \frac{w}{2}$$

which is larger than that emerging in the Nash equilibrium of the game, $g^* = \frac{w}{3}$.

***Intuition:*** Players free-ride off each other's donations in the Nash equilibrium of the game, ultimately contributing a suboptimal amount to the public good. Alternatively, when every donor independently chooses his donation he ignores the positive benefit that the other donor enjoys (Recall that contributions are non-rival, thus providing the same benefit to every individual, whether he made that donation or not!) In contrast, when every player considers the benefit that his donations have on the other donor's utility (as we did in the joint utility maximization problem), he contributes a larger amount to the public good.

*Comparing utility levels.* When every player chooses the socially optimal donation $g_i^{SO} = g_j^{SO} = \frac{w}{2}$, player *i*'s utility becomes

$$\left(w - \frac{w}{2}\right)^{1/2} \left[m\left(\frac{w}{2} + \frac{w}{2}\right)\right]^{1/2} = \frac{w}{\sqrt{2}}\sqrt{m}$$

while when every player chooses the donation in the Nash equilibrium of the game, $g^* = \frac{w}{3}$, player *i*'s utility becomes

$$(w - g^*)^{1/2}[m(g^* + g^*)]^{1/2} = \left(w - \frac{w}{3}\right)^{1/2} \left[m\left(\frac{w}{3} + \frac{w}{3}\right)\right]^{1/2}$$

$$= \frac{2w}{3}\sqrt{m}$$

Comparing his utility when players coordinate choosing the socially optimal donation, $\frac{w}{\sqrt{2}}\sqrt{m}$, and his utility in the Nash equilibrium of the game, $\frac{2w}{3}\sqrt{m}$, we find that their difference

$$\frac{w}{\sqrt{2}}\sqrt{m} - \frac{2w}{3}\sqrt{m} = \frac{3\sqrt{2} - 4}{6} w\sqrt{m}$$

is positive since ratio $\frac{3\sqrt{2}-4}{6} \cong 0.04$. In words, every player obtains a higher utility when they coordinate their donations than when they independently choose their contributions to the public good.

**Part (d).** *Cooperation.* As shown in part (c) of the exercise, when players coordinate their donations to the public good, they donate $g_i^{SO} = g_j^{SO} = \frac{w}{2}$, which yields a per-period cooperating payoff of

$$\left(w - \frac{w}{2}\right)^{1/2} \left[m\left(\frac{w}{2} + \frac{w}{2}\right)\right]^{1/2} = \frac{w}{\sqrt{2}}\sqrt{m}$$

thus yielding a discounted stream of payoffs from cooperation of

$$\frac{w\sqrt{m}}{\sqrt{2}} + \delta \frac{w\sqrt{m}}{\sqrt{2}} + \delta^2 \frac{w\sqrt{m}}{\sqrt{2}} \dots . = \frac{1}{1-\delta} \frac{w\sqrt{m}}{\sqrt{2}}$$

*Optimal deviation.* When player *i* deviates from the cooperative donation $g_i^{SO} = \frac{w}{2}$, while his opponent still cooperates by choosing $g_j^{SO} = \frac{w}{2}$, his optimal deviation is found by inserting $g_j^{SO} = \frac{w}{2}$ into player *i*'s best response function, $g_i(g_j) = \frac{w}{2} - \frac{1}{2}g_j$, as follows

$$g_i\left(\frac{w}{2}\right) = \frac{w}{2} - \frac{1}{2} \cdot \frac{w}{2} = \frac{w}{4}$$

which yields a deviating utility level for player *i* of

$$\left(w - \frac{w}{4}\right)^{1/2} \left[m\left(\frac{w}{4} + \frac{w}{2}\right)\right]^{1/2} = \frac{3w}{4}\sqrt{m}$$

This can be understood as his "deviating payoff" which he enjoys during only one period.

*Punishment.* After player *i* deviates from the cooperative donation $g_i^{SO} = \frac{w}{2}$, his deviation is detected, and punished thereafter by both players, who choose the Nash equilibrium donation in the unrepeated version of the game, that is, $g^* = \frac{w}{3}$, yielding a per-period payoff of

$$(w - g^*)^{1/2}[m(g^* + g^*)]^{1/2} = \left(w - \frac{w}{3}\right)^{1/2} \left[m\left(\frac{w}{3} + \frac{w}{3}\right)\right]^{1/2}$$

$$= \frac{2w}{3}\sqrt{m}$$

Therefore, the stream of discounted payoffs that every player receives after player *i*'s deviation is detected becomes

$$\delta\left(\frac{2w}{3}\sqrt{m}\right) + \delta^2\left(\frac{2w}{3}\sqrt{m}\right) + \ldots = \frac{\delta}{1-\delta} \cdot \left(\frac{2w}{3}\sqrt{m}\right)$$

*Comparing stream of payoffs.* We can now compare the discounted stream of payoffs from cooperation against that from defecting, finding that every player cooperates after a history of cooperation by both players if and only if

$$\frac{1}{1-\delta}\frac{w\sqrt{m}}{\sqrt{2}} \geq \frac{3w}{4}\sqrt{m} + \frac{\delta}{(1-\delta)} \cdot \left(\frac{2w}{3}\sqrt{m}\right)$$

In words, the left-hand side of the inequality represents player *i*'s discounted stream of payoffs from cooperating. The first term in the right-hand side captures the payoff that player *i* enjoys from deviating (which only lasts one period, since his deviation is immediately detected), while the second term reflects the discounted stream of payoffs from being punished in all subsequent periods after his deviation. Multiplying both sides of the inequality by $1 - \delta$, we obtain

$$\frac{w\sqrt{m}}{\sqrt{2}} \geq (1-\delta) \cdot \frac{3w}{4}\sqrt{m} + \delta\left(\frac{2w}{3}\sqrt{m}\right)$$

And solving for discount factor $\delta$, we find that cooperation can be sustained in the Subgame Perfect Nash

Equilibrium (SPNE) of the gam if

$$\delta \geq 9 - 6\sqrt{2} \approx 0.51$$

Intuitively, players must care enough about their future payoffs to sustain their cooperation in the public good game.

**Exercise #3 – Cartel with two asymmetric firms.** Consider two firms competing a la Cournot and facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Firm 1 has more experience in the industry than firm 2, which is reflected in the fact that firm 1's marginal cost of production is $c_1 = 10$ while that of firm 2 is $c_2 = 16$.

a) *Best response function.* Set up each firm's profit-maximization problem and find its best response function. Interpret.

b) *Cournot competition.* Find the equilibrium output each firm produces when competing a la Cournot (that is, when they simultaneously and independently choose their output levels). In addition, find the profits that each firm earns in equilibrium. [*Hint*: You cannot invoke symmetry when solving for equilibrium output levels in this exercise since firms are not cost symmetric.]

c) *Collusive agreement.* Assume now that firms collude to increase their profits. Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the output level that each firm should select to maximize joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

d) *Profit comparison.* Compare the profits that firms obtain when competing a la Cournot (from part b) against their profits when they successfully collude (from part c).

***Answer:***

**Part (a).** Firm 1's profit function is given by:

$$\pi_1(q_1, q_2) = [100 - (q_1 + q_2)]q_1 - 10q_1 \qquad (1)$$

Differentiating with respect to $q_1$, yields

$$100 - 2q_1 - q_2 - 10 = 0$$

Solving for $q_1$, we obtain firm 1's best response function

$$q_1(q_2) = 45 - \frac{1}{2}q_2 \qquad (BRF_1)$$

which originates at 45 and decreases in its rival's output ($q_2$) at a rate of 1/2.

Similarly, firm 2's profit function is given by:

$$\pi_2(q_1, q_2) = [100 - (q_1 + q_2)]q_2 - 16q_2 \qquad (2)$$

Differentiating with respect to $q_2$ yields

$$100 - q_1 - 2q_2 - 16 = 0$$

Solving for $q_2$, we find firm 2's best response function

$$q_2(q_1) = 42 - \frac{1}{2}q_1 \qquad (BRF_2)$$

which originates at 42 and decreases in its rival's output ($q_1$) at a rate of 1/2. Comparing the best response functions of both firms, we find that firm 1 produces a larger output than firm 2 for a given production level of its rival. Intuitively, firm 1 enjoys a cost advantage relative to firm 2, which entails a larger output level for each of its rival's output decisions.

**Part (b).** Inserting firm 2's best response function into firm 1's, we obtain

$$q_1 = 45 - \frac{1}{2}\left(42 - \frac{1}{2}q_1\right)$$

which simplifies to

$$q_1 - \frac{1}{4}q_1 = 45 - 21$$

Solving for firm 1's output, we find $q_1^* = 32$ units. Substituting this output level in firm 2's best response function, we obtain

$$q_2(32) = 42 - \frac{1}{2}32 = 26$$

implying that firm 2 produces $q_2^* = 26$ units in equilibrium. Inserting these equilibrium output levels into each firm's profit function, yields

$$\pi_1(32{,}26) = [100 - (32 + 26)]32 - (10 * 32) = \$1{,}024$$

and

$$\pi_2(32{,}26) = [100 - (32 + 26)]26 - (16 * 26) = \$676$$

The sum of these profits is then $\$1{,}024 + \$676 = \$1700$.

**Part (c).** In this case, firms maximize their joint profits, that is, the sum $\pi_1(q_1, q_2) + \pi_2(q_1, q_2)$, which we can expand as follows

$$\pi_1 + \pi_2 = \{[100 - (q_1 + q_2)]q_1 - 10q_1\} + \{[100 - (q_1 + q_2)]q_2 - 16q_2\}$$

Differentiating joint profits with respect to $q_1$, yields

$$100 - 2q_1 - q_2 - 10 - q_2 = 0$$

Solving for $q_1$, we obtain

$$q_1(q_2) = 45 - q_2$$

Similarly, we now differentiate joint profits with respect to $q_2$, yielding

$$-q_1 + 100 - q_1 - 2q_2 - 16 = 0$$

Solving for $q_2$, we find

$$q_2(q_1) = 42 - q_1$$

Depicting functions $q_1(q_2) = 45 - q_2$ and $q_2(q_1) = 42 - q_1$, we can see that they are parallel, and cross on the axes (at a corner solution), where $q_2 = 0$ and $q_1(0) = 45 - 0 = 45$ units. Intuitively, when firms seek to maximize their joint profits, they assign all output to the firm with the lowest marginal cost (firm 1 in this case), and no output to the firm with the highest marginal cost (that is, the company with a cost disadvantage produces zero units). This leads to joint profits of

$$\pi_1 + \pi_2 = \{[100 - (45 + 0)]45 - (10 * 45)\} + \{[100 - (45 + 0)]0 - (16 * 0)\}$$
$$= 2{,}025 + 0 = \$2{,}025.$$

**Part (d).** The combined profits when firms collude ($2025) exceed those under Cournot competition ($1700).

**Exercise #4. Cartel with firms competing a la Bertrand – Unsustainable in a one-shot game.** Consider two firms competing in prices (a la Bertrand), and facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. For simplicity, assume that firms face a common marginal cost of production $c = 10$.

a) *Bertrand competition.* Find the equilibrium price that each firm sets when competing a la Bertrand (that is, when they simultaneously and independently set their prices). In addition, find the profits that each firm earns in equilibrium.

b) *Collusive agreement.* Assume now that the CEOs from both companies meet for lunch and start talking about how they could increase their profits if they could coordinate their price setting decisions. Beware, firms are trying to collude! Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the price that each firm selects when maximizing joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

c) *Profit comparison.* Compare the profits that firms obtain when competing a la Bertrand (from part a) against their profits when they successfully collude (from part b).

d) *Unsustainable cartel.* Show that the collusive agreement from part (b) cannot be sustained as a Nash equilibrium of the one-shot game, that is, when firms interact only once. Interpret.

***Answer:***

**Part (a).** In the Bertrand competition, the equilibrium price equals to marginal costs, that is,

$$p = MC = \$10.$$

Inserting this equilibrium price of $p = \$10$ into the inverse demand function, $p(Q) = 100 - Q$, we find the aggregate output, as follows

$$10 = 100 - Q$$

which, solving for $Q$, yields $Q = 90$ units. Since both firms are symmetric in costs, each firm produces half of this aggregate output, $q_1 = q_2 = 45$ units. Finally, the profits that each firm earns in equilibrium are

$$\pi_i = (p - MC)q_i = (10 - 10) * 45 = 0.$$

**Part (b).** In a collusive agreement, firms maximize their joint profits by choosing price.

$$\max_{p} \quad \pi_1 + \pi_2 = (p - 10)q_1 + (p - 10)q_2$$

$$= (p - 10)(q_1 + q_2) = (p - 10)Q = (P - 10)(100 - P)$$

Differentiating with respect to $p$, yields

$$100 - 2p + 10 = 0$$

Solving for $p$, we find a collusive price of $p = \$55$.

Inserting this collusive price of $p = \$55$ into the inverse demand function, $p(Q) = 100 - Q$, we find the aggregate output, as follows

$$55 = 100 - Q$$

which, solving for $Q$, yields

$$Q = 100 - 55 = 45 \text{ units.}$$

Since both firms are symmetric in costs, each firm produces half of this aggregate output, $q_1 = q_2 = 22.5$ units. Finally, the profits that each firm earns under collusion are

$$\pi_i = (p - MC)q_i = (55 - 10) * 22.5 = \$1{,}012.5.$$

**Part (c).** While they earn no profit in Bertrand competition, each firm obtains \$1,012.5 when firms collude, implying that firms are better off cooperating.

**Part (d).** In the one-shot game, firms cannot use future punishments to deter deviations from the collusive agreement where every firm produces $q_1 = q_2 = 22.5$ units. In this setting, collusion cannot be sustained in equilibrium since every firm has incentives to charge a price marginally lower than the cartel $p < \$55$ to capture all the market and make a larger profit than in the collusive cartel.

For instance, if firm 1 undercuts the price to $p_1 = \$54$, it becomes the firm with the lowest price, capturing all the market, with sales $Q = 100 - 54 = 46$ units, and earning a deviating profit

$$\pi_1 = (54 - 10) * 46 = \$2{,}024,$$

which exceeds the collusive profit we found in part (c), \$1,012.5.

**Exercise #5. Temporary punishments from deviation.** Consider two firms competing in quantities (a la Cournot), facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. For simplicity, assume that firms face a common marginal cost of production $c = 10$.

a) *Unrepeated game.* Find the equilibrium output each firm produces when competing a la Cournot (that is, when they simultaneously and independently choose their output levels) in the unrepeated version of the game (that is, when firms interact only once). In addition, find the profits that each firm earns in equilibrium.

b) *Repeated game* - *Collusion.* Assume now that the CEOs from both companies meet to discuss a collusive agreement that would increase their profits. Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the output level that each firm should select to maximize joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

c) *Repeated game – Permanent punishment.* Consider a grim-trigger strategy in which every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. Otherwise, every firm deviates to the Cournot equilibrium thereafter (that is, every firm produces the Nash equilibrium of the unrepeated game found in part a forever). In words, this says that the punishment of deviating from the collusive agreement is *permanent*, since firms never return to the collusive outcome. For which discount factors this grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

d) *Repeated game – Temporary punishment.* Consider now a “modified” grim-trigger strategy. Like in the grim-trigger strategy of part (c), every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. However, if a deviation is detected by either firm, every firm deviates to the Cournot equilibrium during only 1 period, and then every firm returns to cooperation (producing the collusive output). Intuitively, this implies that the punishment of deviating from the collusive agreement is now *temporary* (rather than permanent) since it lasts only one period. For which discount factors this “modified” grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

e) Consider again the temporary punishment in part (d) but assume now that it lasts for two periods. How are your results from part (d) affected? Interpret.

f) Consider again the temporary punishment in part (d) but assume now that it lasts for three periods. How are your results from part (d) affected? Interpret.

***Answer:***

**Part (a).** Firm $i$'s profit function is given by:

$$\pi_i(q_i, q_j) = \left[100 - \left(q_i + q_j\right)\right]q_i - 10q_i \qquad (7)$$

Differentiating with respect to output $q_i$ yields,

$$100 - 2q_i - q_i - 10 = 0$$

Solving for $q_i$, we find firm *i*'s best response function

$$q_i\left(q_j\right) = 45 - \frac{q_j}{2} \qquad (\text{BRF}_7)$$

Since this is a symmetric game, firm *j*'s best response function is symmetric. Therefore, in a symmetric equilibrium both firms produce the same output level, $q_i = q_j$, which helps us rewrite the above best response function as follows

$$q_i = 45 - \frac{q_i}{2}$$

Solving this expression yields equilibrium output of $q_i^* = q_1^* = q_2^* = 30$. Substituting these results into profits $\pi_1(q_1, q_2)$ and $\pi_2(q_1, q_2)$, we obtain that

$$\pi_i\left(q_i^*, q_j^*\right) = \pi_i(30{,}30) = [100 - (30 + 30)]30 - 10 \times 30 = \$900$$

where firm's equilibrium profits coincide, that is, $\pi_i\left(q_i^*, q_j^*\right) = \pi_1(q_1^*, q_2^*) = \pi_2(q_1^*, q_2^*)$.

In summary, equilibrium profit is \$900 for each firm and combined profits are \$900+\$900=\$1800.

**Part (b).** In this case, firms maximize the sum $\pi_1(q_1, q_2) + \pi_2(q_1, q_2)$, as follows

$$\left\{\left[100 - \left(q_i + q_j\right)\right]q_i - 10q_i\right\} + \left\{\left[100 - \left(q_i + q_j\right)\right]q_j - 10q_j\right\} \qquad (8)$$

Differentiating with respect to output $q_i$, yields

$$100 - 2q_i - q_j - 10 - q_j = 0$$

Solving for $q_i$, we obtain

$$q_i\left(q_j\right) = 45 - q_j$$

and similarly when we differentiate with respect to $q_j$. In a symmetric equilibrium both firms produce the same output level, $q_i = q_j$, which helps us rewrite the above expression as follows

$$q_i = 45 - q_i$$

Solving for output $q_i$ we obtain equilibrium output $q_i^* = q_1^* = q_2^* = \frac{45}{2} = 22.5$ units. This yields each firm a profit of

$$\pi_i\left(q_i^*, q_j^*\right) = \pi_i(22.5{,}22.5)$$

$$= \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\} + \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\}$$

$$= \$1012.5$$

Thus, each firm makes a profit of \$1,012.5, which yields a joint profit of \$2,025.

**Part (c).** For this part of the exercise, let us first list the payoffs that the firm can obtain from each of its output decisions:

- Cooperation yields a payoff of \$1,012.5 for firm $i$.

  While firm $j$ cooperates $\left(q_j = 22.5\right)$, firm $i$'s optimal deviation is found by inserting output $q_j = 22.5$ into firm $i$'s best response function

  $$q_i(22.5) = 45 - \frac{22.5}{2} = 33.75$$

  which yields a deviating profit for firm $i$ of

  $$\pi_i(33.75{,}22.5) = \{[100 - (33.75 + 22.5)]33.75 - (10 \times 33.75)\} = \$1{,}139.06$$

  However, such defection is punished with Cournot competition in all subsequent periods, which yields a profit of only \$900.

Therefore, firm *i* cooperates as long as

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + \cdots \geq 1139.06 + 900\delta + 900\delta^2 + \cdots$$

which can be simplified to

$$1012.5(1 + \delta + \delta^2 + \cdots) \geq 1139.06 + 900\delta(1 + \delta + \delta^2 + \cdots)$$

and expressed more compactly as

$$\frac{1012.5}{1-\delta} \geq 1139.06 + \frac{900\delta}{1-\delta}$$

Multiplying both sides of the inequality by $1-\delta$, yields

$$1012.5 \geq 1139.06(1-\delta) + 900$$

which rearranging and solving for discount factor $\delta$ entails

$$\delta \geq 0.901$$

Thus, for cooperation to be sustainable in an infinitely repeated game with permanent punishments, firms' discount factor $\delta$ has to be at least 0.901. In words, firms must put a sufficient weight on future payoffs.

**Part (d).** The setup is analogous to that in part (c) of the exercise, but we now write that cooperation is possible if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \ldots . \geq 1139.06 + 900\delta + 1012.5\delta^2 \ldots .$$

Importantly, note that payoffs after the punishment period (in this case, a one period punishment of Cournot competition with \$900 profits) returns to cooperation, explaining that all payoffs in the third term of the left-hand and right-hand side of the inequality coincide. We can therefore cancel them out, simplifying the above inequality to

$$1012.5 + 1012.5\delta \geq 1139.06 + 900\delta$$

Rearranging yields

$$112.5\delta \geq 126.56$$
$$\Rightarrow \delta \geq 1.125$$

ultimately simplifying to $\delta \geq 1.125$. That is, when deviations are only punished during one period cooperation cannot be sustained since discount factor $\delta$ must satisfy $\delta \in [0,1]$ by assumption.

**Part (e).** Following a similar approach as in part (d), we write that cooperation can be sustained if and only if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 \dots . \geq 1139.06 + 900\delta + +900\delta^2 + 1012.5\delta^3 \dots.$$

The stream of payoffs after the punishment period (in this case, a two-period punishment of Cournot competition with $900 profits) returns to cooperation. Hence, the payoffs in both the left- and right-hand side of the inequality after the punishment period coincide and cancel out. The above inequality then becomes

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \geq 1139.06 + 900\delta + 900\delta^2$$

which, after rearranging, yields

$$\delta^2 + \delta - 1.125 \geq 0$$

ultimately simplifying to $\delta \geq 0.67$. In words, when deviations are punished for two periods, cooperation can be sustained if firms' discount factor $\delta$ is at least 0.67.

**Part (f).** Following a similar approach as in part (e), we write that cooperation can be sustained if and only if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 + 1012.5\delta^4 \dots .$$
$$\geq 1139.06 + 900\delta + 900\delta^2 + 900\delta^3 + 1012.5\delta^4 \dots.$$

The stream of payoffs after the punishment period (in this case, a three-period punishment of Cournot competition with $900 profits) returns to cooperation. Hence, the payoffs in both the left- and right-hand side of the inequality after the punishment period coincide and cancel out. The above inequality then becomes

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 \geq 1139.06 + 900\delta + 900\delta^2 + 900\delta^3$$

which, after rearranging, yields

$$\delta^3 + \delta^2 + \delta - 1.125 \geq 0$$

ultimately simplifying to $\delta \geq 0.58$. Therefore, when deviations are punished for two periods, cooperation can be sustained if firms' discount factor $\delta$ is at least 0.58.

**Exercise #6:**

Exercise 12 from Chapter 10 in Harrington's book (new edition).

**Answer:** The unique symmetric BNE is $s_i$=2$w_i$; that is, every student $i$ submits a number equal to twice what is in his wallet. We will not prove uniqueness but will show that it is a BNE. Furthermore, you'll see that it does not depend on the possible values that $w_i$ can take or the probabilities that Nature assigns to those values. Suppose that student 2 uses this strategy, so that $s_2$=2$w_2$, and student 1 submits $s_1$. If student 1 wins, his payoff is

$$w_1 + w_2 – 2s_2 = w_1 + w_2 - 2w_2 = w_1 - w_2$$

which is positive only when $w_1 > w_2$. (Note that student 1's submission does not affect his net payment when he loses; it only affects it when he wins.) Thus, student 1 would like to submit a value for $s_1$ so that he wins if and only if he has more money in his wallet than student 2. Given that $s_2 = 2w_2$, student 1 can do so by setting $s_1 = 2w_1$ as then student 1 wins ($s_1 > s_2$) if and only if $2w_1 > 2w_2$ or $w_1 > w_2$.

This exercise based on the article by Paul Klemperer, "Auctions with Almost Common Values: The 'Wallet Game' and Its Applications," *European Economic Review*, 42 (1998), 757–769.

## BONUS EXERCISE

**Public good game with incomplete information**. Consider a public good game between two players, A and B. The utility function of every player $i$ is

$$u_i(g_i) = (w_i - g_i)^{1/2}\left[m\left(g_i + g_j\right)\right]^{1/2}$$

where the first term, $w_i - g_i$, represents the utility that the player receives from money (i.e., the amount of his wealth not contributed to the public good which he can dedicate to private uses). The second term indicates the utility he obtains from aggregate contributions to the public good, $g_i + g_j$, where $m \geq 0$ denotes the return from aggregate contributions. Since public goods are non-rival in consumption, player $i$'s utility from this good originates from both his own donations, $g_i$, and in those of player $j$, $g_j$.

Consider that the wealth of player A, $w_A > 0$, is common knowledge among the players; but that of player B, $w_B$, is privately observed by player B but not observed by player A. However, player A knows that player B's wealth level $w_B$ can be high $(w_B^H)$ or low $(w_B^L)$ with equal probabilities, where $w_B^H > w_B^L > 0$. Let us next find the Bayesian Nash Equilibrium (BNE) of this public good game where player A is uninformed about the exact realization of parameter $w_B$.

a) Starting with the privately informed player B, set up his utility maximization problem, and separately find his best response function, first, when $w_B = w_B^H$ and, second, when $w_B = w_B^L$.

b) Find the best response function of the uninformed player A.

c) Using your results in parts (a) and (b), find the equilibrium contributions to the public good by each player. For simplicity, you can assume that player A's wealth is $w_A = \$14$, and those of player B are $w_B^H = \$20$ when his wealth is high, and $w_B^L = \$10$ when his wealth is low. [*Hint*: You will find one equilibrium contribution for player A, but two equilibrium contributions for player B as his contribution is dependent on his wealth level $w_B$]

***Answer:***

**Part (a).** *Player B with high wealth.* When $w_B = w_B^H$, player B chooses his contribution to the public good $g_B^H \geq 0$ to solve the following utility maximization problem:

$$\max_{g_B^H \geq 0} \quad u_B(g_B^H) = (w_B^H - g_B^H)^{1/2}[m(g_A + g_B^H)]^{1/2}$$

where the first term represents his utility from private uses (that is, the amount of wealth not contributed to the public good), while the second term reflects his utility from total donations to the public good from him and player A.

Differentiating with respect to $g_B^H$, yields

$$\frac{\partial u_B(g_B^H)}{\partial g_B^H} = -\frac{\sqrt{m}}{2}\sqrt{\frac{g_A + g_B^H}{w_B^H - g_B^H}} + \frac{\sqrt{m}}{2}\sqrt{\frac{w_B^H - g_B^H}{g_A + g_B^H}}$$

Assuming interior solutions $(g_B^H > 0)$, we set the above first order condition equal to zero, which helps us rearrange the above expression as follows

$$g_A + g_B^H = w_B^H - g_B^H$$

Solving for player B's contribution to the public good, $g_B^H$, yields the following best response function,

$$g_B^H(g_A) = \frac{w_B^H}{2} - \frac{1}{2}g_A$$

which originates at half of his wealth, $\frac{w_B^H}{2}$, and decreases in player A's donation by 1/2. In words, when player A does not contribute to the public good ($g_A = 0$), player B donates half of his wealth, $\frac{w_B^H}{2}$, but for every dollar that player A contributes to the public good, player B reduces his donation by 50 cents.

*Player B with low wealth.* When $w_B = w_B^L$, player B chooses his contribution to the public good $g_B^L \geq 0$ to solve the following utility maximization problem

$$\max_{g_B^L \geq 0} \quad u_B(g_B^L) = (w_B^L - g_B^L)^{1/2}[m(g_A + g_B^L)]^{1/2}$$

Differentiating with respect to $g_B^L$, yields

$$\frac{\partial u_B(g_B^L)}{\partial g_B^L} = -\frac{\sqrt{m}}{2}\sqrt{\frac{g_A + g_B^L}{w_B^L - g_B^L}} + \frac{\sqrt{m}}{2}\sqrt{\frac{w_B^L - g_B^L}{g_A + g_B^L}}$$

Assuming interior solutions $(g_B^L > 0)$, we set the above first order condition equal to zero, which helps us simplify the expression as follows

$$g_A + g_B^L = w_B^L - g_B^L$$

After solving for player B's contribution to the public good, $g_B^L$, we find the following best response function

$$g_B^L(g_A) = \frac{w_B^L}{2} - \frac{1}{2}g_A$$

which originates at half of his wealth, $\frac{w_B^L}{2}$, and decreases in player A's contribution to the public good by 1/2.

**Part (b).** Player A chooses his contribution to the public good $g_A \geq 0$ to solve the following expected utility maximization problem,

$$\max_{g_A \geq 0} \quad EU_A(g_A) = \frac{1}{2}(w_A - g_A)^{1/2}[m(g_A + g_B^L)]^{1/2} + \frac{1}{2}(w_A - g_A)^{1/2}[m(g_A + g_B^H)]^{1/2}$$

since he does not observe player B's wealth level, he does not know whether player B contributes $g_B^L$ (which occurs when his wealth is low) or $g_B^H$ (which happens when his wealth is high).
Differentiating with respect to $g_A$, yields

$$\frac{\partial u_A(g_A)}{\partial g_A} = \frac{\sqrt{m}}{4}\left[\sqrt{\frac{w_A - g_A}{g_A + g_B^L}} + \sqrt{\frac{w_A - g_A}{g_A + g_B^H}} - \sqrt{\frac{g_A + g_B^L}{w_A - g_A}} - \sqrt{\frac{g_A + g_B^H}{w_A - g_A}}\right]$$

Assuming interior solutions $(g_A > 0)$, we set the above first order condition equal to zero. Rearranging, we find that player A's best response function, $g_A(g_B^L, g_B^H)$, is implicitly defined by the following expression

$$\sqrt{\frac{g_A + g_B^L}{w_A - g_A}} + \sqrt{\frac{g_A + g_B^H}{w_A - g_A}} = \sqrt{\frac{w_A - g_A}{g_A + g_B^L}} + \sqrt{\frac{w_A - g_A}{g_A + g_B^H}}$$

**Part (c)**. At a Bayesian Nash equilibrium (BNE), the contribution of each player constitutes a best response to one another (mutual best response). Hence, we can substitute the best responses of the two types of player B into the best response of player A, to obtain

$$\sqrt{\frac{g_A + \frac{w_B^L - g_A}{2}}{w_A - g_A}} + \sqrt{\frac{g_A + \frac{w_B^H - g_A}{2}}{w_A - g_A}} = \sqrt{\frac{w_A - g_A}{g_A + \frac{w_B^L - g_A}{2}}} + \sqrt{\frac{w_A - g_A}{g_A + \frac{w_B^H - g_A}{2}}}$$

While the expression is rather large, you can notice that it now depends on player A's contribution to the public good, $g_A$, alone. We can then start to simplify the above expression, obtaining

$$\sqrt{g_A + w_B^L} + \sqrt{g_A + w_B^H} = 2(w_A - g_A)\left(\frac{1}{\sqrt{g_A + w_B^L}} + \frac{1}{\sqrt{g_A + w_B^H}}\right)$$

which is further simplified to

$$\sqrt{(g_A + w_B^L)(g_A + w_B^H)} = 2(w_A - g_A)$$

Squaring both sides yields

$$(g_A + w_B^L)(g_A + w_B^H) = 4(w_A - g_A)^2$$
$$\Rightarrow 3g_A^2 - (8w_A + w_B^L + w_B^H)g_A + 4w_A^2 - w_B^L w_B^H = 0$$

Substituting wealth levels $w_A = \$14$, $w_B^H = \$20$, and $w_B^L = \$10$ into the above quadratic equation, we obtain

$$3g_A^2 - 142g_A + 584 = 0$$

The quadratic equation returns two roots, namely $g_A = 4.55$ or $g_A = 42.78$. However, since player A's contribution to public good cannot exceed his wealth, $w_A = \$14$, we take $g_A^* = 4.55$ as the only feasible solution. Therefore, the contribution made by player 2 when his wealth is high or low are

$$g_B^{H*} = \frac{20 - 4.55}{2} = 7.73$$

and

$$g_B^{L*} = \frac{14 - 4.55}{2} = 4.73$$

As a result, the Bayesian Nash equilibrium (BNE) of this public good game is the triplet of contributions

$$\{g_A^*, g_B^{H*}, g_B^{L*}\} = \{4.55, 7.73, 4.73\}$$

**BONUS EXERCISE**

**Public good game with incomplete information**. Consider a public good game between two players, A and B. The utility function of every player *i* is

$$u_i(g_i) = (w_i - g_i)^{1/2}\left[m\left(g_i + g_j\right)\right]^{1/2}$$

where the first term, $w_i - g_i$, represents the utility that the player receives from money (i.e., the amount of his wealth not contributed to the public good which he can dedicate to private uses). The second term indicates the utility he obtains from aggregate contributions to the public good, $g_i + g_j$, where $m \geq 0$ denotes the return from aggregate contributions. Since public goods are non-rival in consumption, player *i*'s utility from this good originates from both his own donations, $g_i$, and in those of player *j*, $g_j$.

Consider that the wealth of player A, $w_A > 0$, is common knowledge among the players; but that of player B, $w_B$, is privately observed by player B but not observed by player A. However, player A knows that player B's wealth level $w_B$ can be high $(w_B^H)$ or low $(w_B^L)$ with equal probabilities, where $w_B^H > w_B^L > 0$. Let us next find the Bayesian Nash Equilibrium (BNE) of this public good game where player A is uninformed about the exact realization of parameter $w_B$.

a) Starting with the privately informed player B, set up his utility maximization problem, and separately find his best response function, first, when $w_B = w_B^H$ and, second, when $w_B = w_B^L$.
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c) Using your results in parts (a) and (b), find the equilibrium contributions to the public good by each player. For simplicity, you can assume that player A's wealth is $w_A = \$14$, and those of player B are $w_B^H = \$20$ when his wealth is high, and $w_B^L = \$10$ when his wealth is low. [*Hint*: You will find one equilibrium contribution for player A, but two equilibrium contributions for player B as his contribution is dependent on his wealth level $w_B$]

***Answer:***

**Part (a).** *Player B with high wealth.* When $w_B = w_B^H$, player B chooses his contribution to the public good $g_B^H \geq 0$ to solve the following utility maximization problem:

$$\max_{g_B^H \geq 0} \; u_B(g_B^H) = (w_B^H - g_B^H)^{1/2}[m(g_A + g_B^H)]^{1/2}$$

where the first term represents his utility from private uses (that is, the amount of wealth not contributed to the public good), while the second term reflects his utility from total donations to the public good from him and player A.

Differentiating with respect to $g_B^H$, yields

$$\frac{\partial u_B(g_B^H)}{\partial g_B^H} = -\frac{\sqrt{m}}{2}\sqrt{\frac{g_A + g_B^H}{w_B^H - g_B^H}} + \frac{\sqrt{m}}{2}\sqrt{\frac{w_B^H - g_B^H}{g_A + g_B^H}}$$

Assuming interior solutions $(g_B^H > 0)$, we set the above first order condition equal to zero, which helps us rearrange the above expression as follows

$$g_A + g_B^H = w_B^H - g_B^H$$

Solving for player B's contribution to the public good, $g_B^H$, yields the following best response function,

$$g_B^H(g_A) = \frac{w_B^H}{2} - \frac{1}{2}g_A$$

which originates at half of his wealth, $\frac{w_B^H}{2}$, and decreases in player A's donation by 1/2. In words, when player A does not contribute to the public good ($g_A = 0$), player B donates half of his wealth, $\frac{w_B^H}{2}$, but for every dollar that player A contributes to the public good, player B reduces his donation by 50 cents.

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$$\max_{g_B^L \geq 0} \quad u_B(g_B^L) = (w_B^L - g_B^L)^{1/2}[m(g_A + g_B^L)]^{1/2}$$

Differentiating with respect to $g_B^L$, yields

$$\frac{\partial u_B(g_B^L)}{\partial g_B^L} = -\frac{\sqrt{m}}{2}\sqrt{\frac{g_A + g_B^L}{w_B^L - g_B^L}} + \frac{\sqrt{m}}{2}\sqrt{\frac{w_B^L - g_B^L}{g_A + g_B^L}}$$

Assuming interior solutions $(g_B^L > 0)$, we set the above first order condition equal to zero, which helps us simplify the expression as follows

$$g_A + g_B^L = w_B^L - g_B^L$$

After solving for player B's contribution to the public good, $g_B^L$, we find the following best response function

$$g_B^L(g_A) = \frac{w_B^L}{2} - \frac{1}{2}g_A$$

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since he does not observe player B's wealth level, he does not know whether player B contributes $g_B^L$ (which occurs when his wealth is low) or $g_B^H$ (which happens when his wealth is high). Differentiating with respect to $g_A$, yields

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Assuming interior solutions $(g_A > 0)$, we set the above first order condition equal to zero. Rearranging, we find that player A's best response function, $g_A(g_B^L, g_B^H)$, is implicitly defined by the following expression

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which is further simplified to

$$\sqrt{(g_A + w_B^L)(g_A + w_B^H)} = 2(w_A - g_A)$$

Squaring both sides yields

$$(g_A + w_B^L)(g_A + w_B^H) = 4(w_A - g_A)^2$$
$$\Rightarrow 3g_A^2 - (8w_A + w_B^L + w_B^H)g_A + 4w_A^2 - w_B^L w_B^H = 0$$

Substituting wealth levels $w_A = \$14$, $w_B^H = \$20$, and $w_B^L = \$10$ into the above quadratic equation, we obtain

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