ECONS 424 – STRATEGY AND GAME THEORY
MIDTERM EXAM #2

<u>DUE DATE:</u> MONDAY, APRIL 8TH 2019, IN CLASS

Instructions:

- This exam has 6 exercises, plus a bonus exercise at the end.
- Write your answers to each exercise in a different page.
- Show all your work and be as clear as possible in your answer. You can work in groups, but each student must submit his/her exam.
- The due date of this take-home exam is Monday, April 8th, in class. I strongly recommend you work a few exercises every day, rather than trying to solve all exercises in one day.
- Since this is a take-home exam, late submission will be subject to significant grade reduction.

Exercise #1. Hawk-Dove game. Consider the following payoff matrix representing the Hawk-Dove game. Intuitively, Players 1 and 2 compete for a resource, each of them choosing to display an aggressive posture (hawk) or a passive attitude (dove). Assume that payoff $V > 0$ denotes the value that both players assign to the resource, and $C > 0$ is the cost of fighting, which only occurs if they are both aggressive by playing hawk in the top left-hand cell of the matrix.

Player 2

Player 1		*Hawk*	*Dove*
	Hawk	$\frac{V-C}{2}, \frac{V-C}{2}$	$V, 0$
	Dove	$0, V$	$\frac{V}{2}, \frac{V}{2}$

a) Show that if $C \leq V$, the game is strategically equivalent to a Prisoner's Dilemma game.

b) The Hawk-Dove game commonly assumes that the value of the resource is less than the cost of a fight, i.e., $C > V > 0$. Find the set of pure strategy Nash equilibria.

Exercise #2. Collusion in donations – Infinitely repeated public good game. Consider a public good game between two players, A and B. The utility function of every player i={A,B} is

$$u_i(g_i) = (w - g_i)^{1/2}\left[m\left(g_i + g_j\right)\right]^{1/2}$$

where the first term, $w - g_i$, represents the utility that the player receives from money (i.e., the amount of his wealth, w, not contributed to the public good). The second term indicates the utility he obtains from aggregate contributions to the public good, $g_i + g_j$, where $m \geq 0$ denotes the return from aggregate contributions. For simplicity, assume that both players have equal wealth levels w.

a) Find every player i's best response function, $g_i\left(g_j\right)$. Interpret.
b) Find the Nash Equilibrium of this game, g^*.
c) Find the socially optimal contribution level, g^{SO}, that is, the donation that each player should contribute to maximize their joint utilities. Compare it with the Nash equilibrium of the game. Which of them yields the highest utility for every player i?
d) Consider the infinitely repeated version of the game. Show under which values of players' discount factor δ you can sustain cooperation, where cooperation is understood as contributing the socially optimal amount (the one that maximizes joint utilities) you found in part (c). For simplicity, you can consider a "Grim-Triger strategy" where every player starts cooperating (that is, donating g^{SO} to the public good), and continues to do so if both players cooperated in all previous periods. However, if one or both players deviate from g^{SO}, he reverts to the Nash equilibrium of the unrepeated game, g^*, thereafter.

Exercise #3 – Cartel with two asymmetric firms. Consider two firms competing a la Cournot and facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Firm 1 has more experience in the industry than firm 2, which is reflected in the fact that firm 1's marginal cost of production is $c_1 = 10$ while that of firm 2 is $c_2 = 16$.

a) *Best response function.* Set up each firm's profit-maximization problem and find its best response function. Interpret.

b) *Cournot competition.* Find the equilibrium output each firm produces when competing a la Cournot (that is, when they simultaneously and independently choose their output levels). In addition, find the profits that each firm earns in equilibrium. [*Hint*: You cannot invoke symmetry when solving for equilibrium output levels in this exercise since firms are not cost symmetric.]

c) *Collusive agreement.* Assume now that firms collude to increase their profits. Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the output level that each firm should select to maximize joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

d) *Profit comparison.* Compare the profits that firms obtain when competing a la Cournot (from part b) against their profits when they successfully collude (from part c).

Exercise #4. Cartel with firms competing a la Bertrand – Unsustainable in a one-shot game. Consider two firms competing in prices (a la Bertrand), and facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. For simplicity, assume that firms face a common marginal cost of production $c = 10$.

a) *Bertrand competition.* Find the equilibrium price that each firm sets when competing a la Bertrand (that is, when they simultaneously and independently set their prices). In addition, find the profits that each firm earns in equilibrium.

b) *Collusive agreement.* Assume now that the CEOs from both companies meet for lunch and start talking about how they could increase their profits if they could coordinate their price setting decisions. Beware, firms are trying to collude! Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the price that each firm selects when maximizing joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

c) *Profit comparison.* Compare the profits that firms obtain when competing a la Bertrand (from part a) against their profits when they successfully collude (from part b).

d) *Unsustainable cartel.* Show that the collusive agreement from part (b) cannot be sustained as a Nash equilibrium of the one-shot game, that is, when firms interact only once. Interpret.

Exercise #5. Temporary punishments from deviation. Consider two firms competing in quantities (a la Cournot), facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. For simplicity, assume that firms face a common marginal cost of production $c = 10$.

a) *Unrepeated game.* Find the equilibrium output each firm produces when competing a la Cournot (that is, when they simultaneously and independently choose their output levels) in the unrepeated version of the game (that is, when firms interact only once). In addition, find the profits that each firm earns in equilibrium.

b) *Repeated game* **-** *Collusion.* Assume now that the CEOs from both companies meet to discuss a collusive agreement that would increase their profits. Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the output level that each firm should select to maximize joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

c) *Repeated game – Permanent punishment.* Consider a grim-trigger strategy in which every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. Otherwise, every firm deviates to the Cournot equilibrium thereafter (that is, every firm produces the Nash equilibrium of the unrepeated game found in part a forever). In words, this says that the punishment of deviating from the collusive agreement is *permanent*, since firms never return to the collusive outcome. For which discount factors this grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

d) *Repeated game – Temporary punishment.* Consider now a "modified" grim-trigger strategy. Like in the grim-trigger strategy of part (c), every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. However, if a deviation is detected by either firm, every firm deviates to the Cournot equilibrium during only 1 period, and then every firm returns to cooperation (producing the collusive output). Intuitively, this implies that the punishment of deviating from the collusive agreement is now *temporary* (rather than permanent) since it lasts only one period. For which discount factors this "modified" grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

e) Consider again the temporary punishment in part (d) but assume now that it lasts for two periods. How are your results from part (d) affected? Interpret.

f) Consider again the temporary punishment in part (d) but assume now that it lasts for three periods. How are your results from part (d) affected? Interpret.

Exercise #6:

Exercise 12 from Chapter 10 in Harrington’s book (new edition).

BONUS EXERCISE.[1] **Public good game with incomplete information**. Consider a public good game between two players, A and B. The utility function of every player *i* is

$$u_i(g_i) = (w_i - g_i)^{1/2}\left[m\left(g_i + g_j\right)\right]^{1/2}$$

where the first term, $w_i - g_i$, represents the utility that the player receives from money (i.e., the amount of his wealth not contributed to the public good which he can dedicate to private uses). The second term indicates the utility he obtains from aggregate contributions to the public good, $g_i + g_j$, where $m \geq 0$ denotes the return from aggregate contributions. Since public goods are non-rival in consumption, player *i*'s utility from this good originates from both his own donations, g_i, and in those of player j, g_j.

Consider that the wealth of player A, $w_A > 0$, is common knowledge among the players; but that of player B, w_B, is privately observed by player B but not observed by player A. However, player A knows that player B's wealth level w_B can be high (w_B^H) or low (w_B^L) with equal probabilities, where $w_B^H > w_B^L > 0$. Let us next find the Bayesian Nash Equilibrium (BNE) of this public good game where player A is uninformed about the exact realization of parameter w_B.

a) Starting with the privately informed player B, set up his utility maximization problem, and separately find his best response function, first, when $w_B = w_B^H$ and, second, when $w_B = w_B^L$.

b) Find the best response function of the uninformed player A.

c) Using your results in parts (a) and (b), find the equilibrium contributions to the public good by each player. For simplicity, you can assume that player A's wealth is $w_A = \$14$, and those of player B are $w_B^H = \$20$ when his wealth is high, and $w_B^L = \$10$ when his wealth is low. [*Hint*: You will find one equilibrium contribution for player A, but two equilibrium contributions for player B as his contribution is dependent on his wealth level w_B]

[1] This exercise will not hurt your grade. Solving it correctly can bring your grade in the exam to 120 out of 100 points.