Substituting for c and simplifying yields

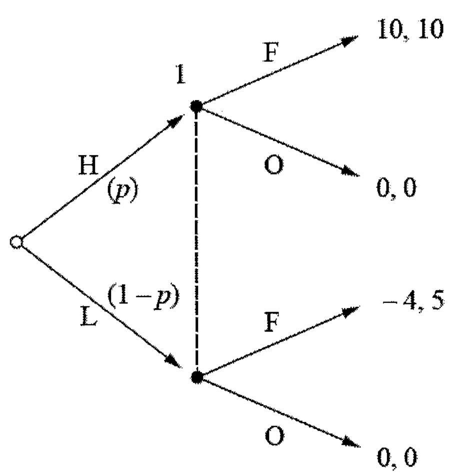
$$p_1 - p_1^2 \frac{4-3\delta}{(2-\delta)^2}.$$

Taking the derivative and solving the first-order condition for p_1 yields

$$p_1 = \frac{(2-\delta)^2}{8-6\delta}.$$

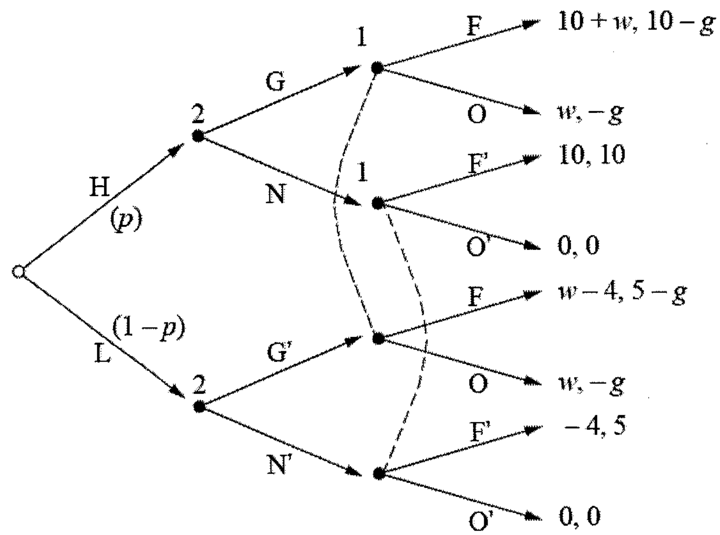
EXERCISE 7.

(a) The extensive form is:



In the Bayesian Nash equilibrium, player 1 forms a firm (F) if $10p - 4(1 - p) \geq 0$, which simplifies to $p \geq 2/7$. Player 1 does not form a firm (O) if $p < 2/7$.

(b) The extensive form is:



(c) Clearly, player 1 wants to choose F with the H type and O with the L type. Thus, there is a separating equilibrium if and only if the types of player 2 have the incentive to separate. This is the case if $10 - g \geq 0$ and $0 \geq 5 - g$, which simplifies to $g \in [5, 10]$.

(d) If $p \geq 2/7$, then there is a pooling equilibrium in which NN′ and F′ are played, player 1's belief conditional on no gift is p, player 1's belief conditional on a gift is arbitrary, and player 1's choice between F and O is optimal given this belief. If, in addition to $p \geq 2/7$, it is the case that $g \in [5, 10]$, then there is also a pooling equilibrium featuring GG′ and FO′. If $p \leq 2/7$, then there is a pooling equilibrium in which NN′ and OO′ are played (and player 1 puts a probability on H that is less than $2/7$ conditional on receiving a gift).

EXERCISE 8.

(a) A player is indifferent between O and F when he believes that the other player will choose O for sure. Thus, (O, O; O, O) is a Bayesian Nash equilibrium.

(b) If both types of the other player select Y, the H type prefers Y if $10p - 4(1-p) \geq 0$, which simplifies to $p \geq 2/7$. The L type weakly prefers Y, regardless of p. Thus, such an equilibrium exists if $p \geq 2/7$.

(c) If the other player behaves as specified, then the H type expects $-g + p(w + 10) + (1 - p)0$ from giving a gift. He expects pw from not giving a gift. Thus, he has the incentive to give a gift if $10p \geq g$. The L type expects $-g+p(9w+5)+(1-p)0$ if he gives a gift, whereas he expects pw if he does not give a gift. The L type prefers not to give if $g \geq 5p$. The equilibrium, therefore, exists if $g \in [5p, 10p]$.

EXERCISE 10.

(a) 1^H selects a_1^H to maximize $4a_1^H + 4a_2 - [a_1^H]^2$, which has a first-order condition of $4 - 2a_1^H \equiv 0$ implying $a_1^H = 2$.

Similarly, 1^L selects a_1^L to maximize $2a_1^L + 2a_2 - [a_1^L]^2$, which has a first-order condition of $2 - 2a_1^L \equiv 0$ implying $a_1^L = 1$.

Player 2 does not observe k and chooses a_2 to maximize $\frac{1}{2}[4a_1^H + 4a_2] + \frac{1}{2}[2a_1^H + 2a_2] - a_2^2$, which has a first-order condition of $2 + 1 - 2a_2 \equiv 0$ implying $a_2 = \frac{3}{2}$.

(b) There is an equilibrium in which both types of player 1 present evidence of their type. This requires that when no evidence is presented, player 2's belief is that $k = 4$. When player 1 shows her type to be H, both player 1 and 2 choose effort of 2, and when player 1 shows her type to be L, both players choose effort of 1. Following the out-of-equilibrium behavior of player 1 not disclosing evidence, player 2 chooses effort of $\frac{3}{2}$, and player 1 chooses effort of 2 when the state is H and 1 when the state is L.

There is also an equilibrium in which player 1 presents evidence in H and does not in L. Upon seeing no evidence presented, player 2 believes that $k = 4$. Player 1 chooses effort of 2 in H and 1 in L. Player 2 chooses effort of 2 when evidence of H is presented and chooses effort of 1 when either no evidence is presented or evidence of L is presented.

In both of these equilibria, player 2 knows the value of k from either direct evidence or from inferring that $k = 4$ due to player 1 not presenting evidence.

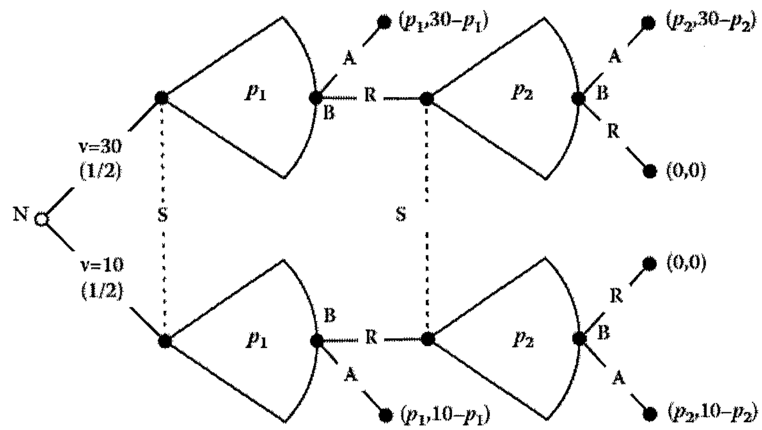
(c) After observing $k = 8$, player 1 would like for player 2 to know the value of k, but after observing $k = 4$, player 1 would like to not be able to convey the value of k.

We can also address this ex ante or prior to the realization of k as follows. When $k = 8$, player 1's payoff is $4[2 + \frac{3}{2}] - 4 = 10$, and when $k = 4$, player 1's payoff is $2[1 + \frac{3}{2}] - 1 = 4$. So player 1's expected payoff is 7. However, when k is known by player 2, player 1's payoffs are the following: when $k = 8$, $u_1 = 4[2 + 2] - 4 = 12$, and when $k = 4$, $u_1 = 2[1 + 1] - 1 = 3$. This yields an expected payoff for player 1 of $7.5 > 7$, so player 1 would prefer that player 2 know the value of k.

EXERCISE #3 [BARGAINING AND PBEs]

Answer:

(a)



(b) We work backwards from the end of the game. In period 2, the 10-buyer accepts if and only if ("iff") $p_2 \leq 10$, and the 30-type buyer accepts iff $p_2 \leq 30$. Given that the seller believes that the buyer is the 30-type with probability 0 in period 2, his sequentially rational response is $p_2 = 10$ so that the 10-buyer will accept.

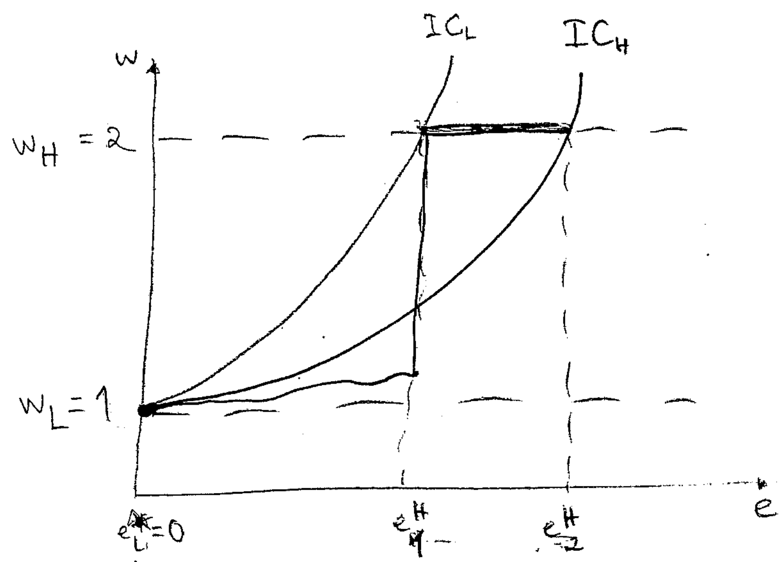
Given that $p_2 = 10$, the 30-buyer will accept in period 1 iff $p_1 \leq 10$, since she could get a price of $p_2 = 10$ in period 2 by rejecting. For the second-period beliefs to be consistent, the 30-buyer must accept in period 1, so we have $p_1 \leq 10$. The 10-buyer also accepts in period 1 iff $p_1 \leq 10$, since she will not get a price lower than 10 in the second period. Note from the extensive form that the seller must believe that the two types of buyers are equally likely in period 1, and so his sequentially rational response to these beliefs (about both the buyer's type and the buyer's strategy) is to offer $p_1 = 10$. Then in equilibrium, both buyer types accept in period 1, allowing us to set period 2 beliefs arbitrarily.

Note that we can apply the same period 2 beliefs (that the buyer is the 30-type with probability 0) for all of the seller's period 2 information sets (there is one for each possible price in period 1). Since none of these are reached in equilibrium, this does not violate consistency. This completes the description of the equilibrium.

EXERCISE #3

$$u(w, e|\theta) = w - \frac{e^3}{\theta}$$

a) Separating PBE when $\theta_1 = 1$ and $\theta_2 = 2$.



The set of all separating PBE is given by:

- low productivity worker chooses $e_L^*(\theta_L) = 0$
- High " " " $e_H^*(\theta_H) \in [e_1^H, e_2^H]$, where the exact ~~values of the upper and lower bound will be determined below~~
- ~~Firms, after observing an education level e, will offer wages:~~

$$w^*(e_L^*(\theta_L)) = \theta_L = 1$$

$$w^*(e_H^*(\theta_H)) = \theta_H = 2$$

For any other education levels $e \neq e_L^*(\theta_L) \neq e_H^*(\theta_H)$ firms will offer a wage schedule $w^*(e)$ that is all the way below the indifference curves of both types of workers for $e < e_1^H$, and below IC_L for all $e \geq e_1^H$. (In the figure, $w(e) = \theta_H$ for all $e \geq e_1^H$.)

In order to determine the exact values of e_1^H and e_2^H we need that the following incentive compatibility conditions are satisfied:

$$EU\left(e_L^*, w^*(e) \mid \theta_L\right) \geq EU\left(e_H^*, w^*(e) \mid \theta_L\right)$$

$$EU\left(e_H^*, w^*(e) \mid \theta_H\right) \geq EU\left(e_L^*, w^*(e) \mid \theta_H\right)$$

That is,

$$1 - \frac{0^3}{1} \geq 2 - \frac{(e^H)^3}{1} \Leftrightarrow 1 \geq 2 - (e^H)^3 \Leftrightarrow e^H \geq 1$$

$$2 - \frac{(e^H)^3}{2} \geq 1 - \frac{0}{2} \Leftrightarrow 1 \geq \frac{(e^H)^3}{2} \Leftrightarrow \sqrt[3]{2} \geq e^H$$

Therefore $e_H^*(\theta_H) \in \left[1, \sqrt[3]{2}\right]$

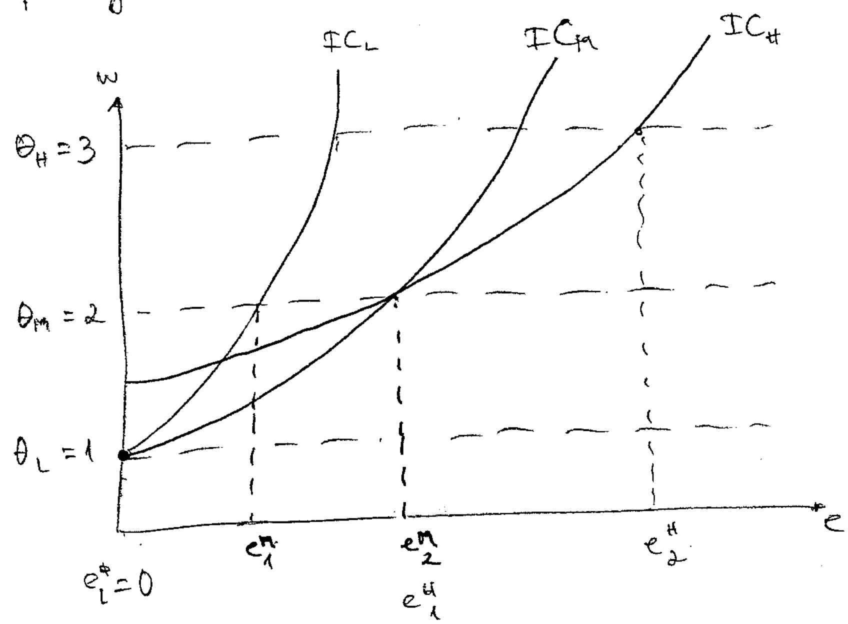
Note that this strategy profile is indeed a PBE:

- firms are optimizing given their beliefs about workers' types and given the equilibrium strategies for the workers.
- workers are optimizing given firms' equilibrium strategies and given firms' beliefs about the workers' types
- firms' beliefs are computed by Bayes' rule when possible.

Note that the θ_L-worker will never acquire more education than $e_L^*(\theta_L) = 0$: by acquiring more education he is still identified as a low-productivity worker and offered a wage of θ_L, and he is incurring a lot of costs in acquiring such additional education.

Even if he acquires e_1^H and he is identified as a high-productivity worker, the increase in wages will not offset his increase in education costs.

b) Separating PBE when $\theta_1 = 1$, $\theta_2 = 2$ and $\theta_3 = 3$.



The set of all separating PBE is given by:

$$\{e_L^*(\theta_L), e_M^*(\theta_M), e_H^*(\theta_H)\} = \{0, e_M^*, e_H^*\} \quad \text{where } e_M^* \in [e_1^M, e_2^M]$$
$$e_H^* \in [e_1^H, e_2^H]$$

$$\{w^*(e_L^*(\theta_L)), w^*(e_M^*(\theta_M)), w^*(e_H^*(\theta_H))\} = \{\theta_L, \theta_M, \theta_H\} = \{1,2,3\}$$

and any wage offer $w^*(e)$ for off-the-equilibrium education levels $e \neq e_L^* \neq e_M^* \neq e_H^*$ such that it is below the IC of all types of workers.

(Other wage schedules are also possible in equilibrium as long as they lie below IC_L for all $e < e_1^M$, and below both IC_L and IC_H for all $e_1^M < e < e_2^M$.)

In order to accurately characterize the set of all separating PBE with $n=3$ types we need to set up the following Incentive Compatibility conditions and Participation Constraints:

$$\boxed{IC_L} \quad u\left(e_L^*, w^*(e) \mid \theta_L\right) \geq u\left(e_M^*, w^*(e) \mid \theta_L\right)$$
$$\geq u\left(e_H^*, w^*(e) \mid \theta_L\right)$$

$$\boxed{IC_M} \quad u\left(e_M^*, w^*(e) \mid \theta_M\right) \geq u\left(e_L^*, w^*(e) \mid \theta_M\right)$$
$$\geq u\left(e_H^*, w^*(e) \mid \theta_M\right)$$

$$\boxed{IC_H} \quad u\left(e_H^*, w^*(e) \mid \theta_H\right) \geq u\left(e_L^*, w^*(e) \mid \theta_H\right)$$
$$\geq u\left(e_M^*, w^*(e) \mid \theta_H\right).$$

$$\boxed{PC_L} \quad u\left(e_L^*, w^*(e) \mid \theta_L\right) \geq 0$$

$$\boxed{PC_M} \quad u\left(e_M^*, w^*(e) \mid \theta_M\right) \geq 0$$

$$\cancel{\boxed{PC_H} \quad u\left(e_H^*, w^*(e) \mid \theta_H\right) \geq 0}$$

That is,

$$\boxed{IC_L} \quad 1 - \frac{0^3}{1} \geq 2 - \frac{(e^M)^3}{1} \iff e^M \geq 1$$
$$\geq 3 - \frac{(e^H)^3}{1} \iff e^H \geq \sqrt[3]{2}$$

$$\boxed{IC_M} \quad 2 - \frac{(e^M)^3}{2} \geq 1 - \frac{0^3}{2} \iff 1 \geq \frac{(e^M)^3}{2} \iff e^M \leq \sqrt[3]{2}$$
$$\geq 3 - \frac{(e^H)^3}{2} \iff \frac{(e^H)^3 - (e^M)^3}{2} \geq 1 \iff (e^H)^3 - (e^M)^3 \geq 2$$

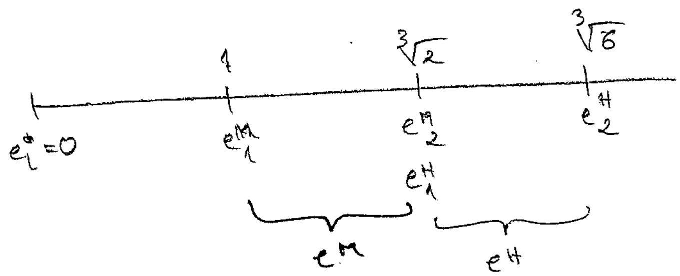
$\boxed{IC_H}$ $\quad 3 - \frac{(e^H)^3}{3} \geq 1 - \frac{0^3}{3} \iff 2 \geq \frac{(e^H)^3}{3} \iff e^H \leq \sqrt[3]{6}$

$\geq 2 - \frac{(e^M)^3}{3} \iff 1 \geq \frac{(e^H)^3 - (e^M)^3}{2} \iff (e^H)^3 - (e^M)^3 \leq 2$

$\boxed{PC_L}$ $\quad 1 - \frac{0^3}{1} \geq 0$

$\boxed{PC_M}$ $\quad 2 - \frac{(e^M)^3}{2} \geq 0 \iff e^M \leq \sqrt[3]{4}$

$\boxed{PC_H}$ $\quad 3 - \frac{(e^H)^3}{3} \geq 0 \iff e^H \leq \sqrt[3]{9}$



As a curiosity, I also include what would happen if we have a continuum of types (rather than 2 or 3 types alone).

c) Continuum of types on $[0,1]$.

Workers will choose the education level that maximizes their expected utility:

$$\max_{e} \quad w(e) - \frac{e^3}{\theta}$$

FOC

$$w'(e^*) - \frac{3(e^*)^2}{\theta} \leq 0$$

and given that firms are competing for labor services in a perfectly competitive labor market, $w(e) = \theta$. Hence,

$$w'(e^*) = \frac{3(e^*)^2}{w(e^*)} \iff w'(e^*)\, w(e) = 3(e^*)^2$$

This is a First Order Differential Equation that can be solved by using integration by parts:

$$\int w'(e)\, w(e)\, de = w(e)\, w(e) - \int w(e)\, w'(e)\, de$$

$$\iff 2\int w'(e)\, w(e)\, de = [w(e)]^2$$

$$\iff 2\int 3e^2\, de = [w(e)]^2$$

$$\iff 6\left[\frac{e^3}{3}\right] = [w(e)]^2 \iff 2e^3 = [w(e)]^2$$

$$\iff w(e) = \sqrt{2}\, e^{3/2}$$

and given that

$$\left.\begin{array}{l} w(e) = \sqrt{2}\ e^{3/2} \\ w(e) = \theta \end{array}\right\} \quad \sqrt{2}\ e^{\frac{3}{2}} = \theta$$

$$\Leftrightarrow e^{3/2} = \frac{\theta}{\sqrt{2}} \quad \Leftrightarrow \quad e = \frac{\theta^{2/3}}{2^{1/2 \cdot 2/3}} = \frac{\theta^{2/3}}{2^{1/3}}$$

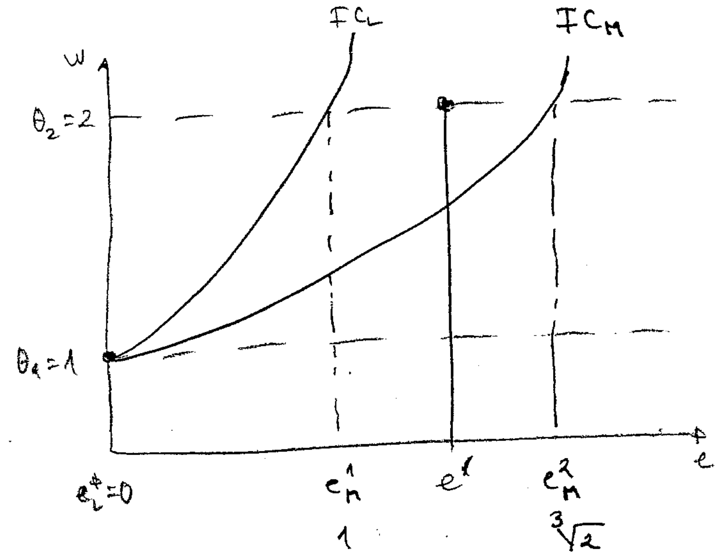
Therefore, the unique separating PBE with a continuum of types is given by:

- workers choose $e^*(\theta) = \dfrac{\theta^{2/3}}{2^{1/3}}$

- firms choose $w(e) = \sqrt{2}\ e^{3/2}$ when observing any e.

  Note that this specifies a complete contingent plan for the firms, even when they observe off-the-equ. messages $e \neq e^*(\theta)$

d) Intuitive Criterion for $n=2$ types.



FIRST STEP

Firms' beliefs when observing messages such as e' need to be restricted to $\Theta^{**}(e') \subset \Theta$ which represents all those types of workers for whom sending e' is never equilibrium dominated:

$$\Theta^{**}(e') = \left\{ \theta \in \Theta \;\middle|\; u^*(\theta_i) \leq \max_{s \in S^*(\Theta, e')} u(s, e', \theta_i) \right\}$$

~~As we can see this inequality is only satisfied for the θ_M-type, but is~~ never satisfied for the low type. ~~That is, even if firms believe that~~ θ_L is in fact a θ_M-worker and pay $w(e')=2$, it is never going to be convenient for a θ_L-worker to send such a message because of the high education costs that he would need to incur

$$1 - \frac{0^3}{1} \geq 2 - \frac{(e')^3}{1} \iff (e')^3 \geq 1 \iff e' \geq \sqrt[3]{1}$$
$$\iff e' \geq 1 \text{ which is true.}$$

So, e' is equilibrium dominated for the θ_L-worker.

However, for θ_H-worker sending message e' is not equilibrium dominated. Therefore, the set of those types of workers for whom sending e' is never equilibrium dominated is:

$$\Theta^{**}(e') = \{\theta_H\}$$

SECOND STEP

Once we have restricted firms' beliefs to $\Theta^{**}(e') = \{\theta_H\}$ we need to find whether there exists a message that can be sent by θ_H that implies a higher level of utility than sending his equilibrium message. That is, we need to find a pair (θ_H, e) for which the following inequality holds:

$$u_i^*(\theta_i) < \min_{s \in S^*(\Theta^{**}(e), e)} u^*(s, e, \theta_i)$$

As we can see, once we have restricted firms beliefs to $\Theta^{**}(e') = \{\theta_H\}$ we have that firms will simply offer wages of $w(e) = \theta_H$ for all those ~~education levels that could never be sent by a θ_L-worker, i.e. any~~ ~~$e > e_1^M$~~

As a consequence, the θ_H-worker will be better-off by sending any message different from (smaller) than the one he was sending in any of the inefficient separating PBE:

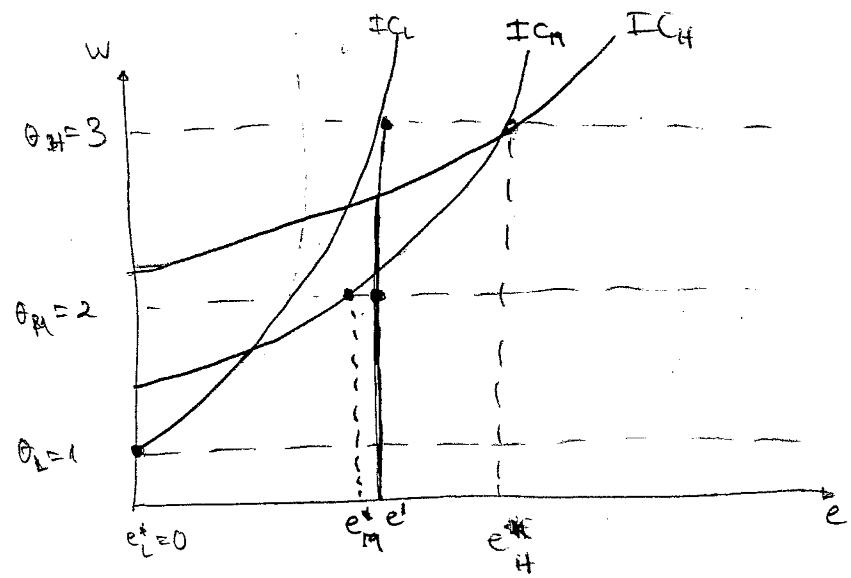
$$\theta_H - c(e_H^*, \theta_H) < \theta_H - c(e, \theta_H)$$

$$\iff c(e_H^*, \theta_H) > c(e, \theta_H) \text{ which is true given that}$$

$$e_H > e \text{ and } c_e(\cdot) > 0$$

So, all the inefficient separating PBE can be eliminated by the Intuitive Criterion. Recall that the Riley outcome (efficient separating PBE) cannot be eliminated by the Intuitive Criterion.

Intuitive Criterion for $n=3$ types.



Let's take one of the (inefficient) separating PBE with $n=3$ and let's check if it can be eliminated by using the Intuitive Criterion:

First Step

Firms' beliefs when observing messages such as e' need to be restricted to those types of workers for whom this message e' is never equilibrium dominated:

$$\Theta^{**}(e') = \left\{ \theta \in \Theta \;\middle|\; u^*(\theta_i) \leq \max_{s \in S^*(\Theta, e')} u(s, e', \theta_i) \right\}$$

As we can see, this message e' could have never been sent by θ_L-workers. Indeed, even if firms believe that e' could only have been sent by θ_H-workers, the cost for θ_L-workers would be too high.

That is,

$$u^*(\theta_L) > \max_{s \in S^*(\Theta, e')} u(s, e', \theta_L)$$

whereas $$u^*(\theta_M) < \max_{s \in S^*(\Theta, e')} u(s, e', \theta_M)$$

$$u^*(\theta_H) < \max_{s \in S^*(\Theta, e')} u(s, e', \theta_H)$$

Hence, the set of types for which message e' is never equil. dominated is:

$$\Theta^{**}(e') = \{\theta_M, \theta_H\}$$

SECOND STEP

Once firms' beliefs have been restricted to $\Theta^{**}(e') = \{\theta_M, \theta_H\}$, then we need to check if there exists a pair (θ, e) of types belonging to $\Theta^{**}(e')$ sending some message e' that strictly improves their equilibrium payoffs. That is,

$$u^*(\theta_i) < \min_{s \in S^*(\Theta^{**}(e), e)} u^*(s, e, \theta_i)$$

But we can easily see that sending e' implies

$$u^*(\theta_M) > \min_{s \in S^*(\Theta^{**}(e'), e')} u^*(s, e', \theta_M)$$

$$u^*(\theta_H) > \min_{s \in S^*(\Theta^{**}(e'), e')} u^*(s, e', \theta_H)$$

Therefore, the above inequality is not satisfied for any type of worker belonging to $\Theta^{**}(e') = \{\theta_M, \theta_H\}$

That is, both θ_M and θ_H would be better off sending their equilibrium messages than sending message e'.

So, all the (inefficient) separating PBE survive the Intuitive Criterion and as a result they cannot be eliminated.

Exercise 8.5

~~See Okuno-Fujiwara, Postlewaite, and Suzumura [1990].~~

Exercise 8.6

(a) Suppose first that Max(p,q) < 1/2.

Let (x,y) denote the strategy profile where the "sane" type (players with the preferences of chapter 1) play strategies x and y (and all types with S or R as a dominant strategy play that strategy).

(S,S) is an equilibrium as

$$Eu_1(S,S) = p \cdot 2 + q \cdot 0 + (1-p-q) \cdot 2$$

$$Eu_1(R,S) = p \cdot 1 + q \cdot 1 + (1-p-q) \cdot 1 \ .$$

It is easy to see that S is a best response when $1-2q > 0$.

(R,R) is also an equilibrium when $1-2p > 0$.

Also, there is a mixed equilibrium where the "sane" types hunt stag with probability r, where r is chosen so that the sane types are indifferent between S and R, i.e.

$$2p + 0 \cdot q + 2r(1-p-q) = 1$$

$$\Rightarrow \quad r = \frac{1-2p}{2(1-p-q)} \ .$$

When $p > 1/2$, the payoff to player i of playing S is at least $2p > 1$ so S is strictly dominant and (S,S) is the unique equilibrium.

When $q > 1/2$, R is strictly dominant for all sane types and (R,R) is the unique equilibrium.

(b) Suppose that player j played S in the first period of a Bayesian equilibrium. Then, player i knows that j is not the type who always prefers

R and in particular applies Bayesian updating to conclude that j is of the type that always plays S with probability

$$\frac{p}{p + (1-p-q)r}$$

where r is the probability that a "sane" player j played S in the first period. Thus, player i's expected payoff to playing S is at least

$$\frac{2\cdot p}{p + (1-p-q)} = \frac{2p}{1-q} > 1 .$$

Hence, the sane player i must play S in period 2.

Similarly, if player j plays R in the first period the sane player i must play R in period 2.

This determines the second period strategies uniquely.

(c) Consider the following strategy profile:

In period 1, the "sane" types of player 1 and 2 each play S with probability β and R with probability $1-\beta$.

In period 2, the "sane" players play the action their opponents used in period 1.

The types who always prefer R or S play their preferred strategy in each period.

By the reasoning of part (b) the second period action is optimal for the sane types. To verify that this is an equilibrium it only remains to show that the sane types are indifferent between their two first period actions.

If a "sane" player 1 plays S in period 1 and then conforms to his strategy, he has

$$Eu_1(S) = p(2+2) + q(0+1) + (1-p-q)(\beta(2+2) + (1-\beta)(0+1)),$$

as (S,S) results in both periods if player 2 is S-loving, (S,R) then (R,R) results if player 2 is R-loving, (S,S) then (S,S) results if the sane player

2 plays S, and (S,R) then (R,S) results if the sane player 2 plays R. If he plays R in period 1, then conforms to his strategy, he has payoff

$$Eu_1(R) = p(1+2) + q(1+1) + (1-p-q)(\beta(1+0) + (1-\beta)(1+1)) .$$

These two expressions are indeed equal for

$$(1-p-q)(4\beta-1) = q-p$$

$$\Rightarrow \quad \beta = \frac{1-2p}{4(1-p-q)} .$$

The condition $\frac{1+2p}{4} \in (p,1-q)$ ensures that $\beta \in (0,1)$.

In this equilibrium, each player plays S with probability

$$\alpha = p + \beta(1-p-q) = \frac{1+2p}{4} .$$

Note that in this equilibrium, player i's expected first period utility from playing S is $2\alpha = \frac{1}{2} + \frac{p}{2} < 1$. Thus, playing S gives less first period utility than R, but the players are nonetheless willing to play S in hopes of earning higher future payoffs.

Two other equilibria are for the sane types to all play S or R in the first period and follow the same second period strategy as above.

To verify that both playing S in the first period is an equilibrium simply compute the payoffs to playing S and R when your opponent plays S if sane. They are

$$(p + (1-p-q))(2+2) + q(0+1)$$

and

$$p(1+2) + (1-p-q)(1+0) + q(1+1)$$

respectively. The first is indeed larger when $\frac{1+2p}{4} < 1-q$.

There are no other equilibria. It is easy to check that (R,S) and (S,R) cannot be the first period strategies in an equilibrium. In an equilibrium where player 1 plays a mixed strategy in period 1, the calculations above show that player 2 must mix with probability β in period 1. Then as player 2 is mixing we conclude player 1 must mix with probability β in period 1. This gives the mixed equilibrium we had before.

Exercise 8.7

Yes, a sequential equilibrium exists in which the specified play has probability 1. The following is one possible choice of strategies and beliefs:

The type h player 1 plays H in each period.

The type s player 1 plays H in period 0, S in period 1 and R in period 2.

The type r player 1 plays R in each period.

In period 0 player 2 has beliefs $(\mu_h, \mu_s, \mu_r) = (1/3, 1/3, 1/3)$ and plays R.

In period 1, player 2:

has beliefs (1/2,1/2,0) and plays S if he saw H in period 1,

has beliefs (1/3,1/3,1/3) and plays S if he saw S,

has beliefs (0,0,1) and plays R if he saw R.

In period 2, player 2's beliefs are:

(0,1,0) after seeing player 1 play S in periods 1 and 2,

(1,0,0) after seeing S then H,

(0,0,1) after seeing S then R,

(0,1,0) after seeing H then S,

(1,0,0) after seeing H then H,

(1/2,0,1/2) after seeing H then R.

In all cases he plays R.

In a history where no one has previously purchased, the short-run player has beliefs p(sane) = 0.99 and thus has expected payoff .01(1) + .99(-1) = -0.98 < 0 if he purchases.

If one or more person has bought and all have received high quality, the short run player believes the producer to be a high quality type and thus wants to purchase.

If one or more person has bought and not all have received high quality, the short run player believes the producer to be sane and thus purchases when high quality is expected. This yields the strategy given. The beliefs are clearly consistent.

Now, consider the sane player 1.

On the first purchase, he gets 2 if he produces low quality and 1 if he produces high quality. His first purchase quality choice does not affect future play so he chooses low quality.

On any subsequent purchase when no low quality has been supplied to a purchaser after the first, player 1 gets 1 in this and all future periods if he supplies high quality and 2 in this period followed by 0 in the future (as purchases stop) if he supplies low quality. He wants to supply high quality if $\delta > 1/2$.

Finally on a purchase after the second or later has received low quality, there will be no future purchases regardless of the sane type's action so again he maximizes payoffs by producing low quality.

Exercise 9.4

Given several assumptions, the entrant should enter the big market first. The reason for this is that the incumbent will be less willing to imitate toughness by fighting in a big market.

Assume the incumbent is tough with probability $p_0 > \frac{b}{b+1}$. In this case, the entrant will choose not to enter in period 2 if no new information about the incumbent's type is revealed. Suppose the entrant enters the "small" market A in period 1. If the weak incumbent accommodates he reveals his type and thus gets 0 in each period. If he fights in period 1, there will be no entry in period 2 so his payoff is $-1 + 2a$.

Suppose the entrant enters the big market in period 1. If the weak incumbent accommodates he again gets 0. If he fights, entry will be deterred in period 2 so his payoff is $-2 + a$.

If we assume $a \in (1/2,2)$ then the incumbent will fight if entry occurs first in A but will accommodate if entry occurs first in B. This gives payoffs of -1 and $3b$ respectively to the entrant so he prefers to enter the big market first.

For other parameter values we need not have a preference. If $a < 1/2$ or $p_0 < b/(b+1)$ the incumbent will never fight so it doesn't matter which market is entered first. If $a > 2$, the incumbent would always fight so neither market should be entered.

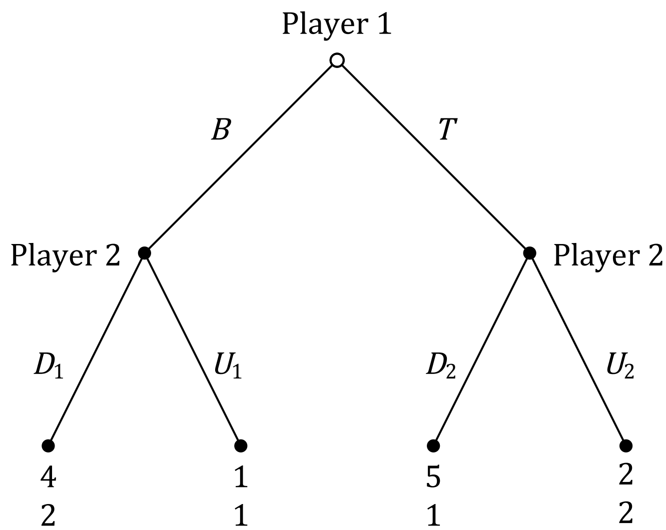
Exercise 9.5

Note that the condition $(1-p)(1-\alpha)D^t - pK > 0$ ensures that either country would threaten default in a one period model. We now consider the possible continuation equilibria after country 1 threatens default in period 1.

No pooling equilibrium is possible. If the soft bank does not lend in period 1, country 2 will threaten default in period 2 anyway so the bank loses by not lending.

2. MWG 9.C.7

Consider the extensive form game depicted below.



a) Find a subgame perfect Nash equilibrium of this game. Is it unique? Are there any other Nash equilibria (not necessarily subgame perfect, or in pure strategies)?

Answer:

The set of pure strategies for player 1 is $S_1 = \{B, T\}$, and for player 2 is

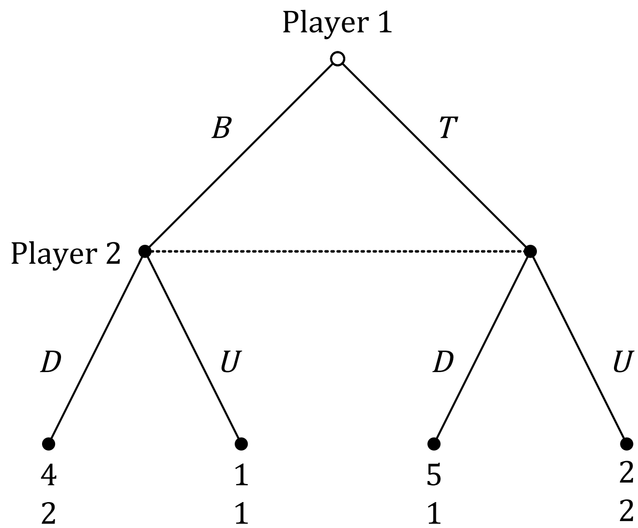
$$S_2 = \{D_1D_2, D_1U_2, U_1D_2, U_1U_2\}.$$

By backward induction it is easy to see that the unique SPNE is (B, D_1U_2). There are two more NE: (i) Player 1 plays T, and player 2 plays U_1U_2 with probability p and D_1U_2 with probability $1-p$, with $p \geq \frac{2}{3}$, and (ii) Player 1 plays B, and player 2 plays D_1U_2 with probability p and D_1D_2 with probability $1-p$, with $p \geq \frac{1}{3}$.

b) Now suppose that player 2 cannot observe player 1's move. Write down the new extensive form. What is the set of Nash equilibria?

Answer:

We can now represent the game as shown below



which is equivaled to a simultaneous move game with the following normal form

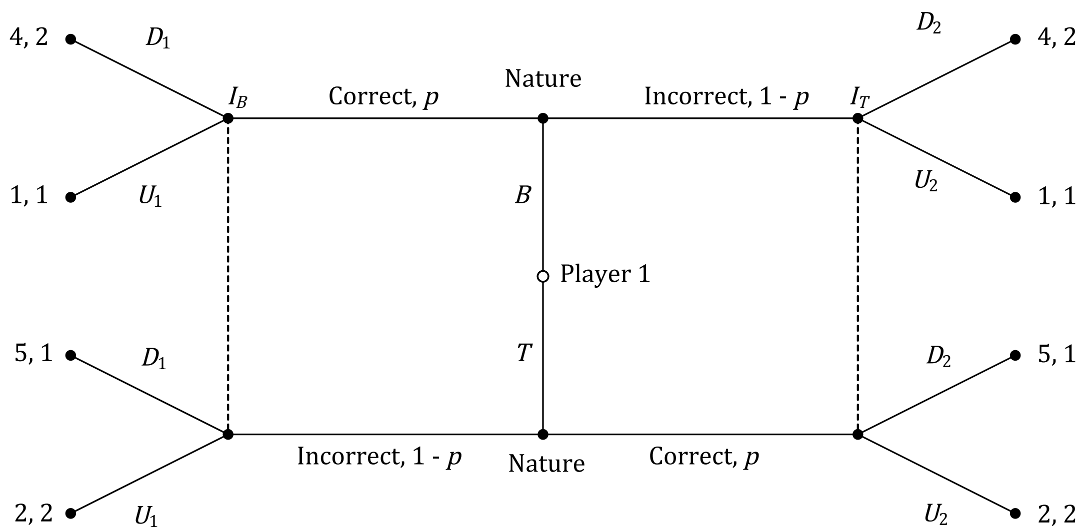
		Player 2	
		D	U
Player 1	B	$4, 2$	$1, 1$
	T	$5, 1$	$2, 2$

It is clear that strategy B is strictly dominated by strategy T for player 1, and then strategy D is strictly dominated by strategy U for player 2 in the reduced form game. Thus, the only Nash equilibrium in this game is (T, U).

c) Now suppose that player 2 observes player 1's move correctly with probability $p \in (0, 1)$ and incorrectly with probability $1 - p$ (e.g., if player 1 plays T, player 2 observes T with probability p and observes B with probability $1 - p$). Suppose that player 2's propensity to observe incorrectly (i.e., given by the value of p) is common knowledge to the two players. What is the extensive form now? Show that there is a unique weak perfect Bayesian equilibrium. What is it?

Answer:

The extensive form of the game now becomes



I_k denotes player 2's information set after she observes $k \in \{B, T\}$, r is the probability she assigns to the event that player 1 played B after she finds herself in formation set I_B, and similarly q is the probability she assigns to the event that player 1 played B after she finds herself in information set I_T. Let $s \in [0,1]$ denote the probability that player 1 plays B. We can have three possible situations in a WPBE: First, player 1 playing $s = 1$; second, player 1 playing $s = 0$; and third, player 1 playing $s \in (0,1)$. Player 1 playing $s = 1$ cannot be part of a WPBE. Indeed, if this were the case we must have $q = r = 1$, which implies that player 2 will always play D. But given that 2 always plays D, player 1 will prefer to deviate and play T. Second, player 1 playing $s = 0$ is part of a WPBE. Indeed, if this is the case we must have $q = r = 0$, which implies that player 2 will always play U, and given that 2 always plays U, player 1 will prefer to play T. Thus, player 1 playing T and player 2 playing U in each of her information sets is a WPBE.
To consider the possibility of a WPBE with $s \in (0,1)$, we first note that this will induce a unique pair of probability beliefs q and r derived by Bayes rule. In particular, in such an equilibrium we must have:

$$\begin{aligned} r &= \frac{s \cdot p}{(1-s)(1-p) + s \cdot p}, \quad \text{and} \\ q &= \frac{s(1-p)}{s(1-p) + p(1-s)}. \end{aligned}$$

Simple algebra shows that $s \underset{<}{\overset{\geq}{=}} p$ if and only if $q \underset{<}{\overset{\geq}{=}} \frac{1}{2}$, and that $s \underset{<}{\overset{\geq}{=}} (1-p)$ if and only if $r \underset{<}{\overset{\geq}{=}} \frac{1}{2}$. This observation allows us to concentrate on 4 cases as follows:

i) $s > p$ and $s > (1-p)$: In this case we must have $q > \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will always play D, which in turn implies that player 1's best response is $s = 0$. Therefore there cannot be a WPBE in this case.

ii) $s < p$ and $s < (1-p)$: In this case we must have $q < \frac{1}{2}$ and $r < \frac{1}{2}$. This implies that player 2 will always play U, which in turn implies that player 1's best response is $s = 0$. This coincides with the pure strategy WPBE described earlier.

iii) $(1-p) < s < p$: (which implies $p > \frac{1}{2}$) In this case we must have $q < \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will play U in information set I_T and will play D in information set I_B. Player 1's best response will now depend on p. Playing B will give player 1 an expected payoff of $4p + 1(1-p)$, and playing T will give him $2p + 5(1-p)$. If $p \neq \frac{2}{3}$ then player 1 will have a unique best response which rules out such WPE. However, if $p = \frac{2}{3}$ then we have a mixed strategy WPBE as follows: player 1 plays B with probability $s \in \left(\frac{1}{3}, \frac{2}{3}\right)$, and player 2 will play U in information set I_T and will play D in information set I_B.

iv) $p < s < (1-p)$: (which implies $p < \frac{1}{2}$) This case is symmetric to case (iii) above. If $p \neq \frac{1}{3}$ then player 1 will have a unique best response which rules out such WPBE. However, if $p = \frac{1}{3}$ the we have a mixed strategy WPBE as follows: player 1 plays B with probability $s \in \left(\frac{1}{3}, \frac{2}{3}\right)$, and player 2 will play D in information set I_T and will play U in information set I_B.

To conclude, there exists a unique pure strategy WPBE as described earlier, and if p is randomly drawn from the interval $(0,1)$ then the pure strategy WPBE is the unique WPBE with probability 1. However, if $p = \frac{1}{3}$ or $p = \frac{2}{3}$ then in addition there exists a mixed strategy WPBE as described in cases (iii) and (iv) above.

3. Signalling with a Spaniard - Based on *The Princess Bride*

In *The Princess Bride*, the Dread Pirate Roberts is climbing up a rocky cliff in pursuit of Princess Buttercup. At the top of the cliff awaits Inigo Montoya, who has been ordered to duel Roberts to the death once he ascends the cliff. Being impatient, Montoya would like Roberts to hurry up the cliff, and has offered to throw down a rope to assist with the climb. The Dread Pirate Roberts does not know if he can trust Inigo Montoya, however, and must decide whether to accept or reject the help. Accepting help from an untrustworthy Montoya would likely injure Roberts since he will likely fall (We'll assume not to his death though, since he is the Dread Pirate Roberts, after all!).

The game proceeds as follows: First, Nature determines whether Inigo Montoya is trustworthy (with probability $p = 0.5$) or not. Montoya can then send a signal to The Dread Pirate Roberts of "I could give you my word as a Spaniard" (denoted as "Spaniard") or "I swear on the soul of my father, Domingo Montoya, you will reach the top alive" (denoted as "Father"). Finally, Roberts chooses whether to accept or reject the help from Montoya.

Inigo Montoya's payoffs are as follows. He receives 2 if his help is accepted and 0 otherwise. Montoya takes great pride in his father and receives a benefit of 2 for invoking his father's honor. If, however, he is untrustworthy, he must pay a cost of 3, as he has brought his father shame. Likewise, if he invokes his father's soul and his help is rejected, he must pay a cost of 2 due to the offense he receives.