# EconS 503 - Microeconomic Theory II
# Homework #5 - Due date: February 25th, 2019

1. **Collusion with probability of being caught - Harrington (2014).**[1] Consider an industry with $N$ firms. For generality, we do not assume whether they compete in quantities or prices yet, nor the inverse demand function or costs they face. Consider that firms are symmetric and in the Nash equilibrium of the unrepeated game, every firm earns profits $\pi^N$, so we label the present value of the noncollusive stream as

$$V^N \equiv \pi^N + \delta\pi^N + ... = \frac{1}{1-\delta}\pi^N.$$

When firms collude, each of them earns profit $\pi^C$, where $\pi^C > \pi^N$. When a firm unilaterally deviates from the collusive outcome, it earns a deviating profit of $\pi^D$, where $\pi^D > \pi^C$ in that period. Consider a standard Grim-Trigger strategy (GTS) where every firm chooses to collude in period $t = 1$, and continues to do so in subsequent periods $t > 1$ if all firms colluded in previous periods. If one firm did not cooperate in previous periods, however, all firms revert to the Nash equilibrium of the unrepeated game, earning $\pi^N$ thereafter (permanent punishment scheme). For simplicity, assume that all firms exhibit the same discount factor $\delta \in (0, 1)$.

   (a) Find the minimal discount factor $\delta$ that sustains this GTS as a subgame perfect equilibrium of the game.

   - After a history of cooperation, every firm $i$ keeps cooperating as long as

$$\underbrace{\frac{1}{1-\delta}\pi^C}_{\text{Cooperation}} \geq \underbrace{\pi^D}_{\text{Deviation}} + \underbrace{\frac{\delta}{1-\delta}\pi^N}_{\text{Permanent punishment}}$$

     which, after solving for discount factor $\delta$, yields

$$\delta \geq \widehat{\delta} \equiv \frac{\pi^D - \pi^C}{\pi^D - \pi^N}$$

     The above number is a positive number since the profit from deviating, $\pi^D$, satisfies $\pi^D > \pi^C$ and $\pi^D > \pi^N$ by definition.

   - *Comparative statics of cutoff* $\widehat{\delta}$:
     - Cutoff $\widehat{\delta}$ is increasing in $\pi^N$. Intuitively, as the profits from reverting to the Nash equilibrium of the unrepeated game $\pi^N$ increase, the punishment from deviation become less severe, ultimately making the deviation more attractive.

[1] Harrington, Joseph E. Jr. (2014) "Penalties and the Deterrence of Unlawful Collusion," *Economic Letters*, 124, pp. 33-36.

– In contrast, cutoff $\widehat{\delta}$ is decreasing in $\pi^C$. In words, this indicates that, when the profits from cooperation $\pi^C$ increase, deviation becomes less attractive and can be sustained under larger values of discount factor $\delta$.

– Finally, cutoff $\widehat{\delta}$ is increasing in the deviating profit $\pi^D$ since $\frac{\partial \widehat{\delta}}{\partial \pi^D} = \frac{\pi^C - \pi^N}{(\pi^D - \pi^N)^2} > 0$, since profits from cooperation $\pi^C$ satisfy $\pi^C > \pi^N$. Intuitively, when deviation becomes more attractive, the GTS can only be sustained for more restrictive conditions on discount factor $\delta$.

(b) For the rest of the exercise, let us assume that the cartel faces a exogeneous probability $p$ of being discovered, prosecuted, and convicted, by a regulatory agency such as the Federal Trade Commission. If caught and convicted in period $t$, a firm must pay a fine $F^t$, where $F^t = \beta F^{t-1} + f$. Parameter $1 - \beta$ can be understood as the depreciation rate, which we assume to satisfy $\beta \in (0, 1)$ to guarantee that the penalty is bounded. In addition, assume that $F^0 = 0$, so that $F^1 = f$, $F^2 = \beta F^1 + f$, and similarly for subsequent periods. Find the collusive value $V^C(F)$ given an accumulated penalty $F$. [*Hint*: Solve for $V^C(F)$ recursively.]

- The collusive value $V^C(F)$ is defined recursively, for period $t = 1$ and $t = 2$, as follows

$$V^C(F^1) = \pi^C + \underbrace{p\left[\frac{\delta}{1-\delta}\pi^N - F^2\right]}_{\text{Detected}} + \underbrace{(1-p)\delta V^C(F^2)}_{\text{Not detected}}$$

where penalties are $F^1 = \beta F^0 + f = f$ and $F^2 = \beta F^1 + f = \beta F + f$. More generally for any two periods $t$ and $t + 1$, the collusive value is defined as

$$V^C(F) = \pi^C + \underbrace{p\left[\frac{\delta}{1-\delta}\pi^N - (\beta F + f)\right]}_{\text{Detected}} + \underbrace{(1-p)\delta V^C(\beta F + f)}_{\text{Not detected}}$$

In words, the collusive value includes the collusive profit, $\pi^C$, and:

– If the cartel is detected, which happens with probability $p$, every firm must pay a penalty $\beta F + f$ after being detected, and a generate stream of Nash equilibrium profits $\pi^N$ thereafter since we assumed that after being detected firms can never form a cartel in the future.

– If the cartel is undetected, which occurs with probability $1 - p$, the stream of collusive payoffs starts in the next period, giving rise to continuation payoff $\delta V^C(\beta F + f)$.

- Since the collusive value $V^C(F)$ shows up in both the left- and right-hand side of the above expression, we can solve for $V^C(F)$ to obtain

$$V^C(F) = \pi^C + p\left[\frac{\delta}{1-\delta}\pi^N - (\beta F + f)\right] + (1-p)\delta V^C(\beta F + f)$$

which can be rearranged as follows

$$
\begin{aligned}
V^C(F) &= \pi^C + p\left[\frac{\delta}{1-\delta}\pi^N - (\beta F + f)\right] + (1-p)\delta\pi^C + p(1-p)\delta\frac{\delta}{1-\delta}\pi^N \\
&- p(1-p)\delta\left(\beta\left(\beta F + f\right) + f\right) + (1-p)^2\delta^2 V^C(\beta\left(\beta F + f\right) + f) \\
&= \left(\pi^C + p\pi^N\frac{\delta}{1-\delta}\right)\left[1 + \delta\left(1-p\right)\right] - pF\left[\beta + (1-p)\delta\beta^2\right] \\
&- pf\left[1 + \delta\left(1-p\right) + \delta\left(1-p\right)\beta\right] + (1-p)^2\delta^2 V^C(\beta\left(\beta F + f\right) + f) \\
&= \left(\pi^C + p\pi^N\frac{\delta}{1-\delta}\right)\sum_{t=0}^{T}\left[\delta\left(1-p\right)\right]^t - p\beta F\sum_{t=0}^{T}\left[\left(1-p\right)\beta\delta\right]^t \\
&- pf\sum_{t=0}^{T}\delta^t\left(1-p\right)^t\left(\sum_{s=0}^{t}\beta^s\right) + (1-p)^{T+1}\delta^{T+1}V^C\left(\beta^{T+1}F + f\sum_{t=0}^{T}\beta^t\right)
\end{aligned}
$$

From the transversality condition that

$$
\lim_{T\to\infty}(1-p)^{T+1}\delta^{T+1}V^C\left(\beta^{T+1}F + f\sum_{t=0}^{T}\beta^t\right) = 0
$$

which means the present discounted value of collusion at infinity is zero. Inserting this result in our above expression for $V^C(F)$ we obtain

$$
\begin{aligned}
V^C(F) &= \left(\pi^C + p\pi^N\frac{\delta}{1-\delta}\right)\frac{1-\left[\delta\left(1-p\right)\right]^{T+1}}{1-\delta\left(1-p\right)} - p\beta F\frac{1-\left[\left(1-p\right)\beta\delta\right]^{T+1}}{1-\beta\delta\left(1-p\right)} \\
&- pf\sum_{t=0}^{T}\delta^t\left(1-p\right)^t\frac{1-\beta^{t+1}}{1-\beta} \\
&= \left(\pi^C + p\pi^N\frac{\delta}{1-\delta}\right)\frac{1-\left[\delta\left(1-p\right)\right]^{T+1}}{1-\delta\left(1-p\right)} - p\beta F\frac{1-\left[\left(1-p\right)\beta\delta\right]^{T+1}}{1-\beta\delta\left(1-p\right)} \\
&- \frac{pf}{1-\beta}\frac{1-\left[\delta\left(1-p\right)\right]^{T+1}}{1-\delta\left(1-p\right)} + \frac{p\beta f}{1-\beta}\frac{1-\left[\beta\delta\left(1-p\right)\right]^{T+1}}{1-\beta\delta\left(1-p\right)}
\end{aligned}
$$

For an infinite horizon game, we take $T\to\infty$, such that

$$
\begin{aligned}
V^C(F) &= \left(\pi^C + p\pi^N\frac{\delta}{1-\delta}\right)\frac{1}{1-\delta\left(1-p\right)} - \frac{p\beta F}{1-\beta\delta\left(1-p\right)} \\
&- \frac{pf}{1-\beta}\frac{1}{1-\delta\left(1-p\right)} + \frac{p\beta f}{1-\beta}\frac{1}{1-\beta\delta\left(1-p\right)} \\
&= \frac{\pi^C + p\pi^N\frac{\delta}{1-\delta}}{1-\delta\left(1-p\right)} - \\
&\frac{p\beta F\left(1-\beta\right)\left[1-\delta\left(1-p\right)\right] + pf\left[1-\beta\delta\left(1-p\right)\right] - p\beta f\left[1-\delta\left(1-p\right)\right]}{\left(1-\beta\right)\left[1-\delta\left(1-p\right)\right]\left[1-\beta\delta\left(1-p\right)\right]} \\
&= \underbrace{\frac{\pi^C + p\pi^N\frac{\delta}{1-\delta}}{1-\delta\left(1-p\right)}}_{\substack{\text{Expected present value}\\ \text{of profits from the product market}}} - \underbrace{\frac{p\left(\beta\left[1-\delta\left(1-p\right)\right]F + f\right)}{\left[1-\delta\left(1-p\right)\right]\left[1-\beta\delta\left(1-p\right)\right]}}_{\text{Expected discounted penalty}}
\end{aligned}
$$

Intuitively, the first term represents the expected present value of profits from the product market, which includes the possibility of earning collusive profits $\pi^C$ for the periods that the cartel is undetected, or Nash equilibrium profits $\pi^N$ for all periods after the cartel is detected. The second term indicates the expected discounted penalty once the firm is discovered, prosecuted, and convicted.

(c) Write down the condition (inequality) expressing that every firm has incentives to collude, obtaining $V^C(F)$ rather than deviating. For simplicity, you can assume that if the cartel is convicted during the deviation period, it has no chances of being caught during the permanent punishment phase.

- Every firm cooperates if and only if

$$\underbrace{V^C(F)}_{\text{Collusion}} \geq \underbrace{\pi^D - p(\beta F + f)}_{\text{Expected profit from deviation}} + \underbrace{\frac{\delta}{1-\delta}\pi^N}_{\text{Permanent punishment}}$$

Intuitively, when the firm deviates from the collusive price the cartel can still be detected by regulatory authorities, which occurs with probability $p$, and thus charged with a penalty $\beta F + f$.

(d) The steady-state penalty is $F = \frac{f}{1-\beta}$, which is found by solving $F = \beta F + f$. Evaluate the collusive value $V^C(F)$ at this penalty, and insert your result in the condition you found in part (c) of the exercise. Rearrange and interpret.

- Evaluating the collusive value $V^C(F)$ at the steady-state penalty $F = \frac{f}{1-\beta}$, we obtain

$$V^C\left(\frac{f}{1-\beta}\right) = \frac{\pi^C + p\pi^N\left(\frac{\delta}{1-\delta}\right) - \frac{pf}{1-\beta}}{1-\delta(1-p)}$$

- Substituting the above results into part (c) yields

$$\frac{\pi^C + \frac{p\delta\pi^N}{1-\delta} - \frac{pf}{1-\beta}}{1-\delta(1-p)} \geq \pi^D - \frac{pf}{1-\beta} + \frac{\delta\pi^N}{1-\delta}$$

Rearranging, we find

$$\pi^C + \delta(1-p)\frac{pf}{1-\beta} + \delta(1-p)\pi^D \geq \pi^D + \frac{\delta\pi^N}{1-\delta}\left[1-\delta(1-p)-p\right]$$

and solving for discount factor $\delta$, we obtain

$$\delta \geq \widehat{\delta}(p) \equiv \frac{\pi^D - \pi^C}{(1-p)\left(\frac{pf}{1-\beta} + \pi^D - \pi^N\right)}$$

Note that cutoff $\widehat{\delta}(p)$ collapses to $\frac{\pi^D-\pi^C}{\pi^D-\pi^N}$ when the probability of cartel detection is zero ($p=0$) as in part (a) of the exercise. In contrast, when detection is perfect, $p=1$, cutoff $\widehat{\delta}(p)$ approaches infinity, thus indicating that condition $\delta \geq \widehat{\delta}(p)$ cannot hold for any admissible discount factor $\delta \in [0,1]$.

- Next, we differentiate the cutoff $\widehat{\delta}(p)$ with respect to $p$,

$$\frac{\partial\widehat{\delta}(p)}{\partial p}=\frac{\left(\pi^D-\pi^C\right)\left[\pi^D-\pi^N-(1-2p)\frac{f}{1-\beta}\right]}{\left[(1-p)\left(\frac{pf}{1-\beta}+\pi^D-\pi^N\right)\right]^2}\geq 0$$

  Intuitively, as detection becomes more likely, the minimal discount factor sustaining collusion $\widehat{\delta}(p)$ increases, since firms are less attracted to collude.

- Lastly, we differentiate the cutoff $\widehat{\delta}(p)$ with respect to $f$,

$$\frac{\partial\widehat{\delta}(p)}{\partial f}=-\frac{p\left(\pi^D-\pi^C\right)}{(1-\beta)(1-p)\left(\frac{pf}{1-\beta}+\pi^D-\pi^N\right)^2}\leq 0$$

  Intuitively, as the penalty becomes more severe, the minimal discount factor sustaining collusion $\widehat{\delta}(p)$ decreases, since firms have to face a larger penalty in expectation and have less to gain from collusion.

(e) *Bertrand competition.* Assume that firms compete a la Bertrand, selling homogeneous products with inverse demand function $p(Q)=1-Q$ where $Q$ denotes aggregate output. All firms face a symmetric marginal cost $c>0$. In this setting, every firm obtains zero profits in the Nash equilibrium of the unrepeated game, entailing $\pi^N=0$. If a firm unilaterally deviates from the collusive price (charging a price infinitely close, but below, the collusive price), it captures all industry sales, earning a profit $\pi^D=N\pi^C$ during the deviating period. Evaluate your results from part (d) of the exercise in this context. Then discuss whether collusion becomes easier to sustain when the penalty $f$ increases; and when the number of firms $N$ increases.

- We first need to find the profits that firms obtain from cooperating by setting a collusive price. Since the inverse demand function is $p(Q)=1-Q$, the direct demand function becomes $Q=1-p$, implying that the joint-profit maximization problem in this setting is

$$\max_{p\geq 0}\ pQ-cQ=p(1-p)-c(1-p)$$

  Differentiating with respect to price $p$ yields $1-2p+c=0$, and solving for $p$ we obtain a collusive price of $p^C=\frac{1+c}{2}$. Therefore, collusive profits for the industry are

$$N\pi^C=p^C(1-p^C)-c(1-p^C)=\left(1-\frac{1+c}{2}\right)\left(\frac{1+c}{2}-c\right)=\frac{(1-c)^2}{4}$$

  which implies that every firm's collusive profits, $\pi^C$, is $\frac{(1-c)^2}{4N}$.

- Evaluating our above results from part (d) at profits $\pi^N=0$, $\pi^C=\frac{(1-c)^2}{4N}$, and $\pi^D=N\pi^C=\frac{(1-c)^2}{4}$, we find that every firm cooperates after a history of cooperation if and only if

$$\delta \geq \overline{\delta} = \frac{\frac{(1-c)^2}{4} - \frac{(1-c)^2}{4N}}{(1-p)\left[\frac{pf}{1-\beta} + \frac{(1-c)^2}{4}\right]}$$

$$= \frac{1}{1-p} \cdot \frac{\frac{(1-c)^2}{4}\left(1-\frac{1}{N}\right)}{\frac{4pf+(1-\beta)(1-c)^2}{4(1-\beta)}}$$

$$= \frac{N-1}{N(1-p)} \cdot \frac{(1-\beta)(1-c)^2}{4pf+(1-\beta)(1-c)^2}$$

- *Comparative statics of the cutoff* $\overline{\delta}$:
  - Differentiating the cutoff $\overline{\delta}$ with respect to $N$, we obtain

$$\frac{\partial \overline{\delta}}{\partial N} = \frac{1}{N^2(1-p)} \cdot \frac{(1-\beta)(1-c)^2}{4pf+(1-\beta)(1-c)^2} \geq 0$$

    so that as the number of firm increases, a higher discount rate is needed to sustain collusion because the firm can capture a larger profit from deviation, $\pi^D$, relative to their collusion profit, $\pi^C$.
  - Differentiating the cutoff $\overline{\delta}$ with respect to $f$, we obtain

$$\frac{\partial \overline{\delta}}{\partial f} = -\frac{N-1}{N(1-p)} \frac{4p(1-\beta)(1-c)^2}{\left[4pf+(1-\beta)(1-c)^2\right]^2} \leq 0$$

    Intuitively, as the penalty becomes more severe, the minimal discount factor sustaining collusion decreases, since firms have to face a larger penalty in expectation and have less to gain from collusion.

2. **Cournot competition when all firms are uninformed - Continuous costs**. Consider two firms competing a la Cournot where every firm $i$ is uninformed about its rival's production costs. Assume that marginal costs are drawn from a continuous, rather than discrete, distribution. In particular, consider that every firm $i$'s marginal cost $c_i$ is drawn from a uniform distribution $c_i \sim U[0, \overline{c}]$ where $\overline{c} > 0$. For simplicity, we assume that the costs of firms $i$ and $j$, that are, $c_i$ and $c_j$, respectively, are independently and identically distributed.

   (a) Find the best response function for firm $i$, $q_i(q_j)$.

   - Every firm $i$ chooses output $q_i \geq 0$ to solve

$$\max_{q_i \geq 0} \ \pi_i(q_i) = \int_0^{\overline{c}} (1 - q_i - q_j(c_j)) q_i dF(c_j) - c_i q_i$$

$$= (1-q_i) q_i \int_0^{\overline{c}} dF(c_j) - q_i \int_0^{\overline{c}} q_j(c_j) dF(c_j) - c_i q_i$$

$$= (1 - q_i - c_i) q_i - q_i \int_0^{\overline{c}} q_j(c_j) dF(c_j)$$

Assuming interior solutions, that is, $q_i > 0$, the first order condition satisfies

$$\frac{\partial \pi_i (q_i)}{\partial q_i} = 1 - 2q_i - c_i - \int_0^{\overline{c}} q_j (c_j)\, dF(c_j) = 0$$

Solving for $q_i$, we find the best response function

$$\begin{aligned} q_i(q_j) &= \frac{1-c_i}{2} - \frac{1}{2}\int_0^{\overline{c}} q_j(c_j)\, dF(c_j) \\ &= \frac{1-c_i}{2} - \frac{1}{2} E_i [q_j(c_j)] \end{aligned}$$

where the last term denotes firm $i$'s expectation of firm $j$'s output.

(b) Use your results from part (a) to find the Bayesian Nash Equilibrium (BNE) of the game.

- Since firms $i$ and $j$ are symmetric, we impose symmetry on their output expectation, as follows

$$E_i [q_j(c_j)] = E_j [q_i(c_i)]$$

which means that firm $i$'s expectation of firm $j$'s output coincides with firm $j$'s expectation of firm $i$'s output, since both firms use the same cost distribution function to assess their expectations. Therefore, applying the expectation operator to the best response function we found in part (a), yields

$$\begin{aligned} E_j [q_i(q_j)] &= E_j \left[\frac{1-c_i}{2}\right] - \frac{1}{2} E_j [E_i [q_j(c_j)]] \\ &= E_j \left[\frac{1-c_i}{2}\right] - \frac{1}{2} E_j [E_j [q_i(c_i)]] \\ &= E_j \left[\frac{1}{2}\right] - E_j \left[\frac{c_i}{2}\right] - \frac{1}{2} E_j [q_i(c_i)] \\ &= \frac{1}{2} - \frac{\overline{c}}{4} - \frac{1}{2} E_j [q_i(c_i)] \end{aligned}$$

where the second displayed equation follows from symmetry in firms' expectations. The second term in the last displayed equation follows from

$$\begin{aligned} E_j \left[\frac{c_i}{2}\right] &= \frac{1}{2}\int_0^{\overline{c}} c_i f(c_i)\, dc_i = \frac{1}{2}\int_0^{\overline{c}} c_i \frac{1}{c_i} dc_i \\ &= \frac{1}{2}\int_0^{\overline{c}} dc_i = \frac{1}{2}[c_i]_0^{\overline{c}} = \frac{\overline{c}}{2}. \end{aligned}$$

Rearranging, we obtain

$$\begin{aligned} \frac{3}{2} E_j [q_i] &= \frac{1}{2} - \frac{\overline{c}}{4} \\ \Rightarrow E_j [q_i] &= \frac{2-\overline{c}}{6} \end{aligned}$$

and by symmetry, $E_i [q_j] = \frac{2-\overline{c}}{6}$.

- Substituting the expected output back into the best response function, we find firm $i$'s equilibrium output

$$\begin{aligned} q_i^* &= \frac{1-c_i}{2} - \frac{1}{2} \times \frac{2-\overline{c}}{6} \\ &= \frac{4+\overline{c}-6c_i}{12} \end{aligned}$$

and similarly, $q_j^* = \frac{4+\overline{c}-6c_j}{12}$.

(c) How do the equilibrium output levels you found in part (b) are affected by changes in $c_i$, $c_j$, and $\overline{c}$? Interpret.

- Note that only $q_i^*$ but not $q_j^*$ is a function of $c_i$, in particular,

$$\begin{aligned} \frac{\partial q_i^*}{\partial c_i} &= -\frac{1}{2} < 0 \\ \frac{\partial q_j^*}{\partial c_i} &= 0 \end{aligned}$$

As firm $i$'s cost increase, it reduces its equilibrium output while firm $j$'s is unaffected since firm $j$ bases its production decision on firm $i$'s average cost rather than on its realization, $c_i$, given that firm $i$ cannot observe such realization.

- Differentiating equilibrium output with respect to the maximum cost $\overline{c}$, yields

$$\begin{aligned} \frac{\partial q_i^*}{\partial \overline{c}} &= \frac{1}{12} > 0 \\ \frac{\partial q_j^*}{\partial \overline{c}} &= \frac{1}{12} > 0 \end{aligned}$$

In words, as the maximum cost increases, both firms increase their equilibrium output.

3. **Stackelberg game under incomplete information**. Consider a Stackelberg game between a leader (firm 1) and a follower (firm 2). Inverse demand function is given by $p(Q) = 1 - Q$, where $Q$ denotes aggregate output. The leader's marginal cost $MC_1 = \frac{1}{4}$ is common knowledge among the players. Intuitively, the leader is the industry incumbent, and all firms can estimate its costs with relative accuracy. However, the follower's marginal costs are either high $\left(MC_2 = \frac{1}{3}\right)$ or low $\left(MC_2 = \frac{1}{5}\right)$ with probabilities $p$ and $1-p$ respectively. The entrant is a newcomer, and thus the leader cannot perfectly observe the follower's costs.

(a) Find the follower's best response function.

- Let $q_1$, $q_2^L$ and $q_2^H$ denote the output of firm 1, low- and high-cost firm 2 respectively.

- When firm 2 has low costs, it chooses $q_2^L \geq 0$ that solves the following expected profit maximization problem:

$$\begin{aligned}\max_{q_2^L \geq 0} \ \pi_2^L\left(q_2^L\right) &= \left(1 - q_1 - q_2^L\right) q_2^L - \frac{1}{5} q_2^L \\ &= \left(\frac{4}{5} - q_1 - q_2^L\right) q_2^L\end{aligned}$$

Assuming interior solutions, that is, $q_2^L > 0$, the first order condition satisfies

$$\frac{\partial \pi_2^L\left(q_2^L\right)}{\partial q_2^L} = \frac{4}{5} - q_1 - 2q_2^L = 0$$

such that the best response function of firm 2 when its costs are low becomes

$$q_2^L\left(q_1\right) = \frac{2}{5} - \frac{1}{2} q_1$$

which originates at $\frac{2}{5}$ and decreases in firm 1's output, $q_1$, at a rate of $1/2$.

- When firm 2 has high costs, it chooses $q_2^H \geq 0$ that solves the following expected profit maximization problem:

$$\begin{aligned}\max_{q_2^H \geq 0} \ \pi_2^H\left(q_2^H\right) &= \left(1 - q_1 - q_2^H\right) q_2^H - \frac{1}{3} q_2^H \\ &= \left(\frac{2}{3} - q_1 - q_2^H\right) q_2^H\end{aligned}$$

Assuming interior solutions, that is, $q_2^H > 0$, the first order condition satisfies

$$\frac{\partial \pi_2^H\left(q_2^H\right)}{\partial q_2^H} = \frac{2}{3} - q_1 - 2q_2^H = 0$$

such that the best response function of firm 2 when its costs are high becomes

$$q_2^H\left(q_1\right) = \frac{1}{3} - \frac{1}{2} q_1$$

which originates at a lower vertical intercept than when its costs are low, but also decreases in $q_1$ at a rate of $1/2$.

(b) Find the leader's profit-maximizing output, and summarize the subgame perfect Nash equilibrium of the game.

- Firm 1 chooses its output $q_1 \geq 0$ that solves the following expected profit maximization problem:

$$\begin{aligned}\max_{q_1 \geq 0} \ \pi_1\left(q_1\right) &= (1-p)\left(1 - q_1 - q_2^L\left(q_1\right)\right) q_1 + p\left(1 - q_1 - q_2^H\left(q_1\right)\right) q_1 - \frac{1}{4} q_1 \\ &= \left(\frac{3}{4} - q_1 - (1-p)\left(\frac{2}{5} - \frac{q_1}{2}\right) - p\left(\frac{1}{3} - \frac{q_1}{2}\right)\right) q_1 \\ &= \left(\frac{7}{20} + \frac{p}{15} - \frac{q_1}{2}\right) q_1\end{aligned}$$

Assuming interior solutions, that is, $q_1 > 0$, the first order condition satisfies

$$\frac{\partial \pi_1(q_1)}{\partial q_1} = \frac{7}{20} + \frac{p}{15} - q_1 = 0$$

such that the output of firm 1(leader) becomes

$$q_1^* = \frac{7}{20} + \frac{p}{15}$$

Substituting the above into the best response functions of firm 2, equilibrium output for the follower is

$$q_2^{L*} = \frac{2}{5} - \frac{1}{2}\left(\frac{7}{20} + \frac{p}{15}\right) = \frac{27 - 4p}{120}$$
$$q_2^{H*} = \frac{1}{3} - \frac{1}{2}\left(\frac{7}{20} + \frac{p}{15}\right) = \frac{19 - 4p}{120}$$

- Therefore, the subgame perfect Nash equilibrium of the game is

$$\{q_1^*, q_2^{L*}, q_2^{H*}\} = \left\{\frac{21 + 4p}{60}, \frac{27 - 4p}{120}, \frac{19 - 4p}{120}\right\}$$

such that when firm 2 is more likely to face high costs (i.e., probability $p$ increases), the output of firm 1 increases but the output of both low-cost and high-cost firm 2 decreases.

(c) How do your results in parts (a) and (b) would change if the leader could perfectly observe the follower's marginal costs when these costs are high? What if the follower's costs are low?

- If the leader could perfectly observe the follower's marginal costs, and these costs are high, we have that $p = 1$, which implies that the above output levels become

$$q_1^* = \frac{5}{12} \text{ and } q_2^{H*} = \frac{19}{120}.$$

- Similarly, if the leader could perfectly observe the follower's marginal costs, and these costs are low, we have that $p = 0$, which implies that the above output levels become

$$q_1^* = \frac{7}{20} \text{ and } q_2^{L*} = \frac{9}{40}.$$

4. **Bargaining under incomplete information.** Consider the following bargaining game between a seller and a buyer. The buyer is privately informed about his value for the good, $v$, drawn from a cumulative distribution function $F_1(v)$ with positive density in its support. The game starts when the seller makes an offer to the buyer in the first period (a price $p_1$), which the buyer chooses to accept or reject. If the buyer accepts, the game is over, with payoff $p_1$ for the seller and $v - p_1$ for the buyer. If he rejects, the seller observes the rejection, and the game proceeds to the second period. In the second period, the seller makes an offer to the buyer (a price $p_2$), and the buyer chooses

to accept or reject it. If the buyer rejects offer $p_2$, both players' payoff is zero. If the buyer accepts offer $p_2$, the seller's payoff is $\delta p_2$, where $\delta \in [0, 1]$ denotes both players' discount factor; and the buyer's payoff is $\delta (v - p_2)$. The seller's prior distribution in the first period is $F_1 (v)$. If the buyer rejects offer $p_1$, the seller updates his beliefs in the second period to $F_2 (v|p_1)$. We next study the PBE of the game by analyzing the second-period game first, and then moving on to the first period.

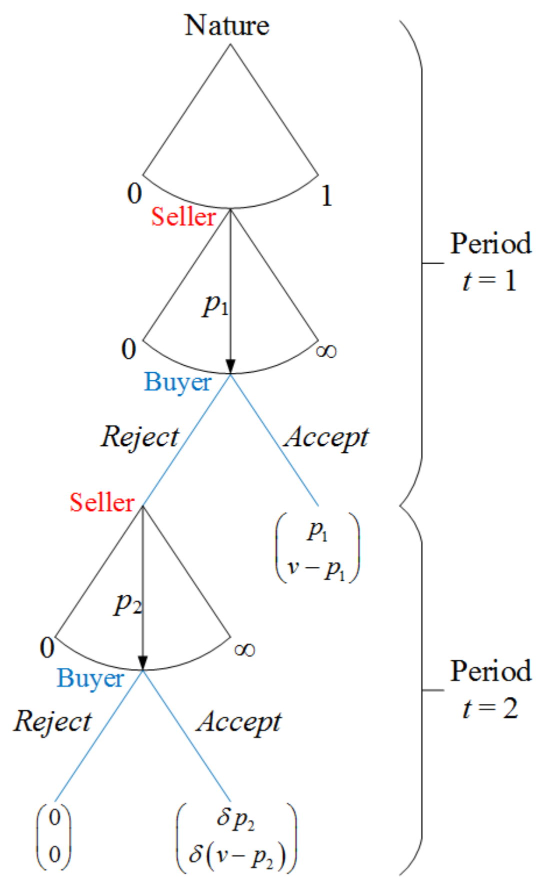


Figure 1. Game tree of the sequential bargaining game.

(a) *Second period.* Find the buyer's acceptance rule in the second period. Anticipating this acceptance rule, find the seller's offer $p_2$ in this period. [*Hint*: Consider the critical type of buyer, who is indifferent between accepting and rejecting the offer in period 1.]

- The buyer's decision rule in period 2 is characterized by accepting price $p_2$ when $v \geq p_2$, but reject it otherwise. Let $t (p_1)$ be the critical type of buyer who rejects offer $p_1$ in period 1, then the buyer's decision rule in period 1 is characterized by accept price $p_1$ if $v \geq t (p_1)$, but reject $p_1$ otherwise; as illustrated in figure 2.

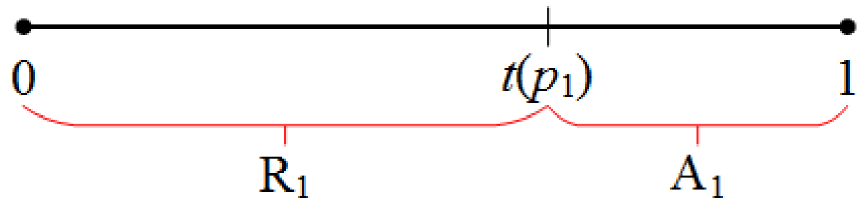


Figure 2. Buyer's decision rule in period 1.

Therefore, the cumulative probability that the buyer rejects offer $p_2$ in period 2, conditional on having rejected offer $p_1$ in period 1, is

$$\begin{aligned} F_2(p_2|p_1) &= \frac{F_1(p_1,p_2)}{F_1(p_1)} \\ &= \frac{\int_0^{p_2}\int_0^{t(p_1)} f_1(p_1,v)\,dp_1 dv}{\int_0^{t(p_1)} f_1(p_1)\,dp_1} \\ &= \frac{\int_0^{1}\int_0^{p_2} f_1(p_1,v)\,dv dp_1}{\int_0^{t(p_1)} f_1(p_1)\,dp_1} \\ &= \frac{F_1(p_2)}{F_1(t(p_1))} \end{aligned}$$

in which the second last line stems from the fact that $p_2 \leq t(p_1)$, because if offer $p_1$ is rejected in period 1, then the seller knows the buyer's critical type so that setting any offer $p_2 > t(p_1)$ would lead to a sure rejection by the same buyer in period 2.

- Then, the seller maximizes the following expected profit maximization problem:

$$\begin{aligned} \max_{p_2\geq 0} E[\pi(p_2|t(p_1))] &= \underbrace{F_2(p_2|p_1)}_{\text{Prob}(v<p_2)} \times \underbrace{0}_{\text{rejection}} + \underbrace{[1-F_2(p_2|p_1)]}_{\text{Prob}(v\geq p_2)} \times \underbrace{p_2}_{\text{acceptance}} \\ &= \left[1-\frac{F_1(p_2)}{F_1(t(p_1))}\right]p_2 \end{aligned}$$

Differentiating the above expression with respect to $p_2$, and assuming interior solutions, that is, $p_2 > 0$, the optimal price $p_2^*$ in period 2 implicitly solves

$$1-\frac{F_1(p_2)}{F_1(t(p_1))} = \frac{f_1(p_2)\,p_2}{F_1(t(p_1))}$$

(b) Assuming a uniformly distributed valuation, where $v \sim F[0,1]$, what is the seller's offer $p_2$ in the second period? [*Hint*: It should be an expression in terms of offer $p_1$.]

- Since $F_1(v) = v$, the pricing formula in part (a) becomes

$$1-\frac{p_2}{t(p_1)} = \frac{p_2}{t(p_1)}$$

which, after rearranging, yields

$$p_2(p_1) = \frac{t(p_1)}{2}$$

- For the critical buyer with valuation $t(p_1)$, he is indifferent between accepting the first period offer and rejecting it in anticipation of the second period offer, yielding

$$\underbrace{t(p_1) - p_1}_{\text{gain from accepting } p_1 \text{ today}} = \underbrace{\delta [t(p_1) - p_2(p_1)]}_{\text{gain from accepting } p_2 \text{ tomorrow}}$$

Substituting the optimal pricing function $p_2(p_1)$ into the above expression, we have

$$t(p_1) - p_1 = \delta \left[ t(p_1) - \frac{t(p_1)}{2} \right]$$

which, after rearranging, yields

$$t(p_1) = \frac{2p_1}{2-\delta}$$

- Substituting the critical type back into the optimal pricing function $p_2(p_1)$, we have

$$p_2(p_1) = \frac{p_1}{2-\delta}$$

which depends on the seller's offer $p_1$ in period 1, as illustrated in figure 3.

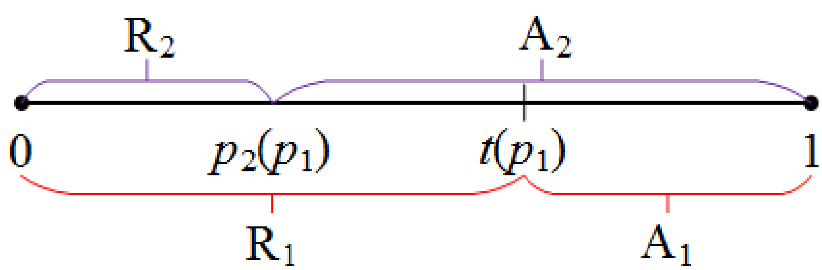


Figure 3. Buyer's decision rule in period 2.

(c) *First period.* Find the buyer's acceptance rule in the first period. (In this case, note that the buyer considers his equilibrium payoff in the second period if he rejects $p_1$ now.) Anticipating the buyer's acceptance rule, find the seller's offer $p_1$ in this period.

- The seller chooses $p_1$ to maximize the following expected profit maximization problem:

$$\max_{p_1 \geq 0} E[\pi(p_1)] = \underbrace{[1 - F_1(t(p_1))]}_{\text{Prob}(v \geq p_1)} \times \underbrace{p_1}_{\text{accept in period 1}}$$

$$+ \underbrace{\delta F_1(t(p_1)) \left[ 1 - \frac{F_1(p_2(p_1))}{F_1(t(p_1))} \right]}_{\text{Prob}(p_2 < v \leq p_1)} \times \underbrace{p_2(p_1)}_{\text{rejected in period 1 but accepted in period 2}}$$

$$+ \underbrace{\frac{F_1(p_2(p_1))}{F_1(t(p_1))}}_{\text{Prob}(v < p_2)} \times \underbrace{0}_{\text{rejected in both periods}}$$

which simplifies the above maximization problem to

$$\begin{aligned}\max_{p_1 \geq 0} E\left[\pi\left(p_1\right)\right] &= \left[1-\frac{2p_1}{2-\delta}\right] p_1+\frac{2\delta p_1}{2-\delta}\left[1-\frac{p_1}{2-\delta} \frac{2-\delta}{2p_1}\right] \frac{p_1}{2-\delta} \\ &= p_1-\frac{2p_1^2}{2-\delta}+\frac{\delta p_1^2}{(2-\delta)^2} \\ &= p_1-\frac{4-3\delta}{(2-\delta)^2} p_1^2\end{aligned}$$

Differentiating the above expression with respect to $p_1$, we find

$$1+2p_1 \frac{4-3\delta}{(2-\delta)^2} \leq 0$$

Assuming an interior solution (that is, $p_1>0$) and solving for $p_1$, we obtain that the optimal price $p_1^*$ in period 1 is

$$p_1^*=\frac{2(2-\delta)^2}{4-3\delta}$$

- Substituting $p_1^*$ into $p_2\left(p_1\right)$, the optimal price in period 2, $p_2^*$, becomes

$$p_2^*=\frac{2(2-\delta)}{4-3\delta}$$

and the critical bidder, $t^*$, is given by

$$t^*=\frac{4(2-\delta)}{4-3\delta}$$

Figure 4a plots the optimal prices as a function of the discount factor $\delta$, while figure 4b illustrates the critical bidder $t^*$ as a function of $\delta$ as well. We interpret the comparative statics of our results in part (d) below.

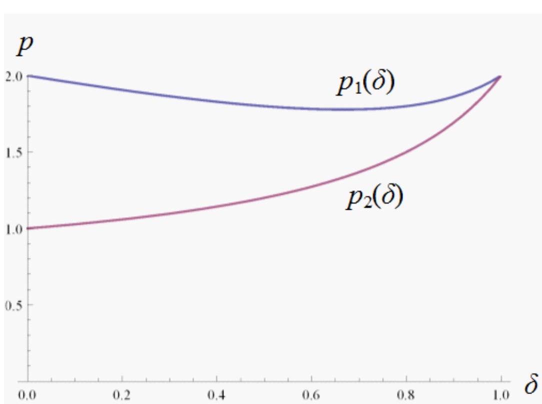


Figure 4a. Optimal prices.

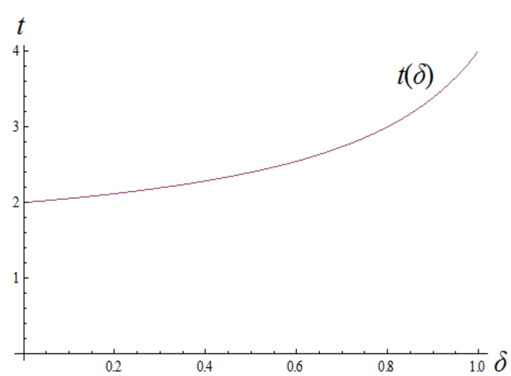


Figure 4b. Cutoff rule.

(d) *Numerical Example.* Evaluate the optimal price in period 1, in period 2, and valuation of the critical bidder when $\delta=0.5$. Then evaluate your results again when $\delta=0$, and when $\delta=1$.

- When $\delta = 0.5$, we find

$$p_1^* = \frac{2\left(2-\frac{1}{2}\right)^2}{4-3\times\frac{1}{2}} = 1.8$$
$$p_2^* = \frac{2\left(2-\frac{1}{2}\right)}{4-3\times\frac{1}{2}} = 1.2$$
$$t^* = \frac{4\left(2-\frac{1}{2}\right)}{4-3\times\frac{1}{2}} = 2.4$$

- When $\delta = 0$, we obtain

$$p_1^* = \frac{2\left(2-0\right)^2}{4-3\times 0} = 2$$
$$p_2^* = \frac{2\left(2-0\right)}{4-3\times 0} = 1$$
$$t^* = \frac{4\left(2-0\right)}{4-3\times 0} = 2$$

- When $\delta = 1$, we have

$$p_1^* = \frac{2\left(2-1\right)^2}{4-3\times 1} = 2$$
$$p_2^* = \frac{2\left(2-1\right)}{4-3\times 1} = 2$$
$$t^* = \frac{4\left(2-1\right)}{4-3\times 1} = 4$$

From the above numerical examples, we see that as discount rate $\delta$ increases, the seller can offer less discount in period 2 (relative to period 1) because the buyer cares more about future payoff. In the extreme case that $\delta = 1$, the first and second period prices coincide, that is, $p_1^* = p_2^* = 2$. However, when $\delta = 0$, the second period price is one-half of the first period price, that is, $p_2^* = 1 = \frac{p_2^*}{2}$. Whereas, when discount rate is in between, the seller offers a larger discount to a less forward-looking buyer (i.e., a lower discount rate) so as to make him indifferent between accepting the offer in period 1 and in period 2, as illustrated in figure 4.

5. **First-price auction with entry fees.** Consider a first-price auction with $N$ bidders. Every bidder $i$'s valuation, $v_i$, is distributed according to a cumulative distribution function $F(v_i)$, with positive support, i.e., $f(v_i) > 0$ for all $v_i \in [0, \overline{v}]$. Consider the following two-stage game: in the first stage, the seller sets an entry fee $E \geq 0$ that every participating bidder must pay, otherwise his bid is ignored; in the second stage, every bidder $i$ independently and simultaneously submit his bid for the object.

   (a) *Second stage.* Starting from the second stage, find the optimal bidding function for bidder $i$, $b_i(v_i)$ [*Hint*: Assume that there exists a critical bidder whose valuation $v_e$ makes him indifferent between participation or not, given a positive entry fee $E$].

- Every bidder $i$'s expected utility maximization problem is

$$\max_{b_i \geq 0} EU_i(b_i) = prob\{win\}(v_i - b_i) - E$$

where the entry fee, $E$, is a constant that bidder $i$ must pay when he participates in the auction, whether he wins the object or not.

- The probability of bidder $i$ winning the object is analogous to the standard First Price Auction without entry fee, which is given by

$$prob\{win\} = [F(v_i)]^{N-1}$$

when his valuation exceeds other $N-1$ bidders, $v_i \geq v_j$ for $j \neq i$, $j \in \{1, \ldots, N\}$. Note that for a given bidding strategy, $b : [0, \overline{v}] \rightarrow \mathbb{R}_+$, that is, $b_i(v_i) = b_i$, we can define its inverse, $b_i^{-1}(b_i) = v_i$, implying that the cumulative distribution function, which represents the probability mass of valuation below his, can be rewritten as

$$F(v_i) = F\left(b_i^{-1}(b_i)\right)$$

such that bidder $i$'s expected utility maximization problem becomes

$$\max_{b_i \geq 0} EU_i(b_i) = \left[F\left(b_i^{-1}(b_i)\right)\right]^{N-1}(v_i - b_i) - E$$

- We assume that the bidding function $b_i(v_i)$ is monotonically increasing in $v_i$; which we will demonstrate later. In addition, let us define a "critical bidder" with valuation $v_e$ and bid $b_e(v_e)$, where $v_e$ solves[2]

$$[F(v_e)]^{N-1}(v_e - b_e) - E = 0$$

which means that his expected utility from participating in the auction (given the entry fee $E$) is zero. Differentiating bidder $i$'s expected utility with respect to his bid $b_i$ yields

$$(N-1)\left[F\left(b_i^{-1}(b_i)\right)\right]^{N-2} f\left(b_i^{-1}(b_i)\right) \frac{\partial b_i^{-1}(b_i)}{\partial b_i}(v_i - b_i) - \left[F\left(b_i^{-1}(b_i)\right)\right]^{N-1} = 0$$

Since the inverse of the bidding function yields the bidder's valuation, $b_i^{-1}(b_i) = v_i$, and the derivative of this inverse can be written as $\frac{db_i^{-1}(b_i)}{db_i} = \frac{1}{b_i'\left(b_i^{-1}(b_i)\right)}$, the above expression becomes

$$(N-1)[F(v_i)]^{N-2} f(v_i)(v_i - b_i) = b_i'(v_i)[F(v_i)]^{N-1}$$

Further rearranging, we obtain

$$(N-1)[F(v_i)]^{N-2} f(v_i) b_i(v_i) + [F(v_i)]^{N-1} b_i'(v_i) = (N-1)[F(v_i)]^{N-2} f(v_i) v_i$$

[2] We need this condition to ensure a one-to-one mapping between the entry fee and the critical bidder's valuation. The monotonically increasing bidding function $b_i(v_i)$ ensures that bidders with valuations above $v_e$ obtain a positive utility from participating in the auction (despite the entry fee $E$) and thus submit a positive bid for the object.

The left-hand side is $\frac{d\left[[F(v_i)]^{N-1}b_i(v_i)\right]}{dv_i}$. Hence,

$$\frac{\partial\left[\left[F\left(v_i\right)\right]^{N-1} b_i\left(v_i\right)\right]}{\partial v_i}=(N-1)\left[F\left(v_i\right)\right]^{N-2} f\left(v_i\right) v_i$$

- Integrating the above expression (right-hand side by parts) with respect to $v_i$, and taking the indifferent bidder's valuation $v_e$ as the lower bound of integration, yields

$$\begin{aligned}
\int_{v_e}^{v_i} \frac{d\left[\left[F(x)\right]^{N-1} b_i(x)\right]}{dv_i} dx &= \int_{v_e}^{v_i}(N-1)\left[F(x)\right]^{N-2} f(x)\, x dx \\
\Rightarrow \left[\left[F(x)\right]^{N-1} b_i(x)\right]_{v_e}^{v_i} &= \left[\left[F(x)\right]^{N-1} x\right]_{v_e}^{v_i}-\int_{v_e}^{v_i}\left[F(x)\right]^{N-1} dx
\end{aligned}$$

  We can then reorder the terms in the above expression as follows:

$$\left[F\left(v_i\right)\right]^{N-1} b_i\left(v_i\right)=\left[F\left(v_i\right)\right]^{N-1} v_i-\left[F\left(v_e\right)\right]^{N-1}\left[v_e-b_e\left(v_e\right)\right]-\int_{v_e}^{v_i}\left[F(x)\right]^{N-1} dx$$

  Substituting the indifferent bidder condition, $\left[F\left(v_e\right)\right]^{N-1}\left(v_e-b_e\right)-E=0$, into the above expression, and rearranging, we obtain

$$b_i\left(v_i\right)=v_i-\underbrace{\frac{E+\int_{v_e}^{v_i}\left[F(x)\right]^{N-1} dx}{\left[F\left(v_i\right)\right]^{N-1}}}_{\text{bid shading}}$$

  Note that when entry fees are absent, $E=0$, the equilibrium bidding function collapses to the standard expression found in previous exercises.

- Lastly, we show that the equilibrium bidding function, $b_i\left(v_i\right)$, is monotonically increasing in the bidder's valuation, $v_i$, since

$$\begin{aligned}
\frac{db_i\left(v_i\right)}{dv_i} &= 1-\frac{\left[F\left(v_i\right)\right]^{2N-2}-(N-1)\left[F\left(v_i\right)\right]^{N-2}\left(E+\int_{v_e}^{v_i}\left[F(x)\right]^{N-1} dx\right)}{\left[F\left(v_i\right)\right]^{2N-2}} \\
&= \frac{N-1}{\left[F\left(v_i\right)\right]^{N}}\left(E+\int_{v_e}^{v_i}\left[F(x)\right]^{N-1} dx\right)>0.
\end{aligned}$$

(b) How are equilibrium bids affected by an increase in the entry fee $E$? Do they limit participation in the auction?

- As in other games on first-price auctions, the second term represents bidder $i$'s bid shading, which is increasing in the entry fee $E$. Intuitively, the entry fee affects all bidders uniformly, inducing those with valuations below $v_e$ to not participate in the auction and, in addition, reducing the bid of those who participate. Intuitively, every participating bidder expects a lower expected utility in equilibrium, both if he wins and if he loses the auction, leading him to decrease his bid (larger bid shading) to compensate for his lower expected utility.

(c) Assume that bidders' valuations are uniformly distributed, that is, $F(v_i) = v_i$ for all $v_i \sim U[0, \overline{v}]$. Evaluate the optimal bidding function found in part (a).

- Since in this context $F(v_i) = v_i$ for all $v_i \in [0, \overline{v}]$, the optimal bidding function $b_i(v_i)$ becomes

$$b_i(v_i) = v_i - \frac{E + \int_{v_e}^{v_i} x^{N-1} dx}{v_i^{N-1}}$$

and solving the integral, we obtain bidder $i$'s bidding function, $b_i(v_i)$, as follows:

$$\begin{aligned} b_i(v_i) &= v_i - \frac{NE + \left[x^N\right]_{v_e}^{v_i}}{N v_i^{N-1}} \\ &= v_i - \underbrace{\frac{NE + v_i^N - v_e^N}{N v_i^{N-1}}}_{\text{Bid shading}} \end{aligned}$$

and the indifferent bidder's bid, $b_e(v_e)$, solves

$$v_e^{N-1}(v_e - b_e(v_e)) = E$$

(d) *First stage.* Anticipating the optimal bidding function $b_i(v_i)$ you found in part (a), what is the optimal entry fee $E^*$ that the seller sets in the first stage to maximize his expected revenue from the auction? For simplicity, assume that the critical bidder, who is indifferent between participation or not, submits a bid, $b_e(v_e) = 0$.

- We first find the seller's expected revenue from the auction, and then differentiate it with respect to the reservation price $r$, to identify the revenue maximizing reservation price $r^*$.
- *Finding the seller's revenue from the auction.* For compactness, let us define $G(x) = (F[x])^{N-1}$ to be the joint cumulative probability density function for $N-1$ bidders, where valuation $x$ satisfies $x \in [0, \overline{v}]$. Then the above optimal bidding function can be rewritten as

$$\begin{aligned} b_i(v_i) &= v_i - \frac{E + \int_{v_e}^{v_i} G(x)\, dx}{G(v_i)} \\ &= \frac{1}{G(v_i)} \left[ G(v_i)\, v_i - E + G(v_e)\, v_e - G(v_e)\, v_e - \int_{v_e}^{v_i} G(x)\, dx \right] \\ &= \frac{1}{G(v_i)} \int_{v_e}^{v_i} x g(x)\, dx \end{aligned}$$

by the fact that $G(v_e)\, v_e = E$ and the opposite of integration by parts.
From an ex-ante point of view (before observing his own valuation for the object), bidder $i$'s expected payment to the seller is given by the probability of winning the auction times the bid he pays for the object upon winning,

that is,

$$\begin{aligned}
\pi_i\left(v_i | v_i \geq v_e\right) &= prob(win) \times b_i\left(v_i\right) + E \\
&= G\left(v_i\right) \times \frac{1}{G\left(v_i\right)} \int_{v_e}^{v_i} x g(x)\, dx + E \\
&= \int_{v_e}^{v_i} x g(x)\, dx + E
\end{aligned}$$

where the second line indicates that, as discussed in previous parts of the exercise, bidder $i$ wins the auction if his valuation $v_i$ is above everyone else's, that is, $v_i \geq v_j$ for every bidder $j \neq i$; and in addition, he participates in the auction and pays a participation fee $E$. The probability of his valuation exceeding that of every other bidder is given by $\left(F\left[v_i\right]\right)^{N-1}$, and we represent it more compactly as $G\left(v_i\right) = \left(F\left[v_i\right]\right)^{N-1}$, and the last line inserts the equilibrium bidding function found above, $b_i\left(v_i\right)$.

- Since the seller cannot observe bidders' values, he finds the expected payment from each bidder $i$, $E\left[\pi_i\left(v_i | v_i \geq v_e\right)\right]$, and then sums up for all $N$ bidders, $\sum_{i=1}^{N} E\left[\pi_i\left(v_i | v_i \geq v_e\right)\right]$, which gives us the seller's revenue from the auction (this is, of course, understood from an *ex-ante* perspective since the seller does not observe bidders' valuations). We find the seller's revenue as follows

$$\begin{aligned}
E\left[\pi\left(v_e\right)\right] &= \sum_{i=1}^{N} E\left[\pi_i\left(v_i | v_i \geq v_e\right)\right] \\
&= N \int_{v_e}^{\overline{v}} \left[E + \int_{v_e}^{v_i} x g(x)\, dx\right] f(z)\, dz
\end{aligned}$$

Since the participation fee, $E$, is a constant, it is unaffected by the integration, helping us to rewrite the seller's revenue as

$$\begin{aligned}
E\left[\pi\left(v_e\right)\right] &= NE \int_{v_e}^{\overline{v}} f(z)\, dz + N \int_{v_e}^{\overline{v}} \left(\int_{v_i}^{\overline{v}} f(z)\, dz\right) x g(x)\, dx \\
&= NE\left[1 - F\left(v_e\right)\right] + N \int_{v_e}^{\overline{v}} \left[1 - F\left(v_i\right)\right] x g(x)\, dx \\
&= N v_e G\left(v_e\right)\left[1 - F\left(v_e\right)\right] + N \int_{v_e}^{\overline{v}} \left[1 - F\left(v_i\right)\right] x g(x)\, dx
\end{aligned}$$

where the first line expands the integral into two parts; and the second line integrates the probability density function into the cumulative distribution function from the critical type's valuation $v_e$ to the upper bound $\overline{v}$, with the second part exchanges the order of integration that considers the density mass of bidders whose valuation are above $v_e$. Finally, the third line stems from the fact that $E = \left[F\left(v_e\right)\right]^{N-1}\left(v_e - b_e\right) = G\left(v_e\right) v_e$ when we assume that the indifferent bidder submits a bid of $b_e\left(v_e\right) = 0$.

- *Revenue-maximizing reservation price.* We can now differentiate the seller's revenue with respect to the critical valuation $v_e$,

$$\frac{dE[\pi(v_e)]}{dv_e} = N[G(v_e)(1-F(v_e)-v_e f(v_e)) + (1-F(v_e))v_e g(v_e) - (1-F(v_e))v_e g(v_e)]$$
$$= NG(v_e)(1-F(v_e))\left[1 - v_e\frac{f(v_e)}{1-F(v_e)}\right]$$

- Assuming interior solutions, we set the above first order condition equal to zero.
$$v_e = \frac{1-F(v_e)}{f(v_e)}$$
The right-hand side of the above inequality is the inverse hazard rate, $\frac{1-F(v_e)}{f(v_e)}$, which measures how sensitive is the distribution of the bidders' valuation $F(\cdot)$ to a change in the critical valuation $v_e$. That is, if the density mass is concentrated in the region above the critical valuation $v_e$, then $1-F(v_e)$ would be large relative to $f(v_e)$ so that the seller can further increase the entry fee $E$ to raise his expected revenue by selling the object to those bidders with higher valuations. Next, we will elaborate on the relationship between the entry fee $E$ and the critical valuation $v_e$.
- Substituting $v_e = \frac{1-F(v_e)}{f(v_e)}$ into the indifferent bidder's valuation function, the optimal entry fee $E^*$ solves
$$E^* = [F(v_e)]^{N-1} v_e$$
$$= [F(v_e)]^{N-1}\frac{1-F(v_e)}{f(v_e)} \tag{4}$$

(e) *Uniformly distributed valuations.* Evaluate the optimal bidding function, $b_i^*(v_i)$, and the optimal entry fee $E^*$ you found in parts (c) and (d) respectively when valuations are uniformly distributed, that is, $F(v_i) = v_i$ for all $v_i \sim U[0,\overline{v}]$. How does the bidding function change with bidder $i$'s valuation, $v_i$, and the number of bidders, $N$?

- Evaluating $v_e = \frac{1-F(v_e)}{f(v_e)}$ for a uniformly distributed valuation $v_i \sim U[0,1]$, we obtain
$$v_e = 1 - v_e, \text{ or } v_e^* = \frac{1}{2}$$
so that half of the bidders would participate in the auction in equilibrium.
- Substituting $v_e^* = \frac{1}{2}$ into the expression of $E^*$, we find
$$E^* = v_e^{N-1}(1-v_e)$$
$$= \frac{1}{2^N}$$
so that the entry fee $E^*$ decreases in the number of bidders $N$ at a decreasing rate. Note that when $N$ becomes infinitely large, that is, $N \to \infty$, the profit-maximizing entry fee that the seller sets approaches zero asymptotically; as depicted in the next figure.

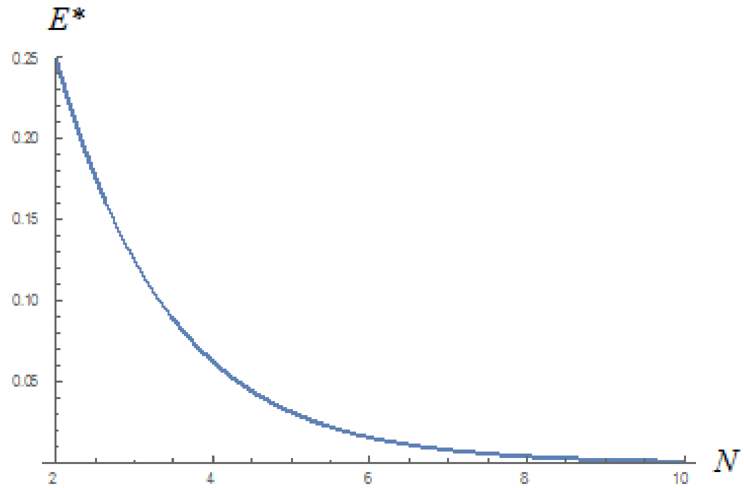


Figure 1. Optimal entry fee $E^*$ as a function of the number of bidders, $N$.

For instance, when only $N = 2$ bidders compete for the object, the optimal entry fee becomes $E^* = \frac{1}{4}$, while when $N = 3$ bidders compete the entry fee decreases to $E^* = \frac{1}{8}$.

- Therefore, the optimal bidding function, $b_i^*(v_i)$, becomes

$$b_i^*(v_i) = v_i - \frac{NE^* + v_i^N - (v_e^*)^N}{Nv_i^{N-1}}$$
$$= v_i - \frac{N\frac{1}{2^N} + v_i^N - \frac{1}{2^N}}{Nv_i^{N-1}}$$
$$= \frac{N-1}{N}\left[1 - (2v_i)^{-N}\right] v_i$$

For instance, when only $N = 2$ bidders compete for the object, this optimal bidding function simplifies to

$$b_i^*(v_i) = \frac{v_i}{2} - \frac{1}{8v_i},$$

when $N = 3$ bidders compete, their optimal bidding function becomes

$$b_i^*(v_i) = \frac{2v_i}{3} - \frac{1}{12v_i^2},$$

while when $N = 10$ bidders compete in the auction, it becomes

$$b_i^*(v_i) = \frac{9v_i}{10} - \frac{9}{10240v_i^9}.$$

The next figure depicts these bidding functions, where valuations are restricted in $v_i \in \left[\frac{1}{2}, 1\right]$ since the bidder indifferent between participating and not participating in the auction when valuations are uniformly distributed is $v_e^* = \frac{1}{2}$, as shown above.

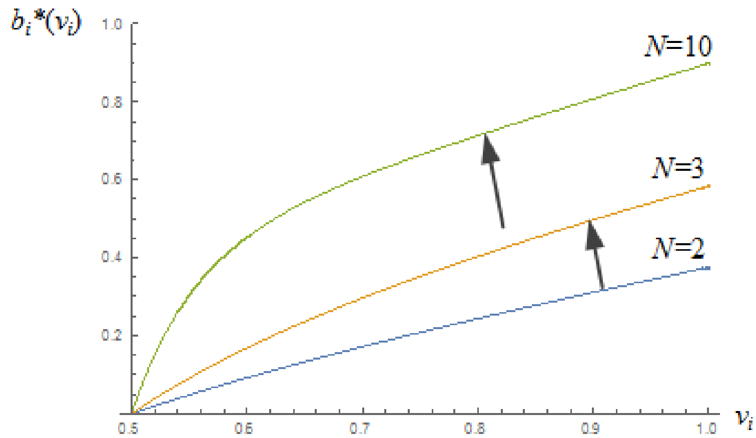


Figure 2. Optimal bidding function shifts up in $N$.

Every player's bids are increasing in his valuation for the object, $v_i$, and shift upwards when he competes against a larger number of bidders, $N$. We next confirm these two points more formally. First, let us differentiate the optimal bidding function, $b_i^*(v_i)$, with respect to valuation $v_i$,

$$\begin{aligned}\frac{db_i^*(v_i)}{dv_i} &= \frac{N-1}{N}\left[1-(2v_i)^{-N}+N(2v_i)^{-N}\right] \\ &= \frac{N-1}{N}\left[1+(N-1)(2v_i)^{-N}\right] > 0\end{aligned}$$

so that bidder $i$'s bid is increasing in his valuation for the object, $v_i$. Second, let us now differentiate the optimal bidding function, $b_i^*(v_i)$, with respect to the number of bidders, $N$,

$$\begin{aligned}\frac{\partial b_i^*(v_i)}{\partial N} &= \frac{1}{N^2}\left[1-(2v_i)^{-N}\right]v_i + \frac{N-1}{N}\left[N\log(2v_i)\right]v_i \\ &= v_i\left[\frac{1-(2v_i)^{-N}}{N^2}+(N-1)\log(2v_i)\right]\end{aligned}$$

and a sufficient condition for $\frac{\partial b_i^*(v_i)}{\partial N} \geq 0$ is $v_i \geq \frac{1}{2}$, which entails that $\log(2v_i) \geq 0$. However, we already showed that bidders who participate in this auction have a private value of $v_i \geq v_e^* = \frac{1}{2}$, such that for those bidders who participate, their equilibrium bids increase when facing competition from more bidders.