ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #3 – ANSWER KEY

Exercises from Harrington: see last pages of this answer key**.**

Exercise 2 – Cournot competition with 3 firms

Consider three firms competing *a la* Cournot, in a market with inverse demand function $P(Q) = 1 - Q$, and production costs normalized to zero.

a. Find the psNE of the game when firms simultaneously and independently choose quantities. Determine the equilibrium profit level for each firm.
b. Consider now that two (out of three) firms merge, and thus choose their output decision in order to maximize their joint profits. Find the psNE in this game for the merged firms and the unmerged firms. Identify the equilibrium profits for each firm, and compare them with your results pre-merger in part (a)
c. Consider now that all three firms merge. Find their profit maximizing output and profits, comparing them with your results in (a) and (b).

Answer:

a. The profits for firm *i* are

$$\pi_i = (1 - Q)q_i = \left(1 - q_i - q_j - q_k\right)q_i$$

Taking first order conditions with respect to q_i, we obtain:

$$1 - 2q_i - q_j - q_k = 0 \quad \Longrightarrow \quad q_i\left(q_j, q_k\right) = \frac{1 - q_j - q_k}{2}$$

And in a symmetric Nash equilibrium in which all firms are producing the same output, i. e., $q_i = q_j = q_k = q$, we find $q_i = \frac{1}{4}$ for every firm i.

Hence, equilibrium prices are $p = 1 - Q = 1 - 3\frac{1}{4} = \frac{1}{4}$

And, therefore, profits for every firm are (recall that there are no production costs)

$$\pi_i = pq_i = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

b. There are only two firms in the market now: the merge of firms 1 and 2, and the (unmerged) firm 3. The profits for either of these *two* firms are:

$$\pi_i = (1 - Q)q_i = \left(1 - q_i - q_j\right)q_i$$

And taking FOCs with respect to q_i, we obtain:

$$1 - 2q_i - q_j = 0 \quad \Longrightarrow \quad q_i\left(q_j\right) = \frac{1 - q_j}{2}$$

and in a symmetric Nash equilibrium in which all firms are producing the same output, i. e., $q_i = q_j = q$, we find $q_i = \frac{1}{3}$ for every firm i.

Hence, equilibrium prices are $p = 1 - Q = 1 - 2\frac{1}{3} = \frac{1}{3}$

And, therefore, profits for every firm are

$$\pi_i = pq_i = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

This implies that

- ✓ Firms 1 and 2 obtain profits of $\frac{\frac{1}{9}}{2} = \frac{1}{18}$ after the merger, which are lower than the pre-merger profits of $\frac{1}{16}$
- ✓ Firm 3 obtains profits of $\frac{1}{9}$, which exceed its pre-merger profits of $\frac{1}{16}$

Intuition: the merged firms internalize part of price reduction that an increase in their aggregate production entails, i. e., they consider the profit loss that the increase in production by one of the firm participating in the merger entails on the other firm that joined the merger. As a consequence, the merged firms reduce their individual production relative to pre-merger levels (in the standard Cournot competition analyzed in part a). However, the unmerged Firm 3 does not take into these price effects, and must responds to a lower output level from both of its competitors by increasing its own production. Ultimately, the firms that merged obtain a lower profit than before the merger, while the merged firm earns a larger profit. This result is often referred as the “merger paradox”.

c. If all firms merge, they form a cartel, acting as a monopolist. [Note that this is only true when they all merge, not when only two of them merge, as we examined in the previous section]. When they all merge their joint profits are

$$\pi_i = (1 - Q)Q = Q - Q^2$$

Taking first order conditions with respect to Q, we obtain

$$1 - 2Q = 0 \quad \Longrightarrow \quad Q = \frac{1}{2}$$

Which implies that equilibrium price is

$$p = 1 - Q = 1 - \frac{1}{2} = \frac{1}{2}$$

And equilibrium profits are

$$\pi = pQ = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Therefore, the individual profits of every firm participating in the merger are $\frac{\frac{1}{4}}{3} = \frac{1}{12}$, which are clearly higher than their profits pre-merger (when all firms compete as Cournot oligopolists) of $\frac{1}{16}$.

Exercise 3 (Bonus Exercise) – Cournot mergers with efficiency gains

a. Each firm *i*={1,2,3} has a profit of

$$\pi_i = (1 - Q - c)q_i.$$

Hence, since $Q \equiv q_1 + q_2 + q_3$, profits can be rewritten as:

$$\pi_i = \big(1 - (q_i + q_j + q_k) - c\big)q_i,$$

The first-order conditions are given by

$$1 - 2q_i - q_j - q_k - c = 0$$

since firms are symmetric $q_i = q_j = q_k = q$ in equilibrium, that is

$$1 - 2q - q - q - c = 0$$

or

$$1 - 4q - c = 0.$$

Solving for q at the symmetric equilibrium yields a Cournot output of,

$$q_c = \frac{1-c}{4}$$

Hence, equilibrium prices are

$$p_c = 1 - \frac{1-c}{4} - \frac{1-c}{4} - \frac{1-c}{4} = 1 - 3\left(\frac{1-c}{4}\right) = \frac{1+3c}{4}$$

and equilibrium profits are $\pi_c = \left(\frac{1-3c}{4} - c\right)\frac{1-c}{4} = \frac{(1-c)^2}{16}$

b.

1. After the merger, two firms are left: firm 1, with cost $e \cdot c$, and firm 3, with cost c.

Hence, the two profit functions are now given by:

$$\pi_1 = (1 - Q - ec)q_1$$

$$\pi_3 = (1 - Q - c)q_3$$

Taking first order conditions of π_1 with respect to q_1 yields

$$1 - 2q_1 - q_3 - ec = 0$$

and, solving for q_1, we obtain firm 1's best response function

$$q_1(q_3) = \frac{1-ec}{2} - \frac{1}{2}q_3$$

Similarly taking first-order conditions of firm 3's profits, π_3, with respect to q_3 yields

$$1 - 2q_3 - q_1 - c = 0$$

which, solving for q_3, provides us with firm 3's best response function

$$q_3(q_1) = \frac{1-c}{2} - \frac{1}{2} q_1$$

Plugging $q_3(q_1)$ into $q_1(q_3)$, yields

$$q_1^* = \frac{1-ec}{2} - \frac{1}{2}\left(\frac{1-c}{2} - \frac{1}{2} q_1^*\right)$$

Rearranging and solving for q_1^*, we obtain firm 1's equilibrium output

$$q_1^* = \frac{1-c(2e-1)}{3}$$

Plugging this output level into firm 3's best response function yields an equilibrium output of

$$q_3^* = \frac{1-c(2-e)}{3}$$

Note that the outsider firm can sell a positive output at equilibrium only if the merger does not give rise to strong cost savings: that is $q_3 \geq 0$ if $e \geq \frac{2c-1}{c}$ (if $c < 1/2$, then the previous payoff becomes $\frac{2c-1}{c} < 0$, implying that $e \geq \frac{2c-1}{c}$ holds for all $e \geq 0$, ultimately implying that the outsider firm will always sell at the equilibrium. We hence concentrate on values of c that satisfy $c > 1/2$.) Figure 1 illustrates cutoff $e > \frac{2c-1}{c}$, where $c > 1/2$, and the region of (e, c)-combinations above this cutoff indicate parameters for which the merger it is sufficiently cost saving to induce the outside firm to produce positive output levels.

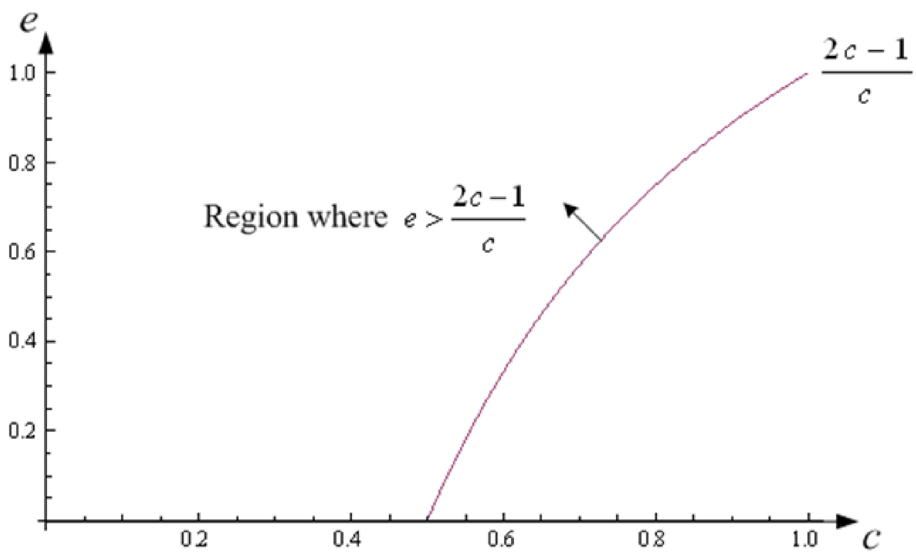


Figure 1. Positive production after the merger.

- The equilibrium price is $p_m = 1 - \frac{1-c(2e-1)}{3} - \frac{1-c(2-e)}{3} = \frac{1+c(1+e)}{3}$, and equilibrium profits are given by $\pi_1 = \frac{(1-c(2e-1))^2}{9}$ and $\pi_3 = \frac{(1-c(2-e))^2}{9}$.

2. Prices decrease after the merger only if there are sufficient efficiency gains: that is, $p_m \leq p_c$ can be rewritten as $e \leq \frac{5c-1}{4c}$. Note that if $c < 1/5$, then $\frac{5c-1}{4c} < 0$, implying that $e \leq \frac{5c-1}{4c}$ cannot hold for any $e \geq 0$. As a consequence, $p_m > p_c$, and prices will never fall no matter how strong efficiency gains, e, are.

3. To see if the merger is profitable, we have to study the inequality $\pi_1 \geq 2\pi_c$, which after some algebra can be seen to correspond to an inequality of the second order whose relevant solution is

$$e \leq \frac{4(1+c) - 3\sqrt{2}(1-c)}{8c}$$

In other words, the merger is profitable only if it gives rise to enough cost savings. Figure 2 depicts the cutoff of e where costs are restricted to $c \in \left[\frac{1}{5}, 1\right]$. Notice that if cost savings are sufficiently strong, i.e., parameter e is sufficiently small as depicted in the region below the cutoff, the merger is profitable.

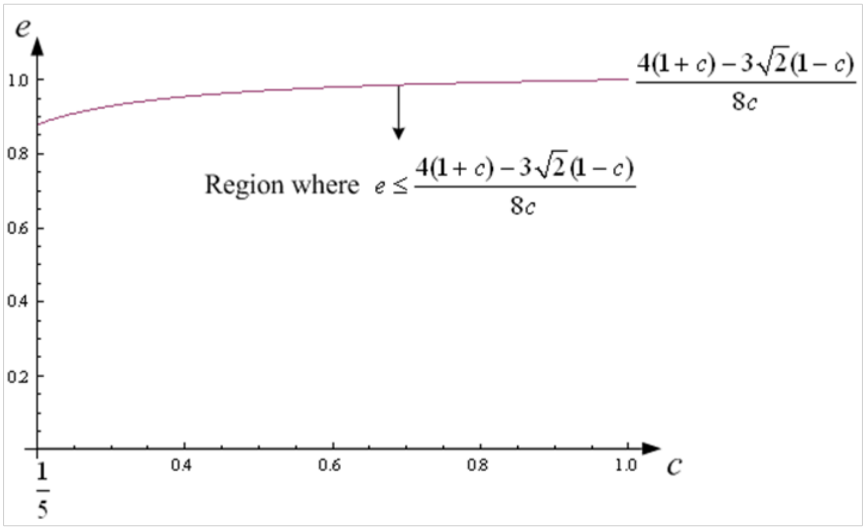


Figure 2. Profitable mergers if *e* is low enough.

is not a Nash equilibrium. If the sum of donations is zero, then all are contributing zero, which is a symmetric strategy profile and thus cannot be an asymmetric Nash equilibrium. (We already proved that it is a symmetric Nash equilibrium.) Thus, the only strategy profiles that remain are those that sum to 2,000,000. Next, note from a donor's best-reply function that a donor is willing to contribute what it takes to get the total to 2,000,000, as long as the amount does not exceed 250,000. Thus, any strategy profile that sums to 2,000,000, and each element is not more than 250,000, is a Nash equilibrium. Furthermore, any strategy profile that has one or more donors donating more than 250,000 is not a Nash equilibrium. In sum, the set of asymmetric Nash equilibria are those asymmetric strategy profiles that sum to 2,000,000 with no player contributing more than 250,000.

7. For a two-player game, the payoff function for player 1 is

$$V_1(x_1, x_2) = x_1 + 10x_1x_2$$

and for player 2 is

$$V_2(x_1, x_2) = x_2 + 20x_1x_2.$$

Player 1's strategy set is the interval [0,100] and player 2's strategy set is the interval [0,50]. Find all Nash equilibria.

ANSWER: Examining player 1's payoff function, note that it is increasing in his strategy:

$$\frac{\partial V_1(x_1, x_2)}{\partial x_1} = 1 + 10x_2 > 0.$$

Hence, regardless of player 2's strategy, player 1 wants to set x_1 at the highest feasible value. We then have the equilibrium value for x_1 as 100. Similarly, player 2's strategy is increasing in her strategy:

$$\frac{\partial V_2(x_1, x_2)}{\partial x_2} = 1 + 20x_1.$$

Therefore, player 2 sets x_2 at its maximal value, which is 50. There is a unique Nash equilibrium in which $x_1 = 100$ and $x_2 = 50$.

EX. 10 CHAPTER 6

8. An arms buildup is thought to have been a contributing factor to World War I. The naval arms race between Germany and Great Britain is particularly noteworthy. In 1889, the British adopted a policy for maintaining naval superiority whereby they required their navy to be at least two-and-a-half times as large as the next-largest navy. This aggressive stance induced Germany to increase the size of its navy, which, according to Britain's policy, led to a yet bigger British navy, and so forth. In spite of attempts at disarmament in 1899 and 1907, this arms race fed on itself. By the start of World War I in 1914, the tonnage of Britain's navy was 2,205,000 pounds, not quite 2.5 times that of Germany's navy, which, as the second largest, weighed in at 1,019,000 pounds. With this scenario in mind, let us model the arms race between two countries, denoted 1 and 2. The arms expenditure of country i is denoted x_i and is restricted to the interval [1,25]. The benefit to a country from investing in arms comes from security or war-making capability, both of which depend on relative arms expenditure. Thus, assume that the benefit to country 1 is $36(\frac{x_1}{x_1 + x_2})$, so it increases with country 1's expenditure relative to total expenditure. The cost is simply x_1, so country 1's payoff function is

$$V_1(x_1, x_2) = 36\left(\frac{x_1}{x_1 + x_2}\right) - x_1,$$

and there is an analogous payoff function for country 2:

$$V_2(x_1, x_2) = 36\left(\frac{x_2}{x_1 + x_2}\right) - x_2.$$

These payoff functions are hill shaped.

a. Derive each country's best-reply function.

ANSWER: Country 1's optimal arms expenditure is that value for x_1 for which the first derivative of its payoff function equals zero:

$$\frac{\partial V_1(x_1, x_2)}{\partial x_1} = 36\left[\frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2}\right] - 1 = 0$$

$$36\left[\frac{x_2}{(x_1 + x_2)^2}\right] - 1 = 0$$

$$36x_2 = (x_1 + x_2)^2$$

$$\sqrt{36x_2} = \sqrt{(x_1 + x_2)^2}$$

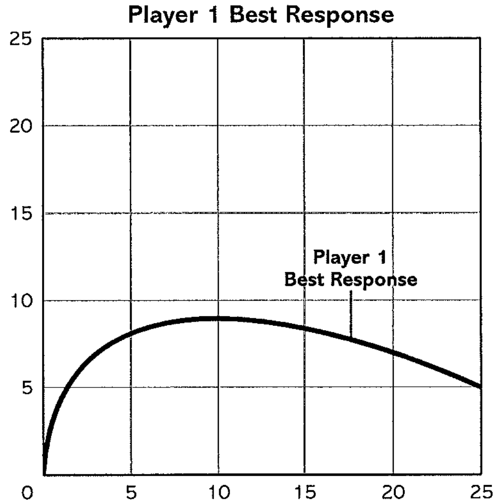
$$6\sqrt{x_2} = x_1 + x_2$$

$$x_1 = 6\sqrt{x_2} - x_2$$

$$BR_1(x_2) = 6\sqrt{x_2} - x_2.$$

That country 1's best-reply function is $6\sqrt{x_2} - x_2$ presumes that whatever value it takes (after specifying a value for x_2) lies in the feasible set of [1,25]. One can show that for all values of x_2 in [1,25], $6\sqrt{x_2} - x_2$ also lies in [1,25], so we're fine. The best-reply function is plotted in **FIGURE SOL6.8.1.**

FIGURE SOL 6.8.1



This can be proven by considering the properties of the best-reply function.

$$\frac{\partial BR_1(x_2)}{\partial x_2} = \frac{\partial[6\sqrt{x_2} - x_2]}{\partial x_2}$$

$$= 3(x_2)^{-\frac{1}{2}} - 1.$$

$$\frac{\partial^2 BR_1(x_2)}{\partial x_2^2} = -\frac{3}{2}(x_2)^{-\frac{3}{2}} < 0.$$

Thus, $BR_1(x_2)$ is hill-shaped and reaches its maximum where its slope is zero:

$$3(x_2)^{-\frac{1}{2}} - 1 = 0$$
$$3(x_2)^{-\frac{1}{2}} = 1$$
$$3 = (x_2)^{\frac{1}{2}}$$
$$(3)^2 = [(x_2)^{\frac{1}{2}}]^2$$
$$9 = x_2.$$

The highest value for country 1's best reply occurs when country 2 spends 9, in which case country 1 spends $9 = (6\sqrt{9} - 9)$, which does lie in [1,25]. The lowest value for country 1's best reply occurs either at 1 or 25 and the associated best replies are 5 and 5, coincidentally, which also lie in [1,25]. We conclude that for all x_2 in [1,25], $6\sqrt{x_2} - x_2$ is in country 1's strategy set. Thus, $6\sqrt{x_2} - x_2$ is country 1's best reply. By symmetry, country 2's best reply function is $BR_2(x_1) = 6\sqrt{x_1} - x_1$.

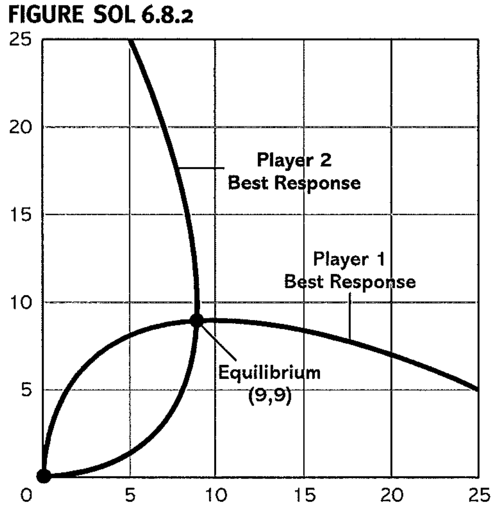
b. Derive a symmetric Nash equilibrium.

ANSWER: A symmetric Nash equilibrium is an arms expenditure, denoted x^*, such that a country finds it optimal to spend x^* when the other country spends x^*. It is then a solution to

$$x^* = 6\sqrt{x^*} - x^*$$
$$2x^* = 6\sqrt{x^*}$$
$$(x^*)^2 = (3\sqrt{x^*})^2$$
$$(x^*)^2 = 9x^*$$
$$x^* = 9.$$

The unique symmetric Nash equilibrium then has each country spend 9. This equilibrium is depicted in **FIGURE SOL6.8.2.**

FIGURE SOL 6.8.2



9. Players 1 and 2 are playing a game in which the strategy of player i is denoted z_i and can be any nonnegative real number. The payoff function for player 1 is

$$V_1(z_1,z_2) = (100 - z_1 - z_2)z_1$$

and for player 2 is

$$V_2(z_1,z_2) = (80 - z_1 - z_2)z_2.$$

These payoff functions are hill shaped. Find all Nash equilibria.

ANSWER: First derive the best-reply functions:

$$\frac{\partial V_1(z_1,z_2)}{\partial z_1} = 100 - 2z_1 - z_2 = 0$$

$$z_1 = 50 - \left(\frac{1}{2}\right)z_2.$$

$$\frac{\partial V_2(z_1,z_2)}{\partial z_2} = 80 - z_1 - 2z_2 = 0$$

$$z_2 = 40 - \left(\frac{1}{2}\right)z_1.$$

(z_1^*, z_2^*) is a Nash equilibrium if it is a solution to this pair of equations:

$$z_1^* = 50 - \left(\frac{1}{2}\right)z_2^*$$

$$z_2^* = 40 - \left(\frac{1}{2}\right)z_1^*.$$

Solving,

$$z_1^* = 50 - \left(\frac{1}{2}\right)\left[40 - \left(\frac{1}{2}\right)z_1^*\right]$$

$$z_1^* = 50 - 20 + \left(\frac{1}{4}\right)z_1^*$$

$$\left(\frac{3}{4}\right)z_1^* = 30$$

$$z_1^* = 40.$$

The next step is to derive player 2's equilibrium strategy:

$$z_2^* = 40 - \left(\frac{1}{2}\right) \times 40$$

$$z_2^* = 20.$$

EXERCISE 12
CHAPTER 6

10. The wedding anniversary of a husband and wife is fast approaching, and each is deciding how much to spend. Let g_H denote the amount that the husband spends on his wife and g_W the amount the wife spends on her husband. Assume that they have agreed that the most each can spend is 500. A players' strategy set is then the interval [0,500]. A spouse enjoys giving a bigger gift, but doesn't like spending money. With that in mind, the husband's payoff function is specified to be

$$V_H(g_H,g_W) = 50g_H + \left(\frac{1}{4}\right)g_Hg_W - \left(\frac{1}{2}\right)(g_H)^2.$$

The payoff function can be understood as follows: The benefit from exchanging gifts is captured by the term $50g_H + (\frac{1}{4})g_Hg_W$. Since "men are boys with bigger toys," this benefit increases with the size of the wife's gift:

$$\frac{\partial(50g_H + (\frac{1}{4})g_Hg_W)}{\partial g_W} = \left(\frac{1}{4}\right)g_H > 0.$$

The "warm glow" the husband gets from giving his wife a gift is reflected in the term $50g_H + (\frac{1}{4})g_Hg_W$, which increases with the size of his gift:

$$\frac{\partial(50g_H + (\frac{1}{4})g_Hg_W)}{\partial g_H} = 50 + \left(\frac{1}{4}\right)g_W > 0.$$

Alas, where there are benefits, there are costs. The personal cost to the husband from buying a gift of size g_H is represented by the term $-g_H \times g_H$, or $-(g_H)^2$, in his payoff function. Thus, we subtract this cost from the benefit, and we have the husband's payoff function as described. The wife's payoff function has the same general form, though with slightly different numbers:

$$V_W(g_H,g_W) = 50g_W + 2g_Hg_W - \left(\frac{1}{2}\right)(g_W)^2.$$

These payoff functions are hill shaped.

a. Derive each spouse's best-reply function and plot it.

ANSWER: The husband's best-reply function is derived below:

$$\frac{\partial V_H(g_H,g_W)}{\partial g_H} = 50 + \frac{1}{4}g_W - g_H = 0 \Rightarrow g_H = 50 + \frac{1}{4}g_W.$$

The wife's best response function is derived below:

$$\frac{\partial V_H(g_H,g_W)}{\partial g_H} = 50 + \frac{1}{4}g_W - g_H = 0 \Rightarrow g_H = 50 + \frac{1}{4}g_W.$$

b. Derive a Nash equilibrium.

ANSWER: Nash equilibrium is the solution of the following two best-reply equations.

$$g_H^* = 50 + \frac{1}{4}g_W^*$$

$$g_W^* = 50 + 2\,g_H^*$$

Solving these two equations:

$$g_H^* = 50 + \frac{1}{4}g_W^* = 50 + \frac{1}{4}(50 + 2g_H^*) = \frac{250}{4} + \frac{g_H^*}{2}.$$

$$g_H^* = 125, g_W^* = 300.$$

c. Now suppose the husband's payoff function is of the same form as the wife's payoff function:

$$V_H(g_H,g_W) = 50g_H + 2g_Hg_W - \left(\frac{1}{2}\right)(g_H)^2.$$

Find a Nash equilibrium. (*Hint:* Don't forget about the strategy sets.)

ANSWER: If the husband's payoff function is of the same form as the wife's payoff function, then his best-reply function is of the same form as the wife's. Using results in (b), the equations that determine Nash equilibrium are then:

$$g_H^* = 50 + 2g_W^*.$$
$$g_W^* = 50 + 2g_H^*.$$

Solving these equations, we have $g_W^* = g_H^* = -50$. However, this cannot be a Nash equilibrium since g_W^* and g_H^* cannot be negative. As we'll show below, $g_W^* = g_H^* = 500$ is a Nash equilibrium. Given $g_H^* = 500$, the wife's payoff function is $V_W(500, g_W) = 1050g_W - \frac{1}{2}g_W^2$, which is an increasing function on (0,1050) since $\frac{\partial V_W(500, g_W)}{\partial g_W} = 1050 - g_W$. Given the constraint that the wife cannot spend more than 500, we have $g_W^* = 500$. Similarly, we can show if the wife spends 500, then the husband's best reply is also 500; that is, he spends the maximum amount that is feasible.

11. Players 1, 2, and 3 are playing a game in which the strategy of player i is denoted x_i and can be any nonnegative real number. The payoff function for player 1 is

$$V_1(x_1, x_2, x_3) = x_1x_2x_3 - \frac{1}{2}(x_1)^2,$$

for player 2 is

$$V_2(x_1, x_2, x_3) = x_1x_2x_3 - \frac{1}{2}(x_2)^2,$$

and for player 3 is

$$V_3(x_1, x_2, x_3) = x_1x_2x_3 - \frac{1}{2}(x_3)^2.$$

These payoff functions are hill shaped. Find a Nash equilibrium.

ANSWER: The trivial Nash equilibrium occurs when all players choose zero. Given the other two players choose zero, a player's payoff is zero regardless of her strategy. Another Nash equilibrium is $(x_1, x_2, x_3) = (1,1,1)$. To prove that claim, note that, at an equilibrium,

$$\frac{\partial V_1(x_1, x_2, x_3)}{\partial x_1} = x_2x_3 - x_1 = 0.$$
$$\frac{\partial V_2(x_1, x_2, x_3)}{\partial x_2} = x_1x_3 - x_2 = 0.$$
$$\frac{\partial V_3(x_1, x_2, x_3)}{\partial x_3} = x_1x_2 - x_3 = 0.$$

From the second condition, we have $x_1x_3 = x_2$. From the first condition, we can then substitute x_2x_3 for x_1 to get $x_2x_3x_3 = x_2$, which implies $x_3^2 = 1$ and thus $x_3 = 1$. From the first condition, $x_3 = 1$ implies $x_1 = x_2$ and, from the third condition, we have $x_1x_2 = 1$. We can then conclude that $x_1 = 1, x_2 = 1$.

12. Players 1, 2, and 3 are playing a game in which the strategy of player i is denoted y_i and can be any nonnegative real number. The payoff function for player 1 is

$$V_1(y_1, y_2, y_3) = y_1 + y_1y_2 - (y_1)^2,$$

for player 2 is

$$V_2(y_1, y_2, y_3) = y_2 + y_1 y_2 - (y_2)^2,$$

and for player 3 is

$$V_3(y_1, y_2, y_3) = (10 - y_1 - y_2 - y_3)y_3.$$

These payoff functions are hill shaped. Find a Nash equilibrium. (*Hint:* The payoff functions are symmetric for players 1 and 2.)

ANSWER: First, derive the players' best-reply functions.

$$\frac{\partial V_1(y_1, y_2, y_3)}{\partial y_1} = 1 + y_2 - 2y_1 = 0 \Rightarrow y_1 = BR_1(y_2, y_3) = \frac{1 + y_2}{2}$$

$$\frac{\partial V_2(y_1, y_2, y_3)}{\partial y_2} = 1 + y_1 - 2y_2 = 0 \Rightarrow y_2 = BR_2(y_1, y_3) = \frac{1 + y_1}{2}$$

$$\frac{\partial V_3(y_1, y_2, y_3)}{\partial y_3} = 10 - y_1 - y_2 - 2y_3 = 0 \Rightarrow y_3 = BR_3(y_1, y_2) = \frac{10 - y_1 - y_2}{2}.$$

A triple of strategies, (y_1^*, y_2^*, y_3^*), is an equilibrium if it is a solution to

$$y_1^* = \frac{1 + y_2^*}{2}$$

$$y_2^* = \frac{1 + y_1^*}{2}$$

$$y_3^* = \frac{10 - y_1^* - y_2^*}{2}.$$

Note that players 1 and 2 are symmetric, as reflected in their best-reply functions having the same form. Thus, let's look for an equilibrium in which $y_1^* = y^* = y_2^*$. This reduces to two equations and two unknowns.

$$y^* = \frac{1 + y^*}{2}$$

$$y_3^* = \frac{10 - 2y^*}{2}$$

Note that the first equation depends only on y^*, so we can solve it for y^*:

$$2y^* = 1 + y^*$$

$$y^* = 1.$$

Substituting it into the best-reply function for player 3, we derive the equilibrium strategy for player 3:

$$y_3^* = \frac{10 - 2}{2}$$

$$y_3^* = 4.$$

There is then a Nash equilibrium in which players 1 and 2 choose 1 and player 3 chooses 4.

For the UN to be content to randomize over its three pure strategies, it must be the case that

$$4 + 5x + 5y = 9 - 5y,$$
$$4 + 5x + 5y = 9 - 5x.$$
$$9 - 5x = 9 - 5y.$$

The last condition implies $x = y$. Letting this common value be denoted w, then the first condition becomes

$$4 + 5w + 5w = 9 - 5w \Rightarrow w = \frac{1}{3}.$$

Therefore, Nash equilibrium has Saddam uniformly randomize over its three pure strategies; assigning $\frac{1}{3}$ to each of them.

CHAPTER 7, EX. 12

12. Consider the two-player game below. Find all of the mixed-strategy Nash equilibria.

Player 2

		Slow	*Fast*
Player 1	*Small*	2,0	3,8
	Medium	3,7	2,1
	Large	3,4	5,6

ANSWER: Since Large strictly dominates Small then we know that all Nash equilibria assign probability zero to Small. Thus, the Nash Equilibria of this game is equivalent to the Nash equilibria of:

2

		Slow	*Fast*
1	*Medium*	3,7	2,1
	Large	3,4	5,6

Let m denote the probability that 1 assigns to Medium and s denote the probability that 2 assigns to Slow. Let us derive each player's best reply. Note that Large weakly dominates Medium. Hence, if 2 assigns any probability to Fast then 1 strictly prefers Large. If s denotes the probability that 2 assigns to Slow then 1's expected payoff from Medium is

$$s \times 3 + (1 - s) \times 2 = 2 + s,$$

and her expected payoff from Large is

$$s \times 3 + (1 - s) \times 5 = 5 - 2s.$$

Large is strictly preferred when

$$5 - 2s > 2 + s \text{ or } 1 > s.$$

Hence, if $s < 1$ then 1's best reply is $m = 0$; that is, the pure strategy Large. If $s = 1$ then either pure strategy gives a payoff of 3 and, in addition, any mixed strategy gives a payoff of 3. Thus, if $s = 1$ then m is a best reply for all values of m, $0 \leq m \leq 1$. Now consider player 2. Given m, player 2's expected payoff from pure strategy Slow is

$$m \times 7 + (1 - m) \times 4 = 4 + 3m,$$

player 2's expected payoff from pure strategy Fast is

$$m \times 1 + (1 - m) \times 6 = 6 - 5m.$$

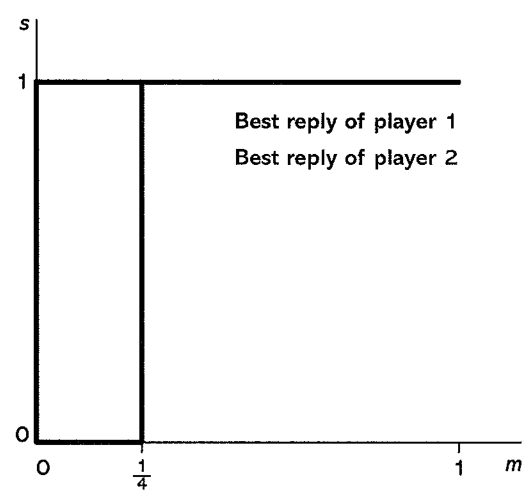
Slow is strictly preferred to Fast when

$$4 + 3m > 6 - 5m \text{ or } m > \frac{1}{4},$$

and Fast is strictly preferred to Slow when

$$4 + 3m < 6 - 5m \text{ or } m < \frac{1}{4}.$$

Player 2 is indifferent between Fast and Slow when $m = \frac{1}{4}$ and, furthermore, all mixed strategies give the same payoff of 4.75. Thus, player 2's best reply is $s = 0$ when $m < \frac{1}{4}$, all values of s when $m = \frac{1}{4}$, and $s = 1$ when $m > \frac{1}{4}$. The best reply functions are plotted in the figure below, and we can see that (m, s) is a Nash equilibrium if $(m, s) = (0, 0)$ or $(m, s) = (m, 1)$ and $m \geq \frac{1}{4}$. Alternatively stated, if $s < 1$ then there is a unique best reply for player 1 of $m = 0$ and the best reply of player 2 is $s = 0$. Hence, $(m, s) = (0, 0)$ is a Nash equilibrium. If $s = 1$ then any mixed strategy for player 1 is a best reply; however, $s = 1$ is a best reply for player 1 if and only if $m \geq \frac{1}{4}$. Thus, $(m, s) = (m, 1)$ is a Nash equilibrium and $m \geq \frac{1}{4}$.



13. Consider the two-player game below. Find all of the mixed-strategy Nash equilibria.

Player 2

Player 1		Left	Right
	Top	1,2	0,2
	Bottom	1,0	3,4

ANSWER: There are two pure-strategy Nash equilibrium: (T,L) and (B,R), where T refers to Top, B to Bottom, L to Left, and R to Right. In fact, these are the only mixed-strategy Nash equilibria.