EconS 503 - Microeconomic Theory II
Homework #2 - Answer key

1. **Strict Nash equilibrium.** Consider the following definition: A strategy profile $s^* \equiv (s_1^*, ..., s_N^*)$ is a *strict Nash equilibrium* (SNE) if it satisfies

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*) \text{ for every player } i, \text{ and every } s_i \in S_i.$$

You probably noticed that this definition is almost identical to the definition of Nash equilibrium (NE) strategy profiles, except for using a strict, rather than weak, inequality. In this exercise we connect both solution concepts, but first examine the relationship between a strict Nash equilibrium and IDSDS.

(a) Show that if a strategy s_i^* is a SNE it must also survive IDSDS.

- Assume, by contradiction, that strategy profile s^* is a SNE, but s^* does not survive IDSDS. Then, there must be at least one player i who can find at least one strategy $s_i' \neq s_i^*$ such that $u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$ during some of the rounds of applying IDSDS. However, this would contradict that strategy s_i^* is a SNE for player i. Then, if strategy s_i^* is a SNE, it must also survive IDSDS.

(b) Show the opposite relationship to that in part (a), that is, if strategy s_i^* survives IDSDS then s_i^* must be a SNE.

- By the same argument as above, assume, by contradiction, that strategy profile s^* survives IDSDS, but s^* is not a SNE. Then, there must be at least one player i and at least one of his strategies $s_i' \neq s_i^*$ for which $u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$, which contradicts strategy s_i^* surviving IDSDS.
- The above argument assumes that a SNE exists, because if it did not, strategy s_i^* could not survive IDSDS.

(c) Show that every player has a unique best response to his rivals' strategies.

- From parts (a) and (b), if strategy profile s^* is a SNE, then, it must be the only strategy profile surviving IDSDS. To show that strategy s_i^* is player i's unique best response, we next operate by contradiction:
- Suppose s^* is the only strategy profile surviving IDSDS, but s_i^* is not a best response to his opponent's equilibrium strategies s_{-i}^*. This implies that there is some other strategy s_i' that was eliminated at some previous round of IDSDS such that $u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$. Then there must be a third strategy s'' that is better than both s_i^* and s_i', more formally, $u_i(s_i'', s_{-i}^*) > u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$, because s_i' was eliminated while s_i^* was not. But this is a contradiction to s_i^* being the unique survivor of IDSDS. In conclusion, if strategy s_i^* is a SNE, then it must be the unique best response of player i to his opponent's equilibrium strategies s_{-i}^*.

(d) Show that if strategy profile s^* is SNE, it must also be a NE.

- If s^* is a SNE, then, it is the only strategy profile surviving IDSDS. In part (c) we showed that strategy s_i^* is also a player i's best response to his opponent's equilibrium strategies s_{-i}^*. Since this argument applies to every player i, strategy is a NE of the game.

(e) Show that if strategy profile s^* is a NE, it doesn't need to be a SNE. An example suffices.

- Consider the following example

		Player 2		
		L	M	R
Player 1	U	$\underline{6},\underline{5}$	$7,3$	$\underline{8},4$
	C	$4,3$	$10,6$	$5,\underline{8}$
	D	$5,2$	$\underline{11},8$	$4,\underline{10}$

- *Nash equilibrium.* First, we underline best response payoffs of each player to find the Nash equilibria of the game. We identify only one Nash equilibrium (U, L), where every player chooses best responses to his opponent's equilibrium strategies.
- *Strict Nash equilibrium.* To see that this game has no SNE, note that there is no strictly dominant strategies, neither for player 1 nor for player 2. From our discussion in parts (a) and (b), this implies that there is no SNE.

(f) Show that if strategy profile s^* is a SNE, the game has no mixed strategy NE.

- If s^* is a SNE, then, it is the only strategy profile surviving IDSDS and, in turn, s^* is the unique pure strategy NE of the game. Both players play their strictly dominant strategies in this NE, and there is no belief that other player will play different strategies. Thus, we cannot sustain a mixed strategy NE.

2. **Contests with N players.** In this exercise we examine equilibrium investment in a contest, where $N \geq 2$ players compete to earn a prize of common value V, and the probability of winning the prize is a function of a player's own investment relative to the aggregate investment of all the players. Hence, contests are often used to model promotions within a firm (where every worker invests time and effort into being selected for a promotion), political campaigns (where candidates invest money and resources to capture a larger share of votes), and R&D races (where firms invest resources into discovering a new product, such as a drug). We consider player i's probability of winning the prize is given by

$$p_i = \frac{x_i^r}{\sum_{j=1}^{N} x_j^r},$$

where x_i denotes his investment and the parameter, $r \geq 1$, represents the effectiveness of his investment. For simplicity, we normalize the cost of every unit of investment to one dollar.

(a) Setup and solve the utility maximization program for player i.

- The expected value of the prize for player i is

$$p_i \times V + (1 - p_i) \times 0 = \frac{x_i^r}{\sum_{j=1}^N x_j^r} V$$

And to contest for the prize, player i has to spend x_i. Therefore, he solves

$$\max_{x_i \geq 0} EU_i [x_i|V]$$
$$= \left(\frac{x_i^r}{\sum_{j=1}^N x_j^r} V - x_i \right)$$

Taking the first order condition of the utility maximization problem with respect to x_i,

$$\frac{\partial EU_i [x_i|V]}{\partial x_i} = \frac{r x_i^{r-1} \left(\sum_{j=1}^N x_j^r \right) - r x_i^{2r-1}}{\left(\sum_{j=1}^N x_j^r \right)^2} V - 1$$

Therefore, setting the above FOC equal to zero, the optimal investment for player i satisfies

$$\frac{r x_i^{r-1} \left(\sum_{j \neq i} x_j^r \right)}{\left(\sum_{j=1}^N x_j^r \right)^2} V = 1 \qquad (1)$$

Due to homogeneity of the players and commonality of prize valuation V, the only possible outcome that satisfies (1) is the symmetric equilibrium, $x^* = x_1^* = x_2^* = \cdots = x_N^*$, such that

$$\frac{r x^{r-1} (N-1) x^r}{(N x^r)^2} V = 1$$

which, solving for x, yields an equilibrium investment of

$$x^* = rV \frac{N-1}{N^2}$$

(b) Verify that the equilibrium investment maximizes the expected utility of player i, that is, check second-order conditions.

- The second-order condition of the utility maximization program is

$$\frac{\partial^2 EU_i [x_i|V]}{\partial x_i^2} = \frac{r(r-1) x_i^{r-2} \left(\sum_{j \neq i} x_j^r \right) \left(\sum_{j=1}^N x_j^r \right) - 2 \left(r x_i^{r-1} \right)^2 \left(\sum_{j \neq i} x_j^r \right)}{\left(\sum_{j=1}^N x_j^r \right)^3}$$
$$= \frac{r x_i^{r-2} \left(\sum_{j \neq i} x_j^r \right)}{\left(\sum_{j=1}^N x_j^r \right)^3} \left[(r-1) \sum_{j=1}^N x_j^r - 2 r x_i^r \right]$$

Substituting the equilibrium investment, x^*, into the above expression,

$$\begin{aligned}\frac{\partial^2 EU_i\left[x_i|V\right]}{\partial x_i^2} &= \frac{rx^{r-2}\left(N-1\right)x^r}{\left(Nx^r\right)^3}\left[\left(r-1\right)Nx^r - 2rx^r\right] \\ &= \frac{N-1}{N^3}\frac{r}{x^2}\left[\left(r-1\right)N - 2r\right]\end{aligned}$$

Therefore, the expected utility is maximized if the above second order condition is negative.

$$\left(r-1\right)N \leq 2r, \text{ which yields } r \leq \frac{N}{N-2}.$$

(c) *Comparative statics.* How does the equilibrium investment of player i change with V, r, and N?

- Differentiate x^* with respect to V,

$$\frac{\partial x^*}{\partial V} = r\frac{N-1}{N^2} \geq 0$$

Therefore, the higher the valuation of the prize, the more player i invests at equilibrium.

- Differentiate x^* with respect to r,

$$\frac{\partial x^*}{\partial r} = V\frac{N-1}{N^2} \geq 0$$

Therefore, the more effective is his investment, the more player i invests at equilibrium.

- Differentiate x^* with respect to N,

$$\frac{\partial x^*}{\partial N} = -rV\frac{N-2}{N^3} \leq 0$$

Since $N \geq 2$, the more players competing for the prize, the less player i invests at equilibrium.

(d) Find the range of the effectiveness parameter r in which all the players participate.

- The expected utility of player i at the equilibrium investment, x^*, is

$$\begin{aligned}EU_i\left[x^*|V\right] &= \frac{\left(x^*\right)^r}{N\left(x^*\right)^r}V - x^* \\ &= \frac{V}{N} - rV\frac{N-1}{N^2} \\ &= \frac{V}{N^2}\left[N - \left(N-1\right)r\right]\end{aligned}$$

Hence, player i would participate if the above participation constraint is positive.

$$N \geq \left(N-1\right)r, \text{ or } r \leq \frac{N}{N-1}$$

Since $\frac{N}{N-1} \leq \frac{N}{N-2}$ for all $N \geq 2$, expression $r \leq \frac{N}{N-1}$ is the most demanding condition. Therefore, the range of r in which all the players participate is $1 \leq r \leq \frac{N}{N-1}$.

(e) Define Rent Dissipation as

$$D \equiv V - \sum_{i=1}^{N} x_i^*,$$

which can be understood as how many resources the society is left with all the players are completed with their investments. Find the Rent Dissipation D and show that it is positive for all the admissible values of r.

- Rent Dissipation is given by

$$\begin{aligned} D &= V - \sum_{i=1}^{N} x_i^* \\ &= V - N x^* \\ &= \left(1 - r\frac{N-1}{N^2}\right) V \end{aligned}$$

Since $r \leq \frac{N}{N-1}$ from part (d), the above expression becomes

$$\begin{aligned} D &= \left(1 - r\frac{N-1}{N^2}\right) V \\ &\geq \left(1 - \frac{N}{N-1}\frac{N-1}{N^2}\right) V \\ &= \left(1 - \frac{1}{N}\right) V \geq 0 \end{aligned}$$

Therefore, rent dissipation must be positive, indicating that, as a whole, players do not invest more than the value of the prize.

(f) Is Rent Dissipation increasing or decreasing in the common value that players assign to the prize, V; in the effectiveness parameter r; and in the number of players N?

- Differentiate D with respect to V, to obtain

$$\frac{\partial D}{\partial V} = 1 - r\frac{N-1}{N^2}$$

Since $r \leq \frac{N}{N-1}$, Rent Dissipation increases with common value of the prize, V.

- Differentiating D with respect to r, yields

$$\frac{\partial D}{\partial r} = -V\frac{N-1}{N^2}$$

Therefore, Rent Dissipation decreases with the effectiveness of investment, r. Intuitively, the more effective is the investment of player i, the more rent he can capture from the investment.

- Finally, differentiating D with respect to N, we find

$$\frac{\partial D}{\partial N} = rV\frac{N-2}{N^3}$$

Therefore, Rent Dissipation increases with the number of players, N. Intuitively, the more players competing for the prize, the less rent is captured by all the players.

(g) What happens when all players coordinate their investment decisions?

- Players $i \in \{1, \ldots, N\}$ collectively choose the profile $(x_1, x_2, ..., x_N)$ to solve the following joint expected utility maximization problem:

$$\begin{aligned}\max_{\{x_1,\ldots,x_N\}\in\mathbb{R}_+^N} & \sum_{i=1}^{N} EU_i\left[x_i|V\right] \\ &= \sum_{i=1}^{N}\left(\frac{x_i^r}{\sum_{j=1}^{N} x_j^r}V - x_i\right) \\ &= V - \sum_{i=1}^{N} x_i\end{aligned}$$

which is maximized at $x_i^* = x_j^* = 0$ for every player $i \neq j$. Intuitively, players i and j should make no investments to compete for the prize but split it equally among themselves, entailing a net utility of $\frac{V}{2}$ for each player. Specifically, when every player simultaneously and independently chooses his investment (as analyzed in part a), he ignores the negative externality that his investment imposes on the other player, i.e., an increase in investment x_i reduces player j's probability of winning the contest. When both players coordinate their investment decisions, however, they internalize this externality, reducing their investment all the way down to zero.

3. **Exercises from Tadelis:**

(a) Chapter 5: Exercises 5.11 and 5.16. (See last pages of this answer key.)

(b) In what way are the best response correspondences different from those in the Cournot game? Why?

Answer: Here the best response function of player i is *increasing* in the choice of player j, whereas in the Cournot model it is *decreasing* in the choice of player j. This is because in this game the choices of the two players are strategic complements while in the Cournot game they are strategic substitutes. ■

(c) Find the Nash equilibrium of this game and argue that it is unique.

Answer: We solve two equations with two unknowns,

$$e_1 = \frac{a + e_2}{2} \text{ and } e_2 = \frac{a + e_1}{2},$$

which yield the solution $e_1 = e_2 = a$. It is easy to see that it is unique because it is the only point at which these two best response functions cross. ■

Ex. 5.11

11. **Wasteful Shipping Costs**. Consider two countries, A and B, each with a monopolist that owns the only coal mine in the country, and it produces coal. Let firm 1 be the one located in country A, and firm 2 the one in country B. Let q_i^j, $i \in \{1, 2\}$ and $j \in \{A, B\}$ denote the quantity that firm i sells in country j. Consequently, let $q_i = q_i^A + q_i^B$ be the total quantity produced by firm $i \in \{1, 2\}$, and let $q^j = q_1^j + q_2^j$ be the total quantity sold in country $j \in \{A, B\}$. The demand for coal in countries A and B is given respectively by,

$$p^j = 90 - q^j,\ j \in \{A, B\},$$

and the costs of production for each firm is given by,

$$c_i(q_i) = 10q_i\,,\ i \in \{1, 2\}.$$

(a) Assume that the countries do not have a trade agreement and, in fact, imports in both countries are prohibited. This implies that $q_2^A = q_1^B = 0$ is set as a political constraint. What quantities q_1^A and q_2^B will both firms produce?

Answer: Each firm is a monopolist in its own country. Let and maximizes,

$$\max_{q_i^j \geq 0} (90 - q_i^j)q_i^j - 10q_i^j$$

where either $i = 1$ and $j = A$, or $i = 2$ and $j = B$ (so that $q_2^A = q_1^B = 0$ is set by assumption.) The first order maximization condition is $90 - 2q_i^j - 10 = 0$, which yields $q_1^A = q_2^B = 40$. The payoff for each firm is $1,600$. ■

Now assume that the two countries sign a free-trade agreement that allows foreign firms to sell in their countries without any tariffs. There are, however shipping costs. If firm i sells quantity q_i^j in the foreign country (i.e., firm 1 selling in B or firm 2 selling in A) then shipping costs are equal to $10q_i^j$. Assume further that *each firm* chooses a pair of quantities q_i^A, q_i^B simultaneously, $i \in \{1, 2\}$, so that a profile of actions consists of four quantity choices.

(b) Model this as a normal form game and find a Nash equilibrium of the game you described. Is it unique?

Answer: This game has two players, $i \in \{1, 2\}$, each choosing a strategy that consists of two non-negative quantities, $(q_i^A, q_i^B) \in \mathbb{R}_+^2$, and the payoff of the two players are given by,

$$\begin{aligned} v_1(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_1^A + (90 - q_1^B - q_2^B)q_1^B - 10(q_1^A + q_1^B) - 10q_1^B, \\ v_2(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_2^A + (90 - q_1^B - q_2^B)q_2^B - 10(q_2^A + q_2^B) - 10q_2^A, \end{aligned}$$

where the first term is the firm's revenue in market A, the second is the revenue in market B, the third is the total production cost and the last is the shipping cost. Given beliefs (q_2^A, q_2^B) about what firm 2 chooses to produce, firm 1's optimization requires two partial derivatives with respect to q_1^A and q_1^B as follows,

$$\frac{\partial v_1(q_1^A, q_1^B, q_2^A, q_2^B)}{\partial q_1^A} = 90 - q_2^A - 2q_1^A - 10 = 0,$$
$$\frac{\partial v_1(q_1^A, q_1^B, q_2^A, q_2^B)}{\partial q_1^B} = 90 - q_2^B - 2q_1^B - 20 = 0,$$

which in turn lead to the two parts of firm 1's best response function,[3]

$$q_1^A = \frac{80 - q_2^A}{2}, \tag{5.5}$$
$$q_1^B = \frac{70 - q_2^B}{2}. \tag{5.6}$$

It is easy to see that the objective of firm 2 is symmetric to that of firm 1 and hence we can directly write down firm 2's best responses as,

$$q_2^A = \frac{70 - q_1^A}{2}, \tag{5.7}$$
$$q_2^B = \frac{80 - q_1^B}{2}. \tag{5.8}$$

The Nash equilibrium is solved by finding a profile of strategies $(q_1^A, q_1^B, q_2^A, q_2^B)$ for which (5.5), (5.6), (5.7) and (5.8) all hold simultaneously. From (5.5) and (5.7) we obtain $q_1^A = 30$ and , $q_2^A = 20$. Similarly, from (5.6) and (5.8) we obtain $q_1^B = 20$ and , $q_2^B = 30$. The payoff of each firms would be equal to $1,300$.

Now assume that before the game you described in (b) is played, the research department of firm 1 discovered that shipping coal with the current ships causes the release of pollutants. If the firm would disclose this report to the World-Trade-Organization (WTO) then the WTO would prohibit the use of the current ships. Instead, a new shipping

[3] Because the payoff function has no interactions between the markets (i.e., it is *separable* in the two markets so that there are no interactions through the cost function) then q_1^A depends only on q_2^A and q_1^B depends only on q_2^B (and vice versa for firm 2). If costs were not linear then this would not be the case and the solution would involve solving four equations wit four unknowns simultaneously.

technology would be offered that would increase shipping costs to $40q_i^j$ (instead of $10q_i^j$ as above).

(c) Would firm 1 be willing to release the information to the WTO? Justify your answer with an equilibrium analysis.

Answer: To answer this we need to solve the Nash equilibrium with the more expensive shipping technology and compare the profits to that of the current cheaper technology. We know that a monopolist (or competitive firm) would never prefer a more expensive technology to a cheaper one, but here there may be interesting strategic effects: the more expensive shipping technology will dampen competition. The new payoff functions are

$$\begin{aligned} v_1(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_1^A + (90 - q_1^B - q_2^B)q_1^B - 10(q_1^A + q_1^B) - 40q_1^B, \\ v_2(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_2^A + (90 - q_1^B - q_2^B)q_2^B - 10(q_2^A + q_2^B) - 40q_2^A, \end{aligned}$$

and following the same arguments in part (b) above, the four equations that will define the best responses of both firms are,

$$q_1^A = \frac{80 - q_2^A}{2}, \tag{5.9}$$

$$q_1^B = \frac{40 - q_2^B}{2}. \tag{5.10}$$

and,

$$q_2^A = \frac{40 - q_1^A}{2}, \tag{5.11}$$

$$q_2^B = \frac{80 - q_1^B}{2}. \tag{5.12}$$

From (5.9) and (5.11) we obtain $q_1^A = 40$ and , $q_2^A = 0$. Similarly, from (5.10) and (5.12) we obtain $q_1^B = 0$ and , $q_2^B = 40$. The payoff of each firms would be equal to $1,600$, as we calculated in part (a) above. Hence, the firm would like to disclose the information and let the WTO impose a ban that would effectively kill cross-border competition. ■

Ex. 5.16

"mass" or quantity of buyers in the interval $[a, b]$ is equal to $b - a$.) Imagine two firms, players 1 and 2 who are located at each end of the interval (player 1 at the 0 point and player 2 at the 1 point.) Each player i can choose its price p_i, and each customer goes to the vendor who offers them the highest value. However, price alone does not determine the value, but distance is important as well. In particular, each buyer who buys the product from player i has a net value of $v - p_i - d_i$ where d_i is the distance between the buyer and vendor i, and represents the transportation costs of buying from vendor i. Thus, buyer $a \in [0, 1]$ buys from 1 and not 2 if $v - p_1 - d_1 > v - p_2 - d_2$, and if buying is better than getting zero. (Here $d_1 = a$ and $d_2 = 1 - a$. The buying choice would be reversed if the inequality is reversed.) Finally, assume that the cost of production is zero.

(a) Assume that v is very large so that all the customers will be served by at least one firm, and that some customer $x^* \in [0, 1]$ is indifferent between the two firms. What is the best response function of each player?

Answer: Because customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, his indifference implies that

$$v - p_1 - x^* = v - p_2 - (1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1 + p_2 - p_1}{2} .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1 + p_2 - p_1}{2} , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1 + p_1 - p_2}{2} . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1 + p_2 - p_1}{2} \right) p_1$$

which yields the first order condition

$$1 + p_2 - 2p_1 = 0 \ ,$$

implying the best response function

$$p_1 = \frac{1}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{2} + \frac{p_1}{2}.$$

■

(b) Assume that $v = 1$. What is the Nash equilibrium? Is it unique?

Answer: If we use the best response functions calculated in part (a) above then we obtain a unique Nash equilibrium $p_1 = p_2 = 1$, and this implies that $x^* = \frac{1}{2}$ so that each firm gets half the market. However, when $v = 1$ then the utility of customer $x^* = \frac{1}{2}$ is $v - p_1 - \frac{1}{2} = 1 - 1 - \frac{1}{2} = -\frac{1}{2}$, implying that he would prefer not to buy, and by continuity, an interval of customers around x^* would also prefer not to buy. his violated the assumptions we used to calculate the best response functions.[5] So, the analysis in part (a) is invalid when $v = 1$. It is therefore useful to start with the monopoly case when $v = 1$ and see how each firm would have priced if the other is absent. Firm 1 maximizes

$$\max_{p_1} \ (1 - p_1)p_1$$

which yields the solution $p_1 = \frac{1}{2}$ so that everyone in the interval $x \in [0, \frac{1}{2}]$ wished to buy from firm 1 and no other customer would buy. By symmetry, if firm 2 were a monopoly then the solution would be $p_2 = \frac{1}{2}$ so that everyone in the interval $x \in [\frac{1}{2}, 1]$ would buy from firm 2 and no other customer would buy. But this implies that if both firms set their

[5] We need $v \geq 1.5$ for customer $x^* = \frac{1}{2}$ to be just indifferent between buying and not buying when $p_1 = p_2 = 1$. All the other customers will strictly prefer buying.

monopoly prices $p_1 = p_2 = \frac{1}{2}$ then each would maximize profits ignoring the other firm, and hence this is the (trivially) unique Nash equilibrium. ■

(c) Now assume that $v = 1$ and that the transportation costs are $\frac{1}{2}d_i$, so that a buyer buys from 1 if and only if $v - p_1 - \frac{1}{2}d_1 > v - p_2 - \frac{1}{2}d_2$. Write the best response function of each player and solve for the Nash Equilibrium.

Answer: Like in part (a), assume that customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, and he is indifferent between buying from either, so his indifference implies that

$$v - p_1 - \frac{1}{2}x^* = v - p_2 - \frac{1}{2}(1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + p_2 - p_1 .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + p_2 - p_1 , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + p_1 - p_2 . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + p_2 - p_1\right) p_1$$

which yields the first order condition

$$\frac{1}{2} + p_2 - 2p_1 = 0 ,$$

implying the best response function

$$p_1 = \frac{1}{4} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{4} + \frac{p_1}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \frac{1}{2}$, and $x^* = \frac{1}{2}$. To verify that this is a Nash equilibrium notice that for customer x^*, the utility form buying from firm 1 is $v - p_1 - \frac{1}{2} = 1 - \frac{1}{2} - \frac{1}{2} = 0$ implying that he is indeed indifferent between buying or not, which in turn implies that every other customer prefer buying over not buying.

■

(d) Following your analysis in (c) above, imagine that transportation costs are αd_i, with $\alpha \in [0, \frac{1}{2}]$. What happens to the Nash equilibrium as $\alpha \to 0$? What is the intuition for this result?

Answer: Using the assumed indifferent customer x^*, his indifference implies that

$$v - p_1 - \alpha x^* = v - p_2 - \alpha(1 - x^*)$$

$$v - p_1 - \alpha x = v - p_2 - \alpha(1 - x)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_1 - p_2) \ . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1)\right) p_1$$

which yields the first order condition

$$\frac{1}{2}+\frac{p_2}{2\alpha}-\frac{p_1}{\alpha}=0 \ ,$$

implying the best response function

$$p_1=\frac{\alpha}{2}+\frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2=\frac{\alpha}{2}+\frac{p_1}{2}.$$

$$p_2=\frac{\alpha}{2}+\frac{\frac{\alpha}{2}+\frac{p_2}{2}}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \alpha$, and $x^* = \frac{1}{2}$. From the analysis in (c) above we know that for any $\alpha \in [0, \frac{1}{2})$ customer x^* will strictly prefer to buy over not buying and so will every other customer. We see that as α decreases, so do the equilibrium prices, so that at the limit of $\alpha = 0$ the prices will be zero. The intuition is that the transportation costs d cause firms 1 and 2 to be differentiated, and this "softens" the Bertrand competition between the two firms. When the transportation costs are higher this implies that competition is less fierce and prices are higher, and the opposite holds for lower transportation costs.■

17. **To vote or not to vote:** Two candidates, D and R, are running for mayoral election in a town with n residents. A total of $0 < d < n$ residents support candidate D while the remainder $r = n - d$ support candidate R. The value for each resident for having their candidate win is 4, for having him tie is 2, and for having him lose is 0. Going to vote costs each resident 1.

    (a) Let $n = 2$ and $d = 1$. Write down this game as a matrix and solve for the Nash equilibrium.

    Answer: The game is between the residents as the candidates seem not