EconS 503 - Microeconomic Theory II
Homework #1 - Answer Key

1. [**Strict dominance and Rationalizability**] Consider the 3x3 matrix at the bottom of page 72 in Tadelis.

Player 2

		L	C	R
	U	3, 3	5, 1	6, 2
Player 1	M	4, 1	8, 4	3, 6
	D	4, 0	9, 6	6, 8

(a) Find the strict dominant equilibrium of this game.

- This game does not have a strict dominant equilibrium because none of the players has a strictly dominant strategy. For a strict dominant equilibrium to exist, we need all players to use a strictly dominant strategy (or strategies). In this context, we need that, every player i, strategy s_i satisfies

$$u_i(s_i, s_j) \geq u_i(s_i', s_j)$$

for every $s_i' \neq s_i$ and for all $s_j \in S_j$.

(b) Which strategy profile/s survive IDSDS?

- Let us start with player 1, who does not have strictly dominated strategy. To see this, note that:
  - $u_1(U, s_2) < u_1(M, s_2)$ when player 2 selects $s_2 = L$ (in the left-hand column) and when he selects $s_2 = C$ (in the center column), but
  - $u_1(U, s_2) > u_1(M, s_2)$ when player 2 selects $s_2 = R$ in the right-hand column.
- A similar argument applies when comparing player 1's payoffs from choosing M and D:
  - $u_1(M, L) = u_1(D, L)$ when player 2 chooses L,
  - $u_1(M, C) < u_1(D, C)$ when player 2 chooses C, and
  - $u_1(M, R) < u_1(D, R)$ when player 2 chooses R.
- We can now move to player 2, where C is strictly dominated by R since $u_2(C, s_1) < u_2(R, s_1)$ for every strategy s_1 chosen by player 1. We can then reproduce the remaining matrix after the first two rounds of IDSDS, i.e., after deleting nothing for player 1 and strategy C for player 2.

Player 2

		L	R
	U	3, 3	6, 2
Player 1	M	4, 1	3, 6
	D	4, 0	6, 8

- We cannot find any more strictly dominated strategies relying on pure strategies. (As a practice, check that allowing for player 1 to randomize would not help us to further reduce the set of strategy profiles surviving IDSDS.) Then, the set of strategies surviving IDSDS is the six strategy profiles in the reduced matrix:

$$\{(U, L), (U, R), (M, L), (M, R), (D, L), (D, R)\}$$

(c) Which strategy profile/s survive rationalizability?

- The following payoff matrix underlines the payoffs that each player obtains when playing a best response to his opponent's strategies.

Player 2

Player 1		L	C	R
	U	3, $\underline{3}$	5, 1	$\underline{6}$, 2
	M	$\underline{4}$, 1	8, 4	3, $\underline{6}$
	D	4, 0	$\underline{9}$, 6	$\underline{6}$, $\underline{8}$

For instance, when player 2 chooses strategy L, player 1's best response is $BR_1(L) = \{M, D\}$. Whereas, when player 2 chooses C, player 1's best response is $BR_1(C) = \{D\}$. Furthermore, when when player 2 chooses R, player 1's best response is $BR_1(R) = \{U, D\}$. In other words, player 1 deploys all of his available strategies as a best response to at least one of his opponent's strategies. Alternatively, player 1 finds that the set of strategies that are never a best response is nil, that is, $NBR_1 = \varnothing$.

- Operating similarly for player 2, we find that his best responses are

$$BR_2(U) = \{L\}, \ \ BR_2(M) = \{R\}, \ \text{ and } BR_2(D) = \{R\},$$

such that the strategy that player 2 never uses as a best response is $NBR_2 = \{C\}$. We can now delete the strategies that are NBR to obtain the following reduced matrix in the first round of the application of rationalizability,

Player 2

Player 1		L	R
	U	3, 3	6, 2
	M	4, 1	3, 6
	D	4, 0	6, 8

Move to player 1 again, we can find that his best responses are now

$$BR_1(L) = \{M, D\} \text{ and } BR_1(R) = \{U, D\}$$

implying that, again, $NBR_1 = \varnothing$. In the second round of applying rationalizability, we can therefore find no more strategies that are NBR to either player, leaving us with the following equilibria from the application of rationalizability:

$$\{(U, L), (U, R), (M, L), (M, R), (D, L), (D, R)\}$$

2. **[Two short proofs]**

(a) If a strategy profile is a Nash equilibrium of a N-player game, it must also survive rationalizability.

- We first recall the definition of Nash equilibrium we discussed in class: Strategy profile $s^* = (s_1^*, \ldots, s_N^*)$ is a Nash equilibrium if every player i finds that his equilibrium strategy $s_i^* \in S_i$ satisfies

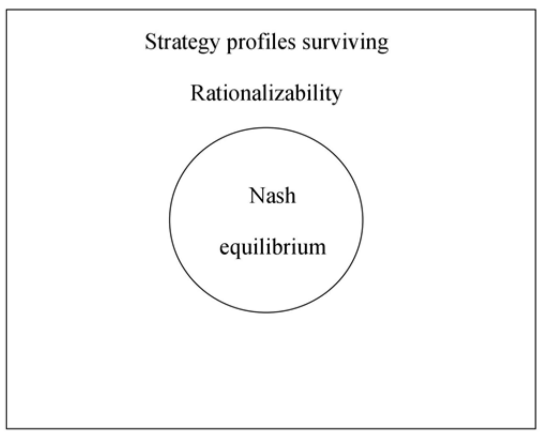
$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_i \in S_i$, $s_i \neq s_i^*$, and $s_{-i} \in S_{-i}$.

- *NE survives rationalizability.* Operating by contradiction, consider that s_i^* is a Nash equilibrium strategy for player i, but suppose that s_i^* is not rationalizable. Then there must be another strategy, $s_i' \neq s_i^*$, where $s_i' \in S_i$, such that

$$u_i(s_i', s_{-i}) > u_i(s_i^*, s_{-i})$$

In words, s_i^* is never a best response (i.e., NBR) to *any* strategy profile $s_{-i} \in S_{-i}$ chosen by player i's opponents. In particular, when $s_{-i} = s_{-i}^*$, the above inequality implies that s_i^* is not a BR to s_{-i}^*, thus contradicting that s_i^* being a Nash equilibrium strategy.



(b) The converse of (a) is not necessarily true. (For this part, an example suffices.)

- *Rationalizable strategies are not necessarily NE.* Consider the following example:

		Player 2	
		L	R
Player 1	U	5, 1	0, 0
	D	0, 0	3, 2

The best responses of both players are

$$BR_1(L) = U \qquad BR_2(U) = L$$
$$BR_1(R) = D \qquad BR_2(D) = R$$

For illustrative purposes, the next matrix underlines best response payoffs for players 1 and 2:

		Player 2	
		L	R
Player 1	U	$\underline{5}, \underline{1}$	$0, 0$
	D	$0, 0$	$\underline{3}, \underline{2}$

According to rationalizability, there is no strategy that is never played by both players, that is, $NBR_1 = \varnothing$ for player 1 and $NBR_2 = \varnothing$ for player 2. Intuitively, player 1 responds with either U or D to player 2's strategy choice (U after L, and D after R), leaving no strategy of player 1 unused. A similar argument applies for player 2, who uses L to respond to U, and R to respond to D, leaving him with no strategy unused.

- Therefore, the set of rationalizable strategies is

$$\textit{Rationalizable} = \{(U, L), (U, R), (D, L), (D, R)\}$$

while the set of Nash equilibria is

$$NE = \{(U, L), (D, R)\}$$

Therefore, the set of rationalizable strategies contains strategy profiles, in particular, $(U, R), (D, L)$, that are not Nash equilibria. As depicted in figure 1, the set of Nash equilibrium strategies is a subset of those strategy profiles surviving rationalizability.

3. **[IDWDS and Sophisticated equilibrium]** Consider a game with N players, where player i's strategy space is denoted as S_i. Assume that the game is solvable by IDWDS, yielding the surviving strategy set S_i^{Sur} for each player i, that is, $S_i^{Sur} \subset S_i$. Therefore, the strategy profile surviving IDWDS is denoted as the Cartesian product of surviving strategy sets $s^{Sur} \in S_1^{Sur} \times ... \times S_N^{Sur}$.

We say that strategy profile surviving IDWDS, s^{Sur}, is a "*sophisticated equilibrium*" if every player i is indifferent between any two of his surviving strategies s_i and s_i', that is,

$$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \text{ for every } s_i, s_i' \in S_i^{Sur} \text{ and all } s_{-i} \in S_{-i}.$$

Consider the following normal-form game.

		Player 2		
		L	M	R
	U	$5, 4$	$5, 4$	$9, 0$
Player 1	C	$1, 7$	$2, 5$	$8, 6$
	D	$2, 3$	$1, 4$	$8, 3$

Find the set of strategy profiles surviving IDWDS. Can you identify any sophisticated equilibria?

- Let us start with player 1, whose strategies C and D do not survive IDWDS. Indeed, U strictly dominates both C and D, since

$$\begin{aligned} u_1(U, s_2) &> u_1(C, s_2) \quad \text{for all } s_2 \in S_2, \text{ and} \\ u_1(U, s_2) &> u_1(D, s_2) \quad \text{for all } s_2 \in S_2 \end{aligned}$$

(As a remark, note that strategies C and D don't survive IDSDS either). We can then C and D from the above matrix, obtainin the following reduced-form matrix:

		Player 2		
		L	M	R
Player 1	U	$5, 4$	$5, 4$	$9, 0$

We can now move to player 2, whose strategy R is strictly dominated by both L and M because

$$u_2(U, R) < u_2(U, L) = u_2(U, M)$$

Therefore, applying IDWDS yields a further reduced-form matrix as follows:

		Player 2	
		L	M
Player 1	U	5, 4	5, 4

As seen from the above, the Cartesian product of surviving strategies for both players becomes

$$S^{Sur} = S_1^{Sur} \times S_2^{Sur} = \{(U, L), (U, M)\}$$

which yields the same payoff to players 1 and 2, that is, $u_1(U, L) = u_1(U, M) = 5$ for player 1, and $u_2(U, L) = u_2(U, M) = 4$ for player 2. Generally, these two expressions can be more compactly written as

$$u_i(U, L) = u_i(U, M) \quad \text{for every player } i = \{1, 2\}.$$

Therefore, the equilibrium strategy profiles surviving IDSDS are "sophisticated" since both players obtain the same payoff in each of these strategy profiles.

4. [**A generalized Battle of the Sexes**] Consider the following, more general, version of the Battle of the Sexes game:

		Wife	
		Football	*Opera*
Husband	*Football*	$h + t, t$	h, w
	Opera	$0, 0$	$t, w + t$

where h (w) denotes the husband's (wife's) payoff from being at his (her) most preferred event, relative to his (her) less preferred event, regardless of whether his (her) spouse attends that event too. Parameter t represents the payoff he (she) obtains when attending the same event as his (her) spouse. For simplicity, assume that all three parameters are positive, $h, w, t > 0$.

(a) Can you find any strictly dominated strategies for either player? Does your result depend on parameter values?

- The Husband finds Opera to be strictly dominated if: (1) $h + t > 0$ which holds by assumption; and (2) $h > t$, which is not implied by our assumptions. Summarizing, we can say that Opera is strictly dominated for the Husband if $h > t$, which intuitively means that he strictly prefers to be at the Football game alone than at the Opera with his wife. Otherwise, he doesn't have a strictly dominated strategy, and chooses the same strategy as his wife.
- A symmetric argument applies for the Wife. She finds Football strictly dominated if: (1) $w > t$, which is not implied by our initial assumptions; and (2) $w + t > 0$, which holds by definition. Therefore, the Wife finds Football to be strictly dominated as long as $w > t$, which means that she strictly prefers to be at the Opera alone than at the Football game with her Husband. Otherwise, she doesn't have a strictly dominated strategy, and chooses the same strategy as her husband.

(b) What strategy profile survives IDSDS?

- Since part (a) identified two conditions, $h > t$ and $w > t$, four different cases emerge:
  - **First case, $h > t$ and $w > t$.** In this case, Opera is strictly dominated for the Husband. After deleting the row corresponding to Opera from the payoff matrix, we are left with a reduced-form matrix containing only the top row. In our second step of IDSDS, the Wife anticipates that only the top row survives, leading her to delete Football since $w > t$ in this first case. Therefore,
  $$IDSDS = (F, O)$$
  is the unique equilibrium prediction according to IDSDS. Intuitively, both players value their most preferred events more than being together, and thus miscoordinate attending (alone) their most preferred event.
  - **Second case, $h > t$ and $w < t$.** In this case, Opera is still strictly dominated for the Husband. After deleting the row corresponding to Opera from the payoff matrix, we are left with a reduced-form matrix containing only the top row. In our second step of IDSDS, the Wife anticipates that only the top row survives, leading her to delete Opera since $w < t$ in this second case. Therefore,
  $$IDSDS = (F, F)$$
  is the unique equilibrium prediction according to IDSDS. Intuitively, the Husband values the Football game more than being together but the Wife values being together more than the Opera, and thus coordinate attending his most preferred event.
  (As a practice, note that the same equilibrium outcome arises if we start deleting strictly dominated strategies for the Wife and then move on the Husband. The Wife, however, doesn't have any strictly dominated strategies in the initial payoff matrix, but when we move to the Husband in the

second step of IDSDS we face the same arguments as in this paragraph, deleting Opera for him in the second step of IDSDS, and finally deleting Opera for the Wife.)

- **Third case,** $h < t$ **and** $w > t$**.** In this case, the Husband does not have strictly dominated strategies. The Wife, however, finds Football to be strictly dominated. After deleting the column corresponding to Football from the payoff matrix, we are left with a reduced-form matrix containing only the right-hand column. In our second step of IDSDS, the Husband anticipates that only the right-hand column survives, leading him to delete Football since $h < t$ in this third case. Therefore,

$$IDSDS = (O, O)$$

is the unique equilibrium prediction according to IDSDS. Intuitively, the Wife values Opera more than being together but her Husband values being together more than Football, and thus coordinate attending her most preferred event.

- **Fourth case,** $h < t$ **and** $w < t$**.** In this case, neither the Husband or the Wife have strictly dominated strategies. As a consequence, we cannot delete any strategy in any step of IDSDS, leaving us with the whole matrix surviving IDSDS. All four strategy profiles are then our equilibrium prediction according to IDSDS:

$$IDSDS = \{(F, F), (F, O), (O, F), (O, O)\}$$

Intuitively, both players value being together more than their most preferred event —as in the standard Battle of the Sexes game. Therefore, they seek to attend the same event as the other player, not letting us delete any row or column as being strictly dominated.

(c) Find the Nash equilibrium of the game when players are restricted to use pure strategies.

- We can underline best response payoffs in the payoff matrix, keeping track that we now face nine different cases. Note that we face more cases than in part (b) since the definition of best response allows for player i to be indifferent between two strategies.
  - **First case,** $h > t$ **and** $w > t$**.** The following payoff matrix underlines best response payoffs in this case.

		Wife	
		Football	*Opera*
Husband	*Football*	$\underline{h+t}, t$	$\underline{h}, \underline{w}$
	Opera	$0, 0$	$t, \underline{w+t}$

Therefore, the unique NE is (F, O). This result was expected, since in case the unique equilibrium prediction according to IDSDS was (F, O) and we know that the set of strategy profiles surviving IDSDS are a superset of NE. (In games with only one strategy surviving IDSDS this property means that IDSDS=NE, as in the Prisoner's Dilemma game.)

- **Second case, $h > t$ and $w < t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife: *Football*	*Opera*
Husband	*Football*	$\underline{h+t},\underline{t}$	$\underline{h}, w$
	Opera	$0, 0$	$t,\underline{w+t}$

Therefore, the unique NE is (F, F). Again, this result was expected since IDSDS provided a unique equilibrium prediction in this case.

- **Third case, $h < t$ and $w > t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife: *Football*	*Opera*
Husband	*Football*	$\underline{h+t}, t$	$h,\underline{w}$
	Opera	$0, 0$	$\underline{t},\underline{w+t}$

Therefore, the unique NE is (O, O). Again, this result was expected since IDSDS provided a unique equilibrium prediction in this case.

- **Fourth case, $h < t$ and $w < t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife: *Football*	*Opera*
Husband	*Football*	$\underline{h+t},\underline{t}$	h, w
	Opera	$0, 0$	$\underline{t},\underline{w+t}$

Therefore, we found two NEs:

$$NE = \{(F, F), (O, O)\}.$$

which coincide with those in the standard Battle of the Sexes game. As expected the set of equilibrium predictions according to IDSDS in this fourth case is a superset of the NEs we just found.

- **Fifth case, $h = t$ and $w > t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife: *Football*	*Opera*
Husband	*Football*	$\underline{h+t}, t$	$\underline{h},\underline{w}$
	Opera	$0, 0$	$\underline{t},\underline{w+t}$

Therefore, we find two NEs:

$$NE = \{(F, O), (O, O)\}.$$

- **Sixth case, $h > t$ and $w = t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife	
		Football	*Opera*
Husband	*Football*	$\underline{h+t}$,$\underline{t}$	$\underline{h}$,$\underline{w}$
	Opera	$0, 0$	t,$\underline{w+t}$

Therefore, we find two NEs:

$$NE = \{(F, F), (F, O)\}.$$

- **Seventh case, $h = t$ and $w < t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife	
		Football	*Opera*
Husband	*Football*	$\underline{h+t}$,$\underline{t}$	$\underline{h}, w$
	Opera	$0, 0$	$\underline{t}$,$\underline{w+t}$

Therefore, we find two NEs:

$$NE = \{(F, F)(O, O)\}.$$

- **Eight case, $h < t$ and $w = t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife	
		Football	*Opera*
Husband	*Football*	$\underline{h+t}$,$\underline{t}$	h,$\underline{w}$
	Opera	$0, 0$	$\underline{t}$,$\underline{w+t}$

Therefore, we find two NEs:

$$NE = \{(F, F), (O, O)\}$$

- **Ninth case, $h = t$ and $w = t$.** The following payoff matrix underlines best response payoffs in this case.

		Wife	
		Football	*Opera*
Husband	*Football*	$\underline{h+t}$,$\underline{t}$	$\underline{h}$,$\underline{w}$
	Opera	$0, 0$	$\underline{t}$,$\underline{w+t}$

Therefore, we find three NEs:

$$NE = \{(F, F), (F, O), (O, O)\}.$$