Exercise 1—From extensive form to normal form representation
Consider the following extensive form game

a) Which are the strategies for player 1?
   Three strategies $S_1=\{H,M,L\}$

b) What are the strategies for player 2?
   Now the second mover can condition his move to the first player’s action, since he is able to observe it (unlike in the prisoners’ dilemma game). Hence, $S_2=\{aaa, aar, arr, rrr, rra, raa, ara, rar\}$ where each of them represents a complete plan of action that specifies what to do in the event that player 1 chooses H, in the event that player 1 chooses M and in the case that he chooses L, respectively.

c) Take your results from a) and b) and construct a matrix representing its normal form game representation.
   If you take the about three strategies for player 1, and the above eight strategies for player 2, we have the following normal form game.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>aa’ a’’</td>
</tr>
<tr>
<td>H</td>
<td>0,10</td>
</tr>
<tr>
<td>M</td>
<td>5,5</td>
</tr>
<tr>
<td>L</td>
<td>10,0</td>
</tr>
</tbody>
</table>
Exercises 2 and 3.
Answer key: See pages at the end of this file.

Exercise 4 – Pure strategies that are only strictly dominated by a mixed strategy
Consider the following normal form game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>4,1</td>
</tr>
<tr>
<td>Middle</td>
<td>0,0</td>
</tr>
<tr>
<td>Down</td>
<td>1,3</td>
</tr>
</tbody>
</table>

Is there some strictly dominated strategy for player 1 involving only the use of pure strategies? No, there is no strategy for player 1 such that the payoff received by player 1 is always higher, regardless of the strategy chosen by player 2. Neither U nor M dominates the other, and D does not dominate these strategies. In addition, neither U nor M dominates D. For instance, although U is better than D when player 2 selects L, D performs better than U when player 2 selects R.

Is there some strictly dominated strategy for player 1 when mixed strategies are allowed? [Hint: you may assign probabilities to two of her strategies, similarly as we did in class]. There is a mixed strategy for player 1 that dominates D. Consider 1’s mixed strategy of selecting U with probability 1/2, M with probability 1/2 and D with probability 0. We represent this mixed strategy as (1/2, 1/2, 0). If player 2 selects L, then this mixed strategy yields an expected payoff to player 1 of

\[ \text{EU(mixed)} = 1/2*4+1/2*0+0*1=2, \]
and player 1 does worse by playing D (a payoff of only 1).
The same is true when player 2 selects R

\[ \text{EU(mixed)} = 1/2*0+1/2*4+0*1=2, \]
and player 1 does worse by playing D (a payoff of only 1).
Therefore, strategy D is strictly dominated by the mixed strategy (1/2, 1/2, 0).

Delete the strictly dominated strategies for player 1 that you found in the previous question. Then, represent the remaining (undeleted) strategies.
Once strategy D is deleted because of being strictly dominated by the above mixed strategy, we have a (reduced) normal form game given by

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>4,1</td>
</tr>
<tr>
<td>Middle</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Proceed with the IDSDS. What is the strategy pair surviving IDSDS?
In the above (reduced) normal form game, player 2 finds Left as being strictly dominated by strategy Right. Note that in the original game (before deleting any of player 1’s strategies, we could not eliminate any of the player 2’s strategies, since none of them were strictly dominated). Hence, the remaining strategies are

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>0,2</td>
</tr>
</tbody>
</table>
That makes strategy Up strictly dominated for player 1, in this iterative process of deletion of strictly dominated strategies. Hence, this game is dominance solvable, and the strategy pair surviving the IDSDS is \{Middle, Right\}.

**Exercise 10 of Chapter 3 – Harrington’s book**

**Part a)**

**ANSWER:** Let us first show that for either player, zero effort strictly dominates effort levels of 3, 4, and 5. If \(x_i = 0\) then \(i\)'s payoff is at least 8, which occurs when she gets a B. By choosing effort \(x_i\), the highest possible payoff is \(10 - x_i\), which occurs when she gets an A. Since \(8 > 10 - x_i\) when \(x_i > 2\), then zero effort strictly dominates effort of 3, 4, or 5. Now consider player 1. We know that strategies 3, 4, and 5 are strictly dominated, but what about strategies 0, 1, and 2? A useful point to note is that if there is some strategy for player 2 such that a particular strategy for player 1 is best (that is, it yields the highest payoff for player 1), then that strategy for 1 is not strictly dominated since strict dominance means there is another strategy that yields a strictly higher payoff for all strategies of the other player. In considering strategy 0, note that \(x_1 = 0\) is the uniquely best strategy for
player 1 when $x_2$ is 0 or 1, as in both cases player 1 receives a payoff of 10, whereas when $x_1 > 0$ his payoff is $10 - x_1$, which is lower. Thus, there is no strategy that strictly dominates zero effort. A similar argument shows that $x_1 = 1$ is not strictly dominated. If $x_2 = 2$, then $x_1 = 1$ yields a payoff of 9, while $x_1 = 0$ yields a payoff of 8 (as player 2’s score is higher in that case) and the payoff is $10 - x_1$ when $x_1 > 1$, which is lower than 9. Finally, note that $x_1 = 2$ is optimal when $x_2 = 3$, as the resulting payoff is 8, whereas the payoff is 8 from $x_1 = 0, 7$ from $x_1 = 1$, and $10 - x_1$ when $x_1 > 2$. Though $x_1 = 2$ generates the same payoff as $x_1 = 0$, we can still conclude that there is no strategy that strictly dominates $x_1 = 2$. It is concluded that the strategies for player 1 that survive the first round of deletion of strictly dominated strategies are $\{0, 1, 2\}$. Now consider player 2. $x_2 = 0$ is not strictly dominated, as it is the optimal strategy for player 2 when $x_1 \geq 4$, as in that case player 1 gets a B regardless of $x_2$. Since his payoff is then $8 - x_2$, it is clearly maximized at $x_2 = 0$. Now consider $x_2 = 1$. I want to show that this strategy is strictly dominated by $x_2 = 0$. Note that regardless of $x_1$, player 2 has the lower score whether he chooses $x_2 = 0$ or $x_2 = 1$. Since $x_2 = 0$ gives a payoff of 8 and $x_2 = 1$ gives a payoff of 7, zero effort strictly dominates effort of player 1. Now consider the remaining strategy of $x_2 = 2$. Suppose $x_1 = 0$. In that case, the payoff to player 2 from $x_2 \leq 1$ is $8 - x_2$ (since he gets a B) and from $x_2 \geq 2$ is $10 - x_2$ (since he gets an A). His payoff is maximized at $x_2 = 2$. Given there is some strategy for player 1 such that $x_2 = 2$ yields the maximum payoff, then $x_2 = 2$ cannot be strictly dominated. It is concluded that the strategies for player 2 that survive the first round of the IDSDS are $\{0, 2\}$. Now move to round 2 of this iterative process. The payoff matrix in Figure SOL3.10.1 shows the surviving strategies. The first number in a cell is player 1’s payoff.

**FIGURE SOL3.10.1**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

$x_1 = 1$ strictly dominates $x_1 = 2$. Neither of the other two strategies for player 1 is dominated. None of player 2’s strategies is strictly dominated since $x_2 = 0$ yields a higher payoff than does $x_2 = 2$ when $x_1 = 1$ or 2 and $x_2 = 2$ yields the same payoff as does $x_2 = 0$ when $x_1 = 0$. It is concluded that the strategies that survive the second round of the IDSDS for player 1 are $\{0, 1\}$ and for player 2 are $\{0, 2\}$. Moving to the third round, the resulting payoff matrix is shown in Figure SOL3.10.2.

**FIGURE SOL3.10.2**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>

Inspection reveals that no strategies are strictly dominated. Thus, the IDSDS predicts that player 1 will exert effort of 0 or 1 and player 2 will exert effort of 0 or 2.

**b.** Derive the strategies that survive the iterative deletion of weakly dominated strategies. (The procedure works the same as the iterative deletion of strictly dominated strategies, except that you eliminate all weakly dominated strategies at each stage.)

**ANSWER:** Now let us repeat the analysis when we are instead iteratively deleting weakly dominated strategies. From part (a), we know that effort of 3, 4, and 5 are all strictly dominated and therefore weakly dominated. For player 1, efforts of 0 and 1 are not weakly dominated for the same reasons given in part (a); there is a strategy for player 2 such that each yields the strictly highest payoff. While $x_1 = 2$ is not strictly dominated, it is weakly dominated by $x_1 = 0$. The latter yields a
Here is a more compact presentation of Harrington, Chapter 3, Exercise 10.

Payoff of at least 8 regardless what player 2 does, while \( x_1 = 2 \) yields a payoff of at most 8. Furthermore, zero effort does strictly better when player 2 chooses effort of 0 or 1. The strategies for player 1 that survive the first round of the IDSDS are [0,1]. With respect to player 2, it was previously shown that strategies 1, 3, 4, and 5 are strictly dominated and, therefore, weakly dominated. Zero effort is not weakly dominated by the same argument as in part (a). \( x_2 = 2 \) is weakly dominated for reasons analogous to those given for player 1. The lone strategy for player 2 that survives the first round of the IDSDS is \( x_1 = 0 \).

Moving to the second round, \( x_1 = 0 \) weakly dominates \( x_1 = 1 \) since both result in player 1’s getting an A (as player 2 is choosing 0 effort). So the former yields a payoff of 10 and the latter a payoff of 9. The IDSDS predicts that both players will exert zero effort.
2. **A variation of the Prisoner’s Dilemma game.** Consider the following Prisoner’s Dilemma game. The game coincides with that we discussed in class, except for the fact that every player sees his payoff decrease by \( m \geq 0 \) when he chooses to confess. For instance, prisoner 1’s payoff decreases by \( m \) in the top row (where he confesses) but is unaffected when he is at the bottom row (where he does not confess). A similar argument applies to prisoner 2, who sees his payoff decrease in the left column (where he confesses) but not in the right-hand column (where he remains silent). Intuitively, \( m \) represents the punishment that the confessing prisoner suffers from other criminals, either in jail (when he serves some time) or on the streets (when he does not serve any time in jail).

\[
\begin{array}{c|cc}
\text{Prisoner 1} & C & NC \\
\hline
C & -5 - m, -5 - m & -m, -15 \\
NC & -15, -m & -1, -1 \\
\end{array}
\]

(a) Find if either player has a strictly dominated strategy. Does your result depend on the value of the punishment, \( m \)?

- For all players, NC is strictly dominated if it yields a lower payoff than C regardless of the strategy that his rival selects, that is, we need that:
  - \(-5 - m > -15\) when your rival picks C, which simplifies to \(10 > m\); and
  - \(-m > -1\) when your rival picks NC, which simplifies to \(m < 1\).

- Putting both conditions together, we have find three cases:
  - Strategy NC is strictly dominated if \(m < 1\). (Note that this condition guarantees both \(m < 10\) and \(m < 1\).) Intuitively, every player \(i\) finds NC to be strictly dominated (as in the standard Prisoner’s Dilemma game) when punishment \(m\) is relatively small. Interestingly, condition \(m < 1\) includes the setting where \(m = 0\) (no punishments, as in the standard Prisoner’s Dilemma game) as a special case.
  - Strategy C is strictly dominated if \(m > 10\) (in this case, both \(m < 10\) and \(m < 1\) are violated). Intuitively, the punishment is so large that all players find C to be strictly dominated, i.e., never choose C regardless of how his opponent behaves.
  - Neither strategy is strictly dominated when the punishment \(m\) is intermediate, that is, \(10 > m > 1\). In this case, every player would like to choose C when his rival chooses C, but NC when his rival chooses NC. Therefore, we cannot delete any strategy (row or column) as being strictly dominated.

(b) Using your results from part (a), which is the strategy profile (or profiles) surviving Iterative Deletion of Strictly Dominated Strategies (IDSDS)?
• **First case:** punishment $m$ satisfies $m < 1$. In this case, we know that NC is strictly dominated for both players (see part a of the exercise) implying that we can delete the row corresponding to strategy NC for Prisoner 1. We are then left with the following reduced-form matrix.

\[
\begin{array}{cc}
\text{Prisoner 2} & \text{C} & \text{NC} \\
\text{Prisoner 1} & C & 5 - m, -5 - m & -m, -15 \\
\end{array}
\]

Analyzing now Prisoner 2, we can delete the column corresponding to NC, since $-5 - m > -15$ simplifies to $m < 10$, a condition that holds in this case. Therefore, outcome (C,C) is the unique equilibrium prediction in the first case.

• **Second case:** punishment $m$ satisfies $m > 10$. In this case, C is strictly dominated for both players. Following a similar approach as in the first case, we find that in this case (NC,NC) is the only outcome surviving IDSDS.

• **Third case:** punishment $m$ satisfies $10 > m > 1$. In this case, no player has strictly dominated strategies. Therefore, when applying IDSDS, we cannot delete any row or column from the original payoff matrix. We are then left with the whole matrix as our equilibrium prediction or, in other words, the following four strategy profiles survive IDSDS:

\[
\text{IDSDS} = \{(C, C), (C, NC), (NC, C), (NC, NC)\}
\]

In this case, we say that IDSDS has no bite.

(c) Find if either player has a strictly dominant strategy? Does your result depend on the value of the punishment, $m$?

• From our analysis in part (a), we can say that every player finds:
  - Strategy C is strictly dominant if $m < 1$.
  - Strategy NC is strictly dominant if $m > 10$.
  - No strategy is strictly dominant otherwise, that is, when $10 > m > 1$.

• As discussed in part (a), when the punishment $m$ is relatively low ($m < 1$) players behave as in the standard Prisoner’s Dilemma game, when the punishment is relatively high ($m > 10$) players never choose confess (choosing NC instead), and when the punishment has an intermediate value ($10 > m > 1$) players do not have any strictly dominated or dominant strategy, implying that IDSDS has no bite.

3. **Unemployment benefits.** Consider the following simultaneous-move game between the government (row player), which decides whether to offer unemployment benefits, and an unemployed worker (column player), who chooses whether to search for a job. As you interpret from the payoff matrix below, the unemployed worker only finds it optimal to search for a job when he receives no unemployment benefit; while the government only finds it optimal to help the worker when he searches for a job.

\[
\begin{array}{cc}
\text{Worker} & \text{Search} & \text{Don’t Search} \\
\text{Government} & \text{Benefit} & 3, 2 & -1, 3 \\
& \text{No benefit} & -1, 1 & 0, 0 \\
\end{array}
\]
(a) Represent this game in its extensive form (game tree), where the government acts first and the worker responds without observing whether the government offered unemployment benefits.

(b) Does government have strictly dominant strategies? How about the worker?
   - There are no strictly dominant strategies for both government and worker. Specifically, when the worker chooses (not) to search for a job, the government will be better off (not) offering unemployment benefits. Whereas, when the government chooses (not) to offer unemployment benefits, the worker will be better off (not) searching for a job.

(c) Find which strategy profile (or profiles) survive the application of IDSDS.
   - In this context, every strategy profile survives IDSDS, which entails

\[
IDSDS = \{(\text{Benefit, Search}), (\text{Benefit, Don't Search}), (\text{No Benefit, Search}), (\text{No Benefit, Don't Search})\}
\]

In this type of games, we say that the application of IDSDS has "no bite" since it does not help us reduce the original set of strategy profiles.