1. **Exercise #6.9, Munoz-Garcia (2017).** Consider a pure-exchange economy with two individuals, A and B, each with utility function \( u^i(x^i, y^i) \) where \( i = \{A, B\} \), whose initial endowments are \( e^A = (10, 0) \) and \( e^B = (0, 10) \), that is, individual A (B) owns all units of good \( x \) (\( y \), respectively).

(a) Assuming that utility functions are \( u^i(x^i, y^i) = \min\{x^i, y^i\} \) for all individuals \( i = \{A, B\} \), find the set of PEAs and the set of WEAs.

- **PEAs.** Since the utility functions are not differentiable we cannot follow the property of \( \text{MRS}^A_{x,y} = \text{MRS}^B_{x,y} \) across consumers. Figure 1 helps us identify the set of PEAs. Points away from the 45°-line, satisfying \( y^A = x^A \), such as \( N \), cannot be pareto efficient since we can still find other points, such as \( M \), where consumer 2 is make better off while consumer 1 reaches the same utility level as under \( N \). Once we are at points on the 45°-line, such as \( M \), we cannot find other points making at least once consumer better off (and keep the other consumer at least as well off). Hence, the set of PEAs is

\[
\{ (x^A, y^A), (x^B, y^B) : y^A = x^A \text{ and } y^B = x^B \}
\]

![Figure 1. Edgeworth box and PEAs.](image)

- **WEAs.** Using good 2 as the numeraire, i.e., \( p_2 = 1 \), the price ratio becomes \( \frac{p_1}{p_2} = p_1 \). The budget line of both consumers therefore has a slope \(-p_1\) and crosses the point representing the initial endowment \( e \) in Figure 2 (where \( e \)
lies at the lower right-hand corner)

(b) Assuming utility functions of $u^A(x^A, y^A) = x^A y^A$ and $u^B(x^B, y^B) = \min\{x^B, y^B\}$, find the set of PEs and WEAs.

- PEs. By the same argument as in question (a), the set of PEs satisfies $y^A = x^A$, as depicted in Figure 3. Point $N$ cannot be efficient as we can still find other feasible points, such as $M$, where at least one consumer is made strictly better off (in this case consumer $A$). At points on the $45^\circ$-line, however, we can no longer find alternatives that would constitute a Pareto improvement.
**WEAs.** Using good \( y \) as the numeraire, \( p_y = 1 \), so that the price vector becomes \( \mathbf{p} = (p_x, 1) \). Hence, **Consumer A’s UMP is**

\[
\max_{x^A, y^A} x^A y^A \\
\text{subject to } p_x x^A + y^A = 10 p_x
\]

Taking first-order conditions

\[
\begin{align*}
y^A - \lambda^A p_x &= 0 \\
x^A - \lambda^A &= 0 \\
p_x x^A + y^A &= 10 p_x
\end{align*}
\]

Combining the first two FOCs and rearranging, we have

\[p_x x^A = y^A\]

and substituting this equation into the third FOC yields

\[p_x x^A + p_y x^A = 10 p_x \implies x^A = 5\]

and substituting this back into \( p_x x^A = y^A \)

\[y^A = 5 p_x\]

**Consumer B’s UMP is not differentiable, but in equilibrium his Walrasian demands satisfy** \( x^B = y^B \). **Substituting this into his budget constraint yields**

\[p_x x^B + x^B = 10 \implies x^B = y^B = \frac{10}{p_x + 1}\]

Furthermore, the feasibility condition for good \( x \) entails

\[5 + \frac{10}{p_x + 1} = 10 + 0, \text{ or } p_x = 1\]

Therefore, the market of good \( x \) will clear at an equilibrium price of \( p_x = 1 \), i.e., \( z_x(p_x, 1) = 0 \) when \( p_x = 1 \). Since market \( y \) clears when market \( x \) does (by Walras’ law), \( z_y(p_x, 1) \) must also be zero when \( p_x = 1 \). Summarizing, the equilibrium price \( p_x = 1 \) yields a WEA

\[\{(5, 5), (5, 5)\}\]

2. **Based on Exercise #2.20 in Hashimzade et al. (2006).** A consumer views two goods as perfect substitutes.

   (a) Sketch the indifference curves of the consumer.
Since consumer A views the two goods as perfect substitutes, one unit of good 1, \( x_1^A \), is worth one unit of good 2, \( x_2^A \), so that the indifference curves are straight lines away from the origin with a gradient of \(-1\), as illustrated in Figure 4.

(b) If an economy is composed of two consumers with these preferences, demonstrate that any allocation is Pareto-efficient,

- For any initial endowment of this economy, the indifference curve of consumer A (solid line) coincides with that of consumer B (dashed line) at every point of the Edgeworth box, so that every allocation is Pareto-efficient, as illustrated in Figure 5.

(c) If an economy has one consumer who views its two goods as perfect substitutes and a second that consider each unit of good 1 to be worth 2 units of good 2, find the Pareto-efficient allocations.
- Without loss of generality, let consumer $A$ views its two goods as perfect substitutes, and consumer $B$ considers each unit of good 1 to be worth 2 units of good 2. As seen in Figure 6, the indifference curves intersect along the $x_1^B$ and $x_2^A$ axes, which represent the set of Pareto-efficient allocations (outer edges of the Edgeworth box, the inverted L-shaped curve bolded in blue color). In this context, consumer $B$ would consume good 1 only (along the $x_1^B$ axis) and no good 2, and not until he trades all units of good 1 with consumer $A$ does he begin to consume good 2 (along the $x_2^A$ axis); while the exact location of the equilibrium outcome depends on the initial endowment of this economy.

![Figure 6. Efficient allocations.](image)

3. Based on Exercises #2.11 and 2.24 in Hashimzade et al. (2006). Consider a pure exchange economy with 2 consumers and 2 goods. Consumer $i$, where $i \in \{A, B\}$, has a utility function of

$$U^i = \gamma \log(x_1^i) + (1 - \gamma) \log(x_2^i),$$

which is the geometric average of his consumption of good 1, $x_1^i$ (with weight $\gamma$), and good 2, $x_2^i$ (with weight $1 - \gamma$).

(a) Let consumer $A$ has an endowment of $(\omega_1^A, \omega_2^A) = (2, 1)$ for goods 1 and 2, and similarly, $(\omega_1^B, \omega_2^B) = (3, 2)$ for consumer $B$. Find the Walrasian demand of both consumers.

- The budget constraint of consumer $i$, where $i \in \{A, B\}$, is

$$p_1 x_1^i + p_2 x_2^i = p_1 \omega_1^i + p_2 \omega_2^i$$

Rearranging, we have

$$x_2^i = \frac{p_1}{p_2} (\omega_1^i - x_1^i) + \omega_2^i \quad (1)$$
Assuming interior solutions for Walrasian allocation,

\[ MRS_{12}^i = \frac{MU_1^i}{MU_2^i} = \frac{\gamma}{x_1^i (1 - \gamma)} = \frac{p_1}{p_2} \]

Rearranging, we have

\[ x_1^i = \frac{\gamma p_2}{1 - \gamma p_1} x_2^i \quad (2) \]

Substituting expression (2) into expression (1),

\[ x_2^i = \frac{p_1}{p_2} \left( \omega_1^i - \frac{\gamma p_2}{1 - \gamma p_1} x_2^i \right) + \omega_2^i \]

Rearranging, we have

\[ \frac{1 - \gamma + \gamma x_2^i}{1 - \gamma} = \frac{p_1}{p_2} \omega_1^i + \omega_2^i \]

\[ \implies x_2^i = \frac{(1 - \gamma) (p_1 \omega_1^i + p_2 \omega_2^i)}{p_2} \]

Substituting the above into expression (2),

\[ x_1^i = \frac{\gamma (p_1 \omega_1^i + p_2 \omega_2^i)}{p_1} \]

Substituting the consumers’ endowments, the Walrasian demand functions become

\[ x_A^1 = \frac{\gamma (2p_1 + p_2)}{p_1} \]
\[ x_A^2 = \frac{(1 - \gamma) (2p_1 + p_2)}{p_2} \]
\[ x_B^1 = \frac{\gamma (3p_1 + 2p_2)}{p_1} \]
\[ x_B^2 = \frac{(1 - \gamma) (3p_1 + 2p_2)}{p_2} \]

(b) Setting the price of good 2 as a numéraire, that is, \( p_2 = 1 \), find the excess demand for good 1, \( z_1(p_1) \), and then plot it as a function of price \( p_1 \).

- Setting \( p_2 = 1 \), the excess demand function for good 1 becomes

\[ z_1(p_1) = x_A^1 + x_B^1 - \omega_A^i - \omega_B^i \]

\[ = \frac{\gamma (2p_1 + 1)}{p_1} + \frac{\gamma (3p_1 + 2)}{p_1} - 2 - 3 \]

\[ = 5 (\gamma - 1) + \frac{3\gamma}{p_1} \]
- Figure 7 plots the excess demand function, $z_1(p_1)$, with the price of good 1, $p_1$, on the horizontal axis, and the excess demand, $z_1$, on the vertical axis.

![Figure 7. Excess demand.](image)

(c) How is the excess demand function $z_1(p_1)$ found in part (b) affected by changes in $\gamma$?

- Differentiating the excess demand function with respect to $\gamma$,

$$\frac{dz_1(p_1)}{d\gamma} = 5 + \frac{3}{p_1} > 0$$

so that when consumers derive a higher utility from consuming good 1 (that is, $\gamma$ increases), excess demand of good 1 increases. In particular, when $\gamma$ is relatively low (e.g., 0.5), $p_1 = 0.6$ clears the market. However, when $\gamma$ is relatively high (e.g., 0.75), good 1 needs to be more expensive at $p_1 = 1.8$ to clear the market.

(d) Calculate the competitive equilibrium allocations, and show that the market clears.

- A competitive equilibrium allocation requires excess demand to be zero, that is,

$$5(\gamma - 1) + \frac{3\gamma}{p_1} = 0$$

After rearranging, we have

$$p_1^* = \frac{3\gamma}{5(1-\gamma)}$$

Substituting the equilibrium price, $(p_1^*, p_2^*) = \left(\frac{3\gamma}{5(1-\gamma)}, 1\right)$, into the demand
functions, the competitive equilibrium allocations become

\[ x_1^A = \frac{\gamma (2p_1^* + 1)}{p_1^*} = 2\gamma + \frac{5(1 - \gamma)}{3} \]
\[ = \frac{5 + \gamma}{3} \]
\[ x_2^A = (1 - \gamma)(2p_1^* + 1) = \frac{3\gamma}{5} + 1 - \gamma \]
\[ = \frac{5 - 2\gamma}{5} \]
\[ x_1^B = \frac{\gamma (3p_1^* + 2)}{p_1^*} = 3\gamma + \frac{10(1 - \gamma)}{3} \]
\[ = \frac{10 - \gamma}{3} \]
\[ x_2^B = (1 - \gamma)(3p_1^* + 2) = \frac{9\gamma}{5} + 2 - 2\gamma \]
\[ = \frac{10 - \gamma}{5} \]

- The excess demand now becomes

\[ z_1^* = x_1^A + x_1^B - \omega_1^A - \omega_1^B \]
\[ = \frac{5 + \gamma}{3} + \frac{10 - \gamma}{3} - 2 - 3 \]
\[ = 5 - 5 = 0 \]
\[ z_2^* = x_2^A + x_2^B - \omega_2^A - \omega_2^B \]
\[ = \frac{5 - 2\gamma}{5} + \frac{10 - \gamma}{5} - 1 - 2 \]
\[ = 3 - 3 = 0 \]

As the excess demand for both goods are zero, the market clears.

(e) Explain how the equilibrium price of good 2 is affected by a change in \( \gamma \) and in \( \omega_1^A \).

- Reconsidering the excess demand function of good 1, but before normalizing the price of good 2,

\[ z_1(p_1) = \frac{\gamma (2p_1 + p_2)}{p_1} + \frac{\gamma (3p_1 + 2p_2)}{p_1} - \omega_1^A - \omega_1^B \]
\[ = 5\gamma + 3\gamma \frac{p_2}{p_1} - \omega_1^A - \omega_1^B \]

Since in equilibrium the market clears, \( z_1(p_1) = 0 \), yielding

\[ p_2 = \frac{p_1}{3\gamma} \left( \omega_1^A + \omega_1^B \right) - \frac{5p_1}{3} \]
• Differentiating $p_2$ with respect to $\gamma$ and $\omega_1^A$,

$$\frac{\partial p_2}{\partial \gamma} = \frac{p_1}{3\gamma^2} (\omega_1^A + \omega_1^B) < 0$$

$$\frac{\partial p_2}{\partial \omega_1^A} = \frac{p_1}{3\gamma} > 0$$

so that as $\gamma$ increases, consumers derive a higher utility from consuming good 1, so that the price of good 1 increases, and relatively speaking, the price of good 2 decreases. On the other hand, as consumer $A$ has a larger endowment of good 1, good 2 becomes relatively scarce, such that the price of good 2 increases.

(f) Can an equal-utility allocation, where both consumers enjoy the same level of utility, be supported as a competitive equilibrium? Calculate the endowments required to make such an allocation.

• For an equal-utility allocation, we need the ratio of consumption equal to the ratio of endowments, that is,

$$\frac{x_1^i}{x_2^i} = \frac{\gamma(p_1\omega_1^i + p_2\omega_2^i)}{p_1} = \frac{(1-\gamma)(p_1\omega_1^i + p_2\omega_2^i)}{p_2}$$

$$= \frac{\gamma}{1-\gamma} \frac{p_2}{p_1} = \frac{\gamma}{1-\gamma} \frac{5(1-\gamma)}{3\gamma} = \frac{5}{3},$$

so that a competitive equilibrium can be supported by the following allocation:

$$x_1^A = x_1^B = x_1^* = \frac{5}{2}$$

$$x_2^A = x_2^B = x_2^* = \frac{3}{2}$$

• Therefore, for an equal-utility allocation, consumer $i$ should have endowments of

$$\omega_i^* = p_i\omega_1^i + \omega_2^i$$

$$= \frac{3\gamma}{5(1-\gamma)} \frac{5}{2} + \frac{3}{2}$$

$$= \frac{3}{2(1-\gamma)}$$

(g) Can a redistribution of endowments support this equilibrium found in part (f)? Discuss your results in light of the Second Fundamental Welfare Theorem.
Consumer $A$ has endowments of
\[ \omega^A = 2p_1 + 1 = \frac{6\gamma}{5(1-\gamma)} + 1 = \frac{5 + \gamma}{5(1-\gamma)} \]

Consumer $B$ has endowments of
\[ \omega^B = 3p_1 + 2 = \frac{9\gamma}{5(1-\gamma)} + 2 = \frac{10 - \gamma}{5(1-\gamma)} \]

Let the social planner implements a wealth redistribution program, where $t_i$ is the transfer to consumer $i$, subject to the balanced-budget condition, that is, $t^A + t^B = 0$, yielding $t^A = -t^B$. Then, post-transfer endowments of the consumers become
\[ \omega^{A*} = \frac{5 + \gamma}{5(1-\gamma)} + t^A \]
\[ \omega^{B*} = \frac{10 - \gamma}{5(1-\gamma)} + t^B \]

To support the equal-utility allocation, set
\[ \omega^{A*} = \omega^{B*} = \omega^* \]
such that the transfer to consumer $A$ becomes
\[ t^A = \omega^* - \frac{5 + \gamma}{5(1-\gamma)} \]
\[ = \frac{3}{5 - 2\gamma} - \frac{5 + \gamma}{5(1-\gamma)} \]
\[ = \frac{10 - 5\gamma}{10(1-\gamma)} \]

and similarly, the transfer to consumer $B$ becomes
\[ t^B = \omega^* - \frac{10 - \gamma}{5(1-\gamma)} \]
\[ = \frac{3}{5 - 2\gamma} - \frac{10 - \gamma}{5(1-\gamma)} \]
\[ = \frac{5 - 2\gamma}{10(1-\gamma)} \]

Since $t^B = -t^A$, the social planner balances the budget. Also, by taxing consumer $B$ and giving the equivalent amount to consumer $A$ as a subsidy, the utility of both consumers are equalized. Intuitively, since consumer $B$ has more endowments than consumer $A$ (that is, having more of both goods 1 and 2), the social planner can tax the wealthier consumer $B$ and subsidize the less well-off consumer $A$ to increase his consumption of both goods. This has the same effect of taking some of the physical stocks (of goods 1 and 2) directly from consumer $B$ and giving them to consumer $A$. Indeed, this wealth redistribution mechanism acts like the social planner taxing consumer $B$, so that consumer $B$ has to sell off some of his endowments to pay the tax. Then, the social planner gives a lump-sum transfer to consumer $A$, so that consumer $A$ can buy the endowments that consumer $B$ sells.
Therefore, any competitive equilibrium allocation, including the equalitarian one (i.e., equal utility for both consumers), can be supported by a redistribution of wealth, thereby satisfying the Second Fundamental Welfare Theorem.