EconS 503 - Advanced Microeconomics II
Handout on General Equilibrium

1. **Exercise #6.9, Munoz-Garcia (2017).** Consider a pure-exchange economy with two individuals, A and B, each with utility function $u^i(x^i, y^i)$ where $i = \{A, B\}$, whose initial endowments are $e^A = (10, 0)$ and $e^B = (0, 10)$, that is, individual A (B) owns all units of good x (y, respectively).

   (a) Assuming that utility functions are $u^i(x^i, y^i) = \min\{x^i, y^i\}$ for all individuals $i = \{A, B\}$, find the set of PEAs and the set of WEAs.

   - *PEAs.* Since the utility functions are not differentiable we cannot follow the property of $MRS^A_{x,y} = MRS^B_{x,y}$ across consumers. Figure 1 helps us identify the set of PEAs. Points away from the 45^o-line, satisfying $y^A = x^A$, such as N, cannot be pareto efficient since we can still find other points, such as M, where consumer 2 is make better off while consumer 1 reaches the same utility level as under N. Once we are at points on the 45^o-line, such as M, we cannot find other points making at least once consumer better off (and keep the other consumer at least as well off). Hence, the set of PEAs is

$$\{(x^A, y^A), (x^B, y^B) : y^A = x^A \text{ and } y^B = x^B\}$$

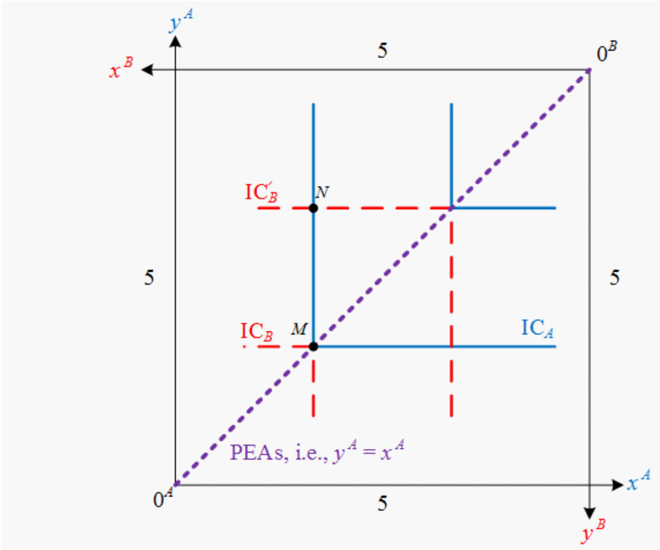

Figure 1. Edgeworth box and PEAs.

   - *WEAs.* Using good 2 as the numeraire, i.e., $p_2 = 1$, the price ratio becomes $\frac{p_1}{p_2} = p_1$. The budget line of both consumers therefore has a slope $-p_1$ and crosses the point representing the initial endowment e in Figure 2 (where e

lies at the lower right-hand corner)

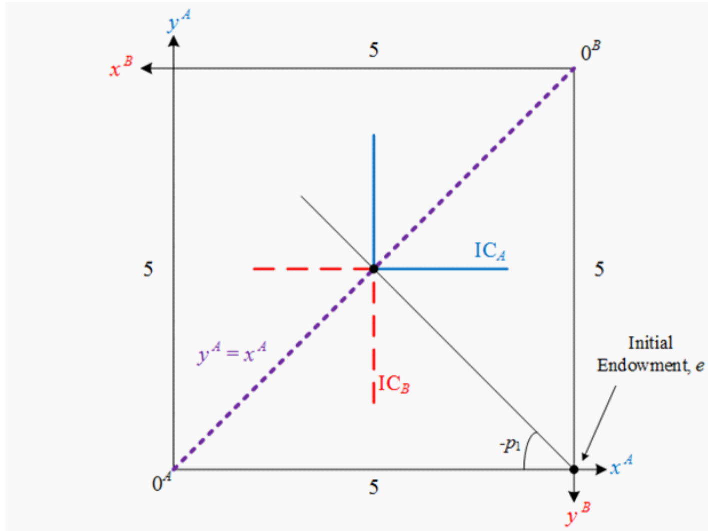

Figure 2. Edgeworth box and WEA.

(b) Assuming utility functions of $u^A(x^A, y^A) = x^A y^A$ and $u^B(x^B, y^B) = \min\{x^B, y^B\}$, find the set of PEAs and WEAs.

- *PEAs.* By the same argument as in question (a), the set of PEAs satisfies $y^A = x^A$, as depicted in Figure 3. Point N cannot be efficient as we can still find other feasible points, such as M, where at least one consumer is made strictly better off (in this case consumer A). At points on the 45^o-line, however, we can no longer find alternatives that would constitute a Pareto improvement.

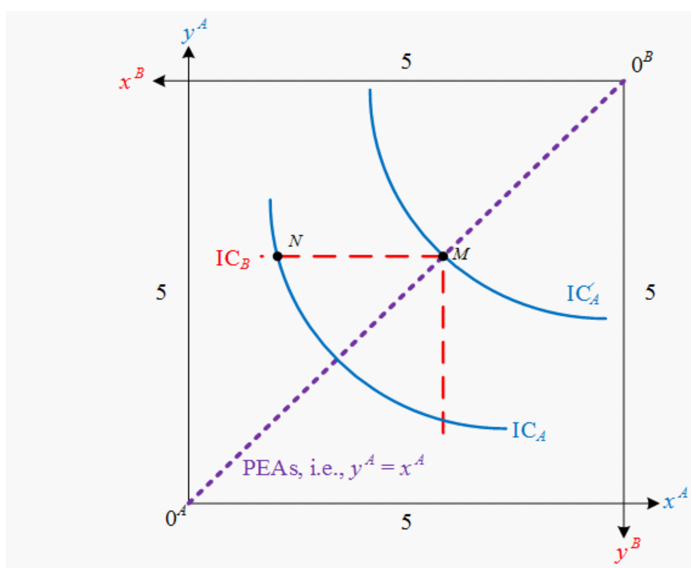

Figure 3. Edgeworth box and PEAs.

- *WEAs.* Using good y as the numeraire, $p_y = 1$, so that the price vector becomes $\mathbf{p} = (p_x, 1)$. Hence, *Consumer A's UMP is*

$$\max_{x^A, y^A} \quad x^A y^A$$
$$\text{subject to } p_x x^A + y^A = 10 p_x$$

  Taking first-order conditions

$$\begin{aligned} y^A - \lambda^A p_x &= 0 \\ x^A - \lambda^A &= 0 \\ p_x x^A + y^A &= 10 p_x \end{aligned}$$

  Combining the first two FOCs and rearranging, we have

$$p_x x^A = y^A$$

  and substituting this equation into the third FOC yields

$$p_x x^A + p_x x^A = 10 p_x \implies x^A = 5$$

  and substituting this back into $p_x x^A = y^A$

$$y^A = 5 p_x$$

  Consumer B's UMP is not differentiable, but in equilibrium his Walrasian demands satisfy $x^B = y^B$. Substituting this into his budget constraint yields

$$p_x x^B + x^B = 10 \implies x^B = y^B = \frac{10}{p_x + 1}$$

  Furthermore, the feasibility condition for good x entails

$$5 + \frac{10}{p_x + 1} = 10 + 0, \text{ or } p_x = 1$$

  Therefore, the market of good x will clear at an equilibrium price of $p_x = 1$, i.e., $z_x(p_x, 1) = 0$ when $p_x = 1$. Since market y clears when market x does (by Walras' law), $z_y(p_x, 1)$ must also be zero when $p_x = 1$. Summarizing, the equilibrium price $p_x = 1$ yields a WEA

$$\{(5, 5), (5, 5)\}$$

2. **Based on Exercise #2.20 in Hashimzade et al. (2006).** A consumer views two goods as perfect substitutes.

   (a) Sketch the indifference curves of the consumer.

- Since consumer A views the two goods as perfect substitutes, one unit of good 1, x_1^A, is worth one unit of good 2, x_2^A, so that the indifference curves are straight lines away from the origin with a gradient of -1, as illustrated in Figure 4.

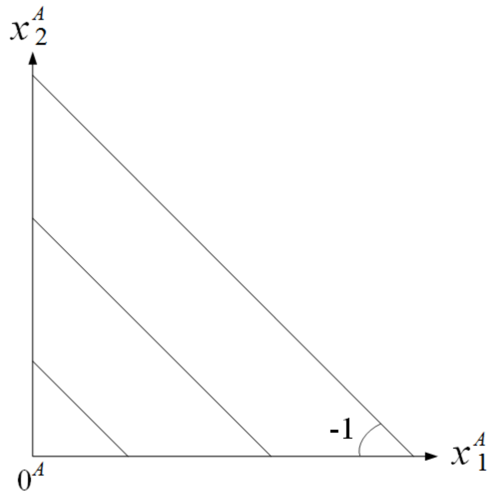

Figure 4. Indifference curves.

(b) If an economy is composed of two consumers with these preferences, demonstrate that any allocation is Pareto-efficient,

- For any initial endowment of this economy, the indifference curve of consumer A (solid line) coincides with that of consumer B (dashed line) at every point of the Edgeworth box, so that every allocation is Pareto-efficient, as illustrated in Figure 5.

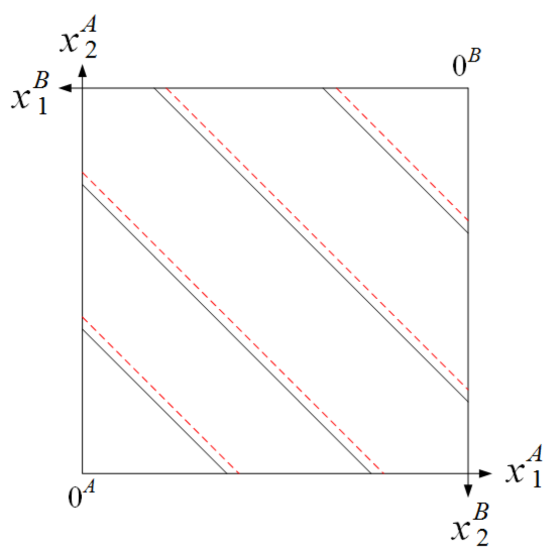

Figure 5. Edgeworth box.

(c) If an economy has one consumer who views its two goods as perfect substitutes and a second that consider each unit of good 1 to be worth 2 units of good 2, find the Pareto-efficient allocations.

- Without loss of generality, let consumer A views its two goods as perfect substitutes, and consumer B considers each unit of good 1 to be worth 2 units of good 2. As seen in Figure 6, the indifference curves intersect along the x_1^B and x_2^A axes, which represent the set of Pareto-efficient allocations (outer edges of the Edgeworth box, the inverted L-shaped curve bolded in blue color). In this context, consumer B would consume good 1 only (along the x_1^B axis) and no good 2, and not until he trades all units of good 1 with consumer A does he begin to consume good 2 (along the x_2^A axis); while the exact location of the equilibrium outcome depends on the initial endowment of this economy.

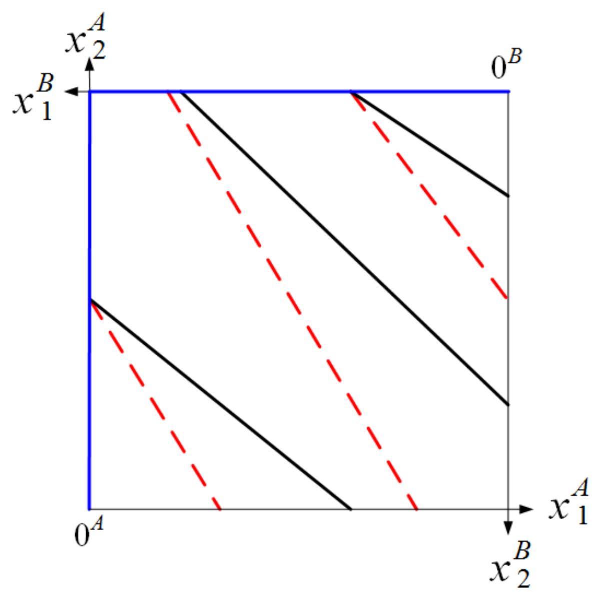

Figure 6. Efficient allocations.

3. **Based on Exercises #2.11 and 2.24 in Hashimzade et al. (2006).** Consider a pure exchange economy with 2 consumers and 2 goods. Consumer i, where $i \in \{A, B\}$, has a utility function of

$$U^i = \gamma \log\left(x_1^i\right) + (1-\gamma)\log\left(x_2^i\right),$$

which is the geometric average of his consumption of good 1, x_1^i (with weight γ), and good 2, x_2^i (with weight $1-\gamma$).

   (a) Let consumer A has an endowment of $\left(\omega_1^A, \omega_2^A\right) = (2,1)$ for goods 1 and 2, and similarly, $\left(\omega_1^B, \omega_2^B\right) = (3,2)$ for consumer B. Find the Walrasian demand of both consumers.

   - The budget constraint of consumer i, where $i \in \{A, B\}$, is

$$p_1 x_1^i + p_2 x_2^i = p_1 \omega_1^i + p_2 \omega_2^i$$

   Rearranging, we have

$$x_2^i = \frac{p_1}{p_2}\left(\omega_1^i - x_1^i\right) + \omega_2^i \tag{1}$$

Assuming interior solutions for Walrasian allocation,

$$MRS_{12}^i = \frac{MU_1^i}{MU_2^i} = \frac{\gamma}{x_1^i}\frac{x_2^i}{1-\gamma} = \frac{p_1}{p_2}$$

Rearranging, we have

$$x_1^i = \frac{\gamma}{1-\gamma}\frac{p_2}{p_1}x_2^i \tag{2}$$

Substituting expression (2) into expression (1),

$$x_2^i = \frac{p_1}{p_2}\left(\omega_1^i - \frac{\gamma}{1-\gamma}\frac{p_2}{p_1}x_2^i\right) + \omega_2^i$$

Rearranging, we have

$$\begin{aligned} \frac{1-\gamma+\gamma}{1-\gamma}x_2^i &= \frac{p_1}{p_2}\omega_1^i + \omega_2^i \\ \implies x_2^i &= \frac{(1-\gamma)\left(p_1\omega_1^i + p_2\omega_2^i\right)}{p_2} \end{aligned}$$

Substituting the above into expression (2),

$$x_1^i = \frac{\gamma\left(p_1\omega_1^i + p_2\omega_2^i\right)}{p_1}$$

Substituting the consumers' endowments, the Walrasian demand functions become

$$\begin{aligned} x_1^A &= \frac{\gamma\left(2p_1 + p_2\right)}{p_1} \\ x_2^A &= \frac{(1-\gamma)\left(2p_1 + p_2\right)}{p_2} \\ x_1^B &= \frac{\gamma\left(3p_1 + 2p_2\right)}{p_1} \\ x_2^B &= \frac{(1-\gamma)\left(3p_1 + 2p_2\right)}{p_2} \end{aligned}$$

(b) Setting the price of good 2 as a numéraire, that is, $p_2 = 1$, find the excess demand for good 1, $z_1(p_1)$, and then plot it as a function of price p_1.

- Setting $p_2 = 1$, the excess demand function for good 1 becomes

$$\begin{aligned} z_1(p_1) &= x_1^A + x_1^B - \omega_1^A - \omega_1^B \\ &= \frac{\gamma\left(2p_1+1\right)}{p_1} + \frac{\gamma\left(3p_1+2\right)}{p_1} - 2 - 3 \\ &= 5(\gamma - 1) + \frac{3\gamma}{p_1} \end{aligned}$$

- Figure 7 plots the excess demand function, $z_1(p_1)$, with the price of good 1, p_1, on the horizontal axis, and the excess demand, z_1, on the vertical axis.

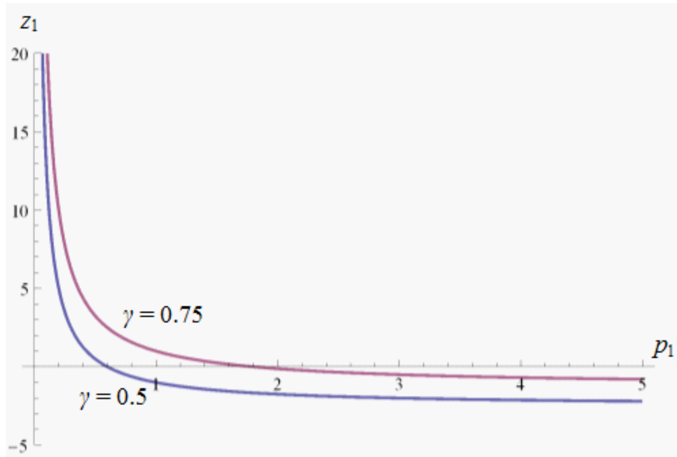

Figure 7. Excess demand.

(c) How is the excess demand function $z_1(p_1)$ found in part (b) affected by changes in γ?

- Differentiating the excess demand function with respect to γ,

$$\frac{dz_1(p_1)}{d\gamma} = 5 + \frac{3}{p_1} > 0$$

so that when consumers derive a higher utility from consuming good 1 (that is, γ increases), excess demand of good 1 increases. In particular, when γ is relatively low (e.g., 0.5), $p_1 = 0.6$ clears the market. However, when γ is relatively high (e.g., 0.75), good 1 needs to be more expensive at $p_1 = 1.8$ to clear the market.

(d) Calculate the competitive equilibrium allocations, and show that the market clears.

- A competitive equilibrium allocation requires excess demand to be zero, that is,

$$5(\gamma - 1) + \frac{3\gamma}{p_1} = 0$$

After rearranging, we have

$$p_1^* = \frac{3\gamma}{5(1-\gamma)}$$

Substituting the equilibrium price, $(p_1^*, p_2^*) = \left(\frac{3\gamma}{5(1-\gamma)}, 1\right)$, into the demand functions, the competitive equilibrium allocations become

$$\begin{aligned}
x_1^{A*} &= \frac{\gamma\left(2p_1^*+1\right)}{p_1^*} = 2\gamma + \frac{5\left(1-\gamma\right)}{3} \\
&= \frac{5+\gamma}{3} \\
x_2^{A*} &= \left(1-\gamma\right)\left(2p_1^*+1\right) = \frac{3\gamma}{5} + 1 - \gamma \\
&= \frac{5-2\gamma}{5} \\
x_1^{B*} &= \frac{\gamma\left(3p_1^*+2\right)}{p_1^*} = 3\gamma + \frac{10\left(1-\gamma\right)}{3} \\
&= \frac{10-\gamma}{3} \\
x_2^{B*} &= \left(1-\gamma\right)\left(3p_1^*+2\right) = \frac{9\gamma}{5} + 2 - 2\gamma \\
&= \frac{10-\gamma}{5}
\end{aligned}$$

- The excess demand now becomes

$$\begin{aligned}
z_1^* &= x_1^{A*} + x_1^{B*} - \omega_1^A - \omega_1^B \\
&= \frac{5+\gamma}{3} + \frac{10-\gamma}{3} - 2 - 3 \\
&= 5 - 5 = 0 \\
z_2^* &= x_2^{A*} + x_2^{B*} - \omega_2^A - \omega_2^B \\
&= \frac{5-2\gamma}{5} + \frac{10-\gamma}{5} - 1 - 2 \\
&= 3 - 3 = 0
\end{aligned}$$

  As the excess demand for both goods are zero, the market clears.

(e) Explain how the equilibrium price of good 2 is affected by a change in γ and in ω_1^A.

- Reconsidering the excess demand function of good 1, but before normalizing the price of good 2,

$$\begin{aligned}
z_1\left(p_1\right) &= \frac{\gamma\left(2p_1+p_2\right)}{p_1} + \frac{\gamma\left(3p_1+2p_2\right)}{p_1} - \omega_1^A - \omega_1^B \\
&= 5\gamma + 3\gamma\frac{p_2}{p_1} - \omega_1^A - \omega_1^B
\end{aligned}$$

  Since in equilibrium the market clears, $z_1\left(p_1\right) = 0$, yielding

$$p_2 = \frac{p_1}{3\gamma}\left(\omega_1^A + \omega_1^B\right) - \frac{5p_1}{3}$$

- Differentiating p_2 with respect to γ and ω_1^A,

$$\frac{\partial p_2}{\partial \gamma} = -\frac{p_1}{3\gamma^2}\left(\omega_1^A + \omega_1^B\right) < 0$$
$$\frac{\partial p_2}{\partial \omega_1^A} = \frac{p_1}{3\gamma} > 0$$

so that as γ increases, consumers derive a higher utility from consuming good 1, so that the price of good 1 increases, and relatively speaking, the price of good 2 decreases. On the other hand, as consumer A has a larger endowment of good 1, good 2 becomes relatively scarce, such that the price of good 2 increases.

(f) Can an equal-utility allocation, where both consumers enjoy the same level of utility, be supported as a competitive equilibrium? Calculate the endowments required to make such an allocation.

- For an equal-utility allocation, we need the ratio of consumption equal to the ratio of endowments, that is,

$$\begin{aligned}\frac{x_1^i}{x_2^i} &= \frac{\frac{\gamma\left(p_1\omega_1^i + p_2\omega_2^i\right)}{p_1}}{\frac{(1-\gamma)\left(p_1\omega_1^i + p_2\omega_2^i\right)}{p_2}} \\ &= \frac{\gamma}{1-\gamma}\frac{p_2}{p_1} \\ &= \frac{\gamma}{1-\gamma}\frac{5\left(1-\gamma\right)}{3\gamma} \\ &= \frac{5}{3},\end{aligned}$$

so that a competitive equilibrium can be supported by the following allocation:

$$x_1^A = x_1^B = x_1^* = \frac{5}{2}$$
$$x_2^A = x_2^B = x_2^* = \frac{3}{2}$$

- Therefore, for an equal-utility allocation, consumer i should have endowments of

$$\begin{aligned}\omega^* &= p_1\omega_1^i + \omega_2^i \\ &= \frac{3\gamma}{5\left(1-\gamma\right)}\frac{5}{2} + \frac{3}{2} \\ &= \frac{3}{2\left(1-\gamma\right)}\end{aligned}$$

(g) Can a redistribution of endowments support this equilibrium found in part (f)? Discuss your results in light of the Second Fundamental Welfare Theorem.

- Consumer A has endowments of

$$\omega^A = 2p_1 + 1 = \frac{6\gamma}{5(1-\gamma)} + 1 = \frac{5+\gamma}{5(1-\gamma)}$$

- Consumer B has endowments of

$$\omega^B = 3p_1 + 2 = \frac{9\gamma}{5(1-\gamma)} + 2 = \frac{10-\gamma}{5(1-\gamma)}$$

- Let the social planner implements a wealth redistribution program, where t_i is the transfer to consumer i, subject to the balanced-budget condition, that is, $t^A + t^B = 0$, yielding $t^A = -t^B$. Then, post-transfer endowments of the consumers become

$$\omega^{A*} = \frac{5+\gamma}{5(1-\gamma)} + t^A$$
$$\omega^{B*} = \frac{10-\gamma}{5(1-\gamma)} + t^B$$

- To support the equal-utility allocation, set

$$\omega^{A*} = \omega^{B*} = \omega^*$$

such that the transfer to consumer A becomes

$$\begin{aligned} t^A &= \omega^* - \frac{5+\gamma}{5(1-\gamma)} \\ &= \frac{3}{2(1-\gamma)} - \frac{5+\gamma}{5(1-\gamma)} \\ &= \frac{5-2\gamma}{10(1-\gamma)} \end{aligned}$$

and similarly, the transfer to consumer B becomes

$$\begin{aligned} t^B &= \omega^* - \frac{10-\gamma}{5(1-\gamma)} \\ &= \frac{3}{2(1-\gamma)} - \frac{10-\gamma}{5(1-\gamma)} \\ &= -\frac{5-2\gamma}{10(1-\gamma)} \end{aligned}$$

Since $t^B = -t^A$, the social planner balances the budget. Also, by taxing consumer B and giving the equivalent amount to consumer A as a subsidy, the utility of both consumers are equalized. Intuitively, since consumer B has more endowments than consumer A (that is, having more of both goods 1 and 2), the social planner can tax the wealthier consumer B and subsidize the less well-off consumer A to increase his consumption of both goods. This has the same effect of taking some of the physical stocks (of goods 1 and 2) directly from consumer B and giving them to consumer A. Indeed, this wealth redistribution mechanism acts like the social planner taxing consumer B, so that consumer B has to sell off some of his endowments to pay the tax. Then, the social planner gives a lump-sum transfer to consumer A, so that consumer A can buy the endowments that consumer B sells.

- Therefore, any competitive equilibrium allocation, including the equalitarian one (i.e., equal utility for both consumers), can be supported by a redistribution of wealth, thereby satisfying the Second Fundamental Welfare Theorem.