EconS 503 - Microeconomic Theory II
Midterm Exam #2 - Answer Key

1. **Revenue comparison in two auction formats.** Consider a sealed-bid auction with N bidders. Every bidder i privately observes his valuation v_i for the object, drawn from uniform distribution $U[0,1]$, which is common knowledge among players. Assume that all bidders are risk neutral.

   (a) Find equilibrium bidding functions if the auction is: (1) first-price auction, and (2) second-price auction.

   - *First-price auction.* From previous exercises, we know that the equilibrium bidding function in this auction with N bidders and with uniformly distributed valuations is

   $$b_i(v_i) = \frac{N-1}{N} v_i \text{ for every bidder } i.$$

   For instance, in the case of $N = 3$ bidders, this bidding function becomes $b_i(v_i) = \frac{2}{3} v_i$. If bidder j is the individual with the highest valuation for the object, he submits the highest bid (since bids are increasing in valuations) and pays a share $\frac{N-1}{N}$ of his true valuation (this is what we normally denote as "bid shading").

   - *Second-price auction.* From previous exercises, we also know that bidding according to his valuation is a weakly dominant strategy in the second-price auction, and also the Bayesian Nash equilibrium of the game, that is, $b(v_i) = v_i$ for every bidder i. Therefore, if bidder i has the highest valuation for the object, that is $v_i > \max_{j \neq i} v_j$, he submits the highest bid, winning the auction, and paying a price that coincides with the bid of the second highest bidder, $b(v_k) = v_k$ where $v_k = \max_{j \neq i} v_j$ denotes the highest valuation among all bidder i's rivals.

   (b) Evaluate the seller's *ex-post* revenue in each auction format (which is a function of the realizations of players' valuations). Does the revenue equivalence theorem hold?

   - *Revenue from first-price auction.* From part (a), we know that $b_i(v_i) = \frac{2}{3} v_i$, implying that the expected revenue of the seller in this auction format is

   $$\frac{N-1}{N} \max\{v_1, v_2, ..., v_N\}$$

   since the winning bid (and thus the price that the seller receives) is $\frac{2}{3}$ of the highest bidder's valuation.

   - *Revenue from second-price auction.* Recall from part (a) that the individual i with the highest valuation submits the highest bid, $b(v_i) = v_i$, winning the object, but he only pays a price equal to the bid of the individual with the second-highest valuation, $b(v_j) = v_j$, implying that the seller's expected revenue is also v_j.

- *Revenue comparison.* Depending on the specific realization of players' valuations, the actual revenue that the seller receives can be different across auction formats. However, this is not contradictory with the revenue equivalence theorem. This theorem only tells us that, in expectation (that is, before valuations are drawn from the uniform distribution), revenues should coincide across these auction formats.

(c) Find the seller's *ex-ante* (or expected) revenue in each auction format (which is not a function of the realization of players' valuations but, instead, of the expected value of these valuations). Does the revenue equivalence theorem hold now?

- The *ex-post* profit of first-price auction for the winning bidder is

$$\begin{aligned}\pi_i(v_i) &= \underbrace{v_i^{N-1}}_{\text{probability of winning}} \times \underbrace{\frac{N-1}{N}v_i}_{\text{value of the bid}} \\ &= \frac{N-1}{N}v_i^N\end{aligned}$$

  such that the *ex-ante* profit of first-price auction for all bidders is

$$\begin{aligned}E[\pi(v)] &= \sum_{i=1}^{N} E[\pi_i(v_i)] \\ &= \sum_{i=1}^{N}\int_0^1 \frac{N-1}{N}v_i^N f(v_i)\,dv_i \\ &= \frac{N-1}{N}\sum_{i=1}^{N}\int_0^1 v_i^N dv_i \\ &= \frac{N-1}{N}N\left[\frac{1}{N+1}v^{N+1}\right]_0^1 \\ &= \frac{N-1}{N+1}\end{aligned}$$

- The *ex-post* profit of second-price auction for the winning bidder is

$$\begin{aligned}\pi_i(v_i) &= \underbrace{v_i^{N-1}}_{\text{probability of winning}} \times \underbrace{v_j}_{\text{value of the second-highest bid}} \\ &= v_i^{N-1}\frac{N-1}{N}v_i \\ &= \frac{N-1}{N}v_i^N\end{aligned}$$

  where the second line originates from the fact of uniform distribution that sets v_j to be $\frac{N-1}{N}$ below v_i, which establishes the equivalence between the *ex-post* profit of first-price and second-price auctions. Applying the same calculations as above, we can show that the *ex-ante* profit of second-price auction is also $\frac{N-1}{N+1}$.

2. **Selten's horse.** Consider the "Selten's Horse" game depicted in Figure 1. Player 1 is the first mover in the game, choosing between C and D. If he chooses C, player 2 is called on to move between C' and D'. If player 2 selects C' the game is over. If player 1 chooses D or player 2 chooses D', then player 3 is called on to move without being informed whether player 1 chose D before him or whether it was player 2 who chose D'. Player 3 can choose between L and R, and then the game ends.

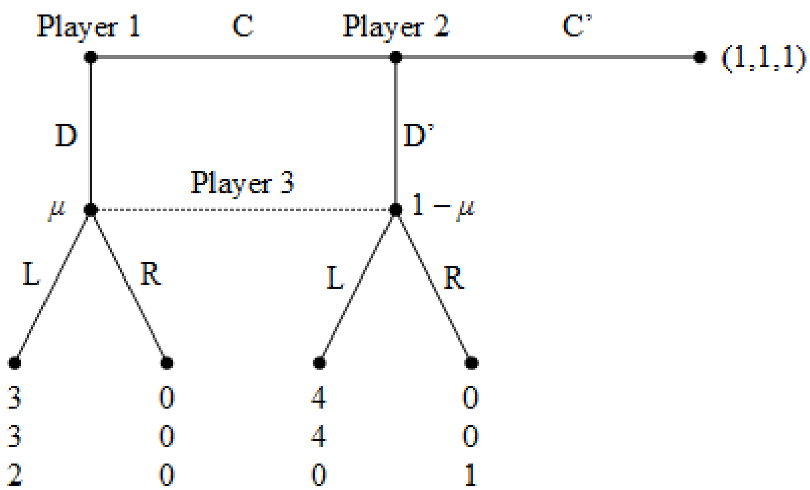

Figure 1. Selten's horse.

(a) Define the strategy spaces for each player. Then find all pure strategy Nash equilibria (psNE) of the game. [*Hint*: This is a three-player game, so you can consider that player 1 chooses rows, player 2 columns, and player 3 chooses matrices.]

- The strategy spaces of the players are as follows:

$$S_1 = \{C, D\}$$
$$S_2 = \{C', D'\}$$
$$S_3 = \{L, R\}$$

In Figure 2, we represent the strategies and payoffs of the three players in the following normal form representation of the game, where Player 1 chooses between the rows, Player 2 chooses between the columns, and Player 3 chooses between the matrixes.

Player 3 choosing L

Player 1 \ Player 2	C'	D'
C	1, 1, 1	4, 4, 0
D	3, 3, 2	3, 3, 2

Player 3 choosing R

Player 1 \ Player 2	C'	D'
C	1, 1, 1	0, 0, 1
D	0, 0, 0	0, 0, 0

Figure 2. Selten's horse - Matrix representation.

- We next underline the best responses of the three players in Figure 3, and identify that (C, C', R), (D, C', L), and (D, D', L) are the pure strategy Nash equilibria of this game.

Player 3 choosing L

Player 1 \ Player 2	C'	D'
C	$1, 1, \underline{1}$	$\underline{4}, \underline{4}, 0$
D	$\underline{3}, \underline{3}, \underline{2}$	$3, \underline{3}, \underline{2}$

Player 3 choosing R

Player 1 \ Player 2	C'	D'
C	$\underline{1}, \underline{1}, \underline{1}$	$\underline{0}, 0, \underline{1}$
D	$0, \underline{0}, 0$	$\underline{0}, \underline{0}, 0$

Figure 3. Selten's horse - Underlining best response payoffs.

(b) Argue that one of the three psNEs you found in part (a) is not sequentially rational. A short verbal explanation suffices.

- (D, C', L) is not sequentially rational. If Player 1 chooses D, then Player 3's belief is $\mu = 1$, responding with L (see left-hand side at the bottom of the tree). Anticipating that Player 3 choosing L, Player 2 compares his payoff from C', 1, against that from D' (which is followed by Player 3 responding with L), 4, and thus chooses D'. Therefore, Player 2 choosing C' is not sequentially rational.

(c) Show that strategy profile $\{C, C', R\}$ can be sustained as a PBE of the game. (You don't need to show that this is actually the unique PBE we can sustain in this game.) Discuss that this strategy profile is based on credible beliefs by player 3.

- We check the pooling strategy profile, C, C', where Player 1 chooses C and Player 2 selects C'.
  As depicted in Figure 4, since player 1 chooses C (as illustrated by the blue horizontal arrow) and player 2 chooses C' (as illustrated by the green horizontal arrow), messages D and D' are off-the-equilibrium path, leaving the

beliefs of Player 3 unrestricted, that is, $\mu \in [0,1]$. In other words, Player 3's information set should never be reached in this strategy profile.

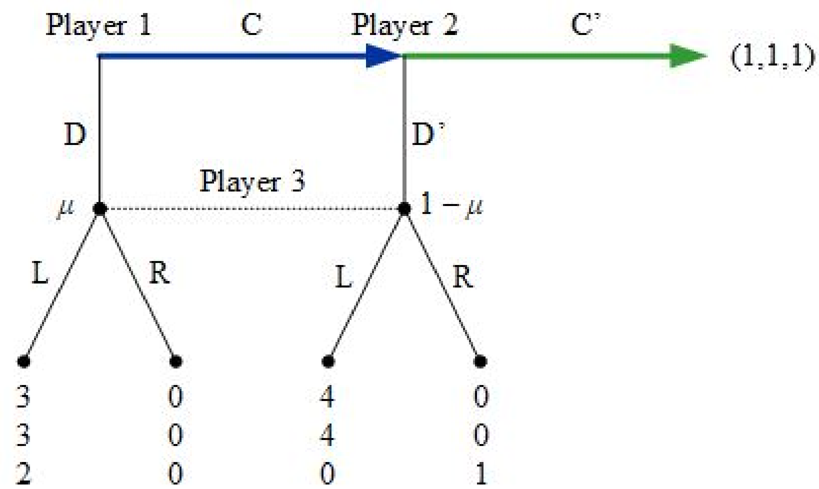

Figure 4. Pooling Strategy Profile C, C'

Therefore, if Player 3 is ever called out to move, he compares the expected payoff from responding with L and R, as follows:

$$EU_3(L) = 2 \times \mu + 0 \times (1-\mu) = 2\mu$$
$$EU_3(R) = 0 \times \mu + 1 \times (1-\mu) = 1-\mu$$

Player 3 then responds with L if $2\mu > 1-\mu$, which simplifies to $\mu > \frac{1}{3}$. Otherwise, he responds with R. This gives rise to two cases (one in which $\mu > \frac{1}{3}$, and Player 3 responds with L; and another in which $\mu \leq \frac{1}{3}$ and Player 3 responds with R), which we separately analyze below.

- *Case 1,* $\mu > \frac{1}{3}$. As depicted in Figure 5a, Player 3 responds with L (as illustrated by the red arrows) since $\mu > \frac{1}{3}$. In this context, Player 2 can improve his payoff by deviating from C', which yields a payoff of 1, to D', which yields a payoff of 4. Therefore, the pooling strategy profile C, C' cannot be supported as a PBE of this game when Player 3's beliefs satisfy $\mu > \frac{1}{3}$.

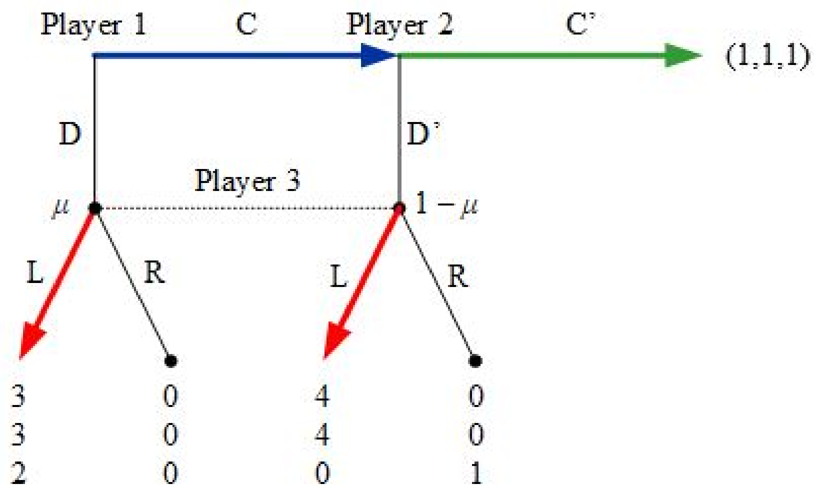

Figure 5a. Pooling Strategy Profile C, C' when $\mu > \frac{1}{3}$.

- *Case 2,* $\mu \leq \frac{1}{3}$. As depicted in Figure 5b, Player 3 responds with R (as illustrated by the red arrows) given that his beliefs are $\mu \leq \frac{1}{3}$. In this context, Player 2 does not deviate because his prescribed strategy, C', gives him a payoff of 1, while deviating to D' would give him a payoff of 0. Similarly, Player 1 does not deviate because his prescribed strategy, C, gives him a payoff of 1, exceeds his payoff from deviating to D, zero. Therefore, strategy profile C, C' can be supported as a PBE of this game when Player 3's beliefs satisfy $\mu \leq \frac{1}{3}$.

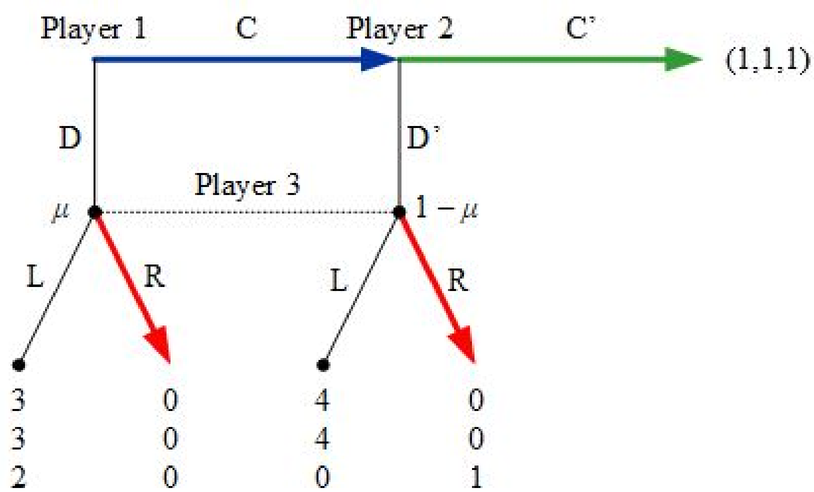

Figure 5b. Pooling Strategy Profile C, C' when $\mu \leq \frac{1}{3}$.

3. **[Principal-agent problem]** Consider a situation where a principal has profit function

$$u^p(e, w) = \beta e - w$$

where e denotes the effort that agent (e.g., employee) exerts, which is transformed into profits at a rate $\beta > 1$, and w represents the salary that principal pays the agent.

While the principal cannot observe the agent's type θ_i, where $i = \{L, H\}$ and $\theta_L = 1$ and $\theta_H = 2$, he knows the frequency of worker with high-type, p, and low-type, $1 - p$. Utility function of each agent is

$$u^i(e, w) = w - \theta_i e^2$$

where $i = \{L, H\}$. Intuitively, the agent's utility increases in the salary that he receives, but decreases in the effort he exerts. Hence, the second term $\theta_i e^2$ can be interpreted as the agent's cost of effort, which is increasing and convex in effort, and where the (absolute and marginal) cost of effort is larger for the high type than for the low-type since $\theta_H > \theta_L$. The reservation utility of the agents is zero.

(a) *Symmetric Information*: Find the contract(s) that will be offered by the principal when he can observe the agent's type.

- Since the principal can observe each agent's type, he solves, for every worker type θ_i,

$$\max_{w_i, e_i} \quad \beta e_i - w_i$$
$$\text{subject to } w_i - \theta_i e_i^2 \geq 0 \qquad (PC)$$

As usual, the P.C. constraint must be binding, i.e, $w_i = \theta_i e_i^2$ for all $i = \{L, H\}$. Otherwise, the firm manager could still lower salaries and extract a larger surplus. We can thus substitute the binding PC constraint into the principal's objective function, to obtain the following unconstrained problem

$$\max_{e_i} \quad \beta e_i - \theta_i e_i^2$$

Taking the FOC with respect to e_i, yields

$$\beta - 2\theta_i e_i = 0$$

which, solving for e_i, helps us obtain the optimal effort level under symmetric information

$$e_i = \frac{\beta}{2\theta_i} \quad \text{for all } i = \{L, H\}$$

That is, since $\theta_L = 1$ and $\theta_H = 2$, effort levels are $e_L = \frac{\beta}{2}$ and $e_H = \frac{\beta}{4}$, thus prescribing a higher effort level for the worker with a low disutility from effort, i.e., $e_L > e_H$. Salaries in this context become $w_i = \theta_i e_i^2$, which entail

$$w_H = 2\left(\frac{\beta}{4}\right)^2 = \frac{\beta^2}{8} \quad \text{and} \quad w_L = \left(\frac{\beta}{2}\right)^2 = \frac{\beta^2}{4} \text{ and}$$

(b) *Asymmetric Information*: Find the contract(s) that the principal offers when he cannot observe the types of each agent.

- **Remark:** For generality, I first solve the incomplete information game without assuming specific values for parameters p and β, and then evaluate the results at the parameters given in the exercise $p = \frac{1}{3}$ and $\beta = 32$.

- Under incomplete information, the principal maximizes his expected utility

$$\max_{w_L, w_H, e_L, e_H} \quad p(\beta e_H - w_H) + (1-p)(\beta e_L - w_L)$$

subject to the participation constraints for each agent

$$w_L - e_L^2 \geq 0 \qquad (PC_L)$$

$$w_H - 2e_H^2 \geq 0 \qquad (PC_H)$$

and the incentive compatibility constraints for each agent

$$w_L - e_L^2 \geq w_H - e_H^2 \qquad (IC_L)$$

$$w_H - 2e_H^2 \geq w_L - 2e_L^2 \qquad (IC_H)$$

From the above equations

$$w_L - e_L^2 \underbrace{\geq}_{\text{From } IC_L} w_H - e_H^2 > w_H - 2e_H^2.$$

This indicates that the PC_H is binding. So we can now set up our Lagrangean as follows:

$$\begin{aligned}\mathcal{L} \; = \; & p(\beta e_H - w_H) + (1-p)(\beta e_L - w_L) \\ & + \lambda_1(w_H - 2e_H^2) \\ & + \lambda_2(w_L - e_L^2 - w_H + e_H^2) + \lambda_3(w_H - 2e_H^2 - w_L + 2e_L^2)\end{aligned}$$

Taking the FOCs

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial w_H} &= -p + \lambda_1 - \lambda_2 + \lambda_3 = 0 \Longrightarrow \lambda_3 - \lambda_2 = p - \lambda_1 \\ \frac{\partial \mathcal{L}}{\partial w_L} &= -(1-p) + \lambda_2 - \lambda_3 = 0 \Longrightarrow \lambda_2 - \lambda_3 = 1 - p \\ \frac{\partial \mathcal{L}}{\partial e_H} &= p\beta - 4\lambda_1 e_H + 2\lambda_2 e_H - 4\lambda_3 e_H = 0 \Longrightarrow e_H = \frac{p\beta}{2\left(2\lambda_1 - \lambda_2 + 2\lambda_3\right)} \\ \frac{\partial \mathcal{L}}{\partial e_L} &= (1-p)\beta + 4\lambda_3 e_L - 2\lambda_2 e_L = 0 \Longrightarrow e_L = \frac{(1-p)\beta}{2\left(\lambda_2 - 2\lambda_3\right)}\end{aligned}$$

In addition, from the first and second FOC, we obtain that $p - \lambda_1 = p - 1$, or $\lambda_1 = 1$, thus confirming that its associated constraint, PC_H, binds, i.e., $w_H - 2e_H^2 = 0$. We can then use this result into the expression of IC_H to obtain that

$$\underbrace{w_H - 2e_H^2}_{0} \geq w_L - 2e_L^2$$

which means that the high-type agent would receive a negative utility should he select the contract meant for the low-type. Hence, the IC_H must slack (i.e., hold strictly) entailing that its associated Lagrange multiplier is nil, $\lambda_3 = 0$. Using the second FOC, $\lambda_2 - \lambda_3 = 1 - p$, we then find that $\lambda_2 = 1 - p > 0$.

- We can now evaluate the effort levels found in the last two FOCs at $\lambda_1 = 1$, $\lambda_3 = 0$, and $\lambda_2 = 1 - p$, which yields

$$e_H = \frac{p\beta}{2(2-(1-p))} = \frac{p\beta}{2(1+p)} \quad \text{and}$$
$$e_L = \frac{(1-p)\beta}{2(1-p)} = \frac{\beta}{2}$$

Let us now compare these effort levels against those under complete information found in part (a), where we obtained that $e_H = \frac{\beta}{4}$ for the high type and $e_L = \frac{\beta}{2}$ for the low type. Therefore, the effort level of the high-type agent is lower under incomplete than complete information since $\frac{p\beta}{2(1+p)} < \frac{\beta}{4}$ simplifies to $p \leq 1$, which holds by definition; but the effort level of the low-type agent coincides across information settings. In other words, there is "no distortion" for the low-type worker (the most efficient worker), while there is distortion in the effort required from the high disutility of effort worker. (In addition, note that while $e_H = \frac{p\beta}{2(1+p)}$ increases in p, it lies below $e_L = \frac{\beta}{2}$ for all values of p, including at $p = 1$.) Regarding salaries, we find that

$$w_L = e_H^2 + e_L^2 = \left[\frac{p\beta}{2(1+p)}\right]^2 + \frac{\beta^2}{4}, \text{ and}$$
$$w_H = 2e_H^2 = \frac{p^2\beta^2}{2(1+p)^2}$$

In addition, note that when $p = 1$, our results converge to the finding under symmetric information of part (a), i.e., effort levels become $e_H = \frac{\beta}{4}$ and $e_L = \frac{\beta}{2}$, and salaries are $w_L = \frac{\beta^2}{16} + \frac{\beta^2}{4} = \frac{5\beta^2}{16}$ and $w_H = \frac{\beta^2}{8}$.

- *Numerical example*: Consider that $p = \frac{1}{3}$, and $\beta = 32$. Then the optimal efforts under symmetric information are

$$e_L = 16 \text{ and } e_H = 8$$

while under asymmetric information are

$$e_L = 16 \text{ and } e_H = 4$$

Similarly, salaries under symmetric information are

$$w_L = 256 \text{ and } w_H = 128$$

whereas under asymmetric information

$$w_L = 272 \text{ and } w_H = 32.$$

(c) Find the information rents of the worker with low cost of effort and of the worker with high cost of effort. Interpret.

- We can measure the information rents of each type of agent, by comparing his utility under asymmetric information (where it can be positive) and symmetric information (where it was zero). In particular, since PC_H binds, the high-type does not retain information rents. However, the low-type agent obtains a utility level of

$$u_L = w_L - e_L^2 = \underbrace{\left(e_H^2 + e_L^2\right)}_{w_L} - e_L^2 = e_H^2 = \left[\frac{p\beta}{2(1+p)}\right]^2$$

which is positive under all parameter values, and thus larger than his zero utility level under symmetric information. His information rent is thus $\left[\frac{p\beta}{2(1+p)}\right]^2$, which is increasing in the probability of the agent being of high type, p. Intuitively, as the proportion of low-cost agents decreases, they require a higher salary for the principal to distinguish them from the high-cost agents. In the above numerical example, this information rent becomes $\left[\frac{\frac{1}{3}32}{2\left(1+\frac{1}{3}\right)}\right]^2 = 16$.