EconS 503 - Microeconomic Theory II
Midterm Exam #1 - Answer key

1. **[IDWDS and Sophisticated equilibrium]** Consider a game with N players, where player i's strategy space is denoted as S_i. Assume that the game is solvable by IDWDS, yielding the surviving strategy set S_i^{Sur} for each player i, that is, $S_i^{Sur} \subset S_i$. Therefore, the strategy profile surviving IDWDS is denoted as the Cartesian product of surviving strategy sets $s^{Sur} \in S_1^{Sur} \times ... \times S_N^{Sur}$.

   We say that strategy profile surviving IDWDS, s^{Sur}, is a "*sophisticated equilibrium*" if every player i is indifferent between any two of his surviving strategies s_i and s_i', that is,

   $$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \text{ for every } s_i, s_i' \in S_i^{Sur} \text{ and all } s_{-i} \in S_{-i}.$$

   Consider the following normal-form game.

Player 1 \ Player 2	L	M	R
U	$5, 4$	$5, 4$	$9, 0$
C	$1, 7$	$2, 5$	$8, 6$
D	$2, 3$	$1, 4$	$8, 3$

   (a) Find the set of strategy profiles surviving IDWDS.

   - Let us start with player 1, whose strategies C and D do not survive IDWDS. Indeed, U strictly dominates both C and D, since

     $$\begin{aligned} u_1(U, s_2) &> u_1(C, s_2) \text{ for all } s_2 \in S_2, \text{ and} \\ u_1(U, s_2) &> u_1(D, s_2) \text{ for all } s_2 \in S_2 \end{aligned}$$

     (As a remark, note that strategies C and D don't survive IDSDS either). We can then C and D from the above matrix, obtainin the following reduced-form matrix:

Player 1 \ Player 2	L	M	R
U	$5, 4$	$5, 4$	$9, 0$

     We can now move to player 2, whose strategy R is strictly dominated by both L and M because

     $$u_2(U, R) < u_2(U, L) = u_2(U, M)$$

     Therefore, applying IDWDS yields a further reduced-form matrix as follows:

Player 1 \ Player 2	L	M
U	$5, 4$	$5, 4$

     As seen from the above, the Cartesian product of surviving strategies for both players becomes

     $$S^{Sur} = S_1^{Sur} \times S_2^{Sur} = \{(U, L), (U, M)\}$$

(b) Can you identify any sophisticated equilibria?

- The strategy profiles surviving IDWDS found in part (a) yield the same payoff to players 1 and 2, that is,

$$u_1(U, L) = u_1(U, M) = 5$$

for player 1, and

$$u_2(U, L) = u_2(U, M) = 4$$

for player 2. Generally, these two expressions can be more compactly written as $u_i(U, L) = u_i(U, M)$ for every player $i = \{1, 2\}$. Therefore, the equilibrium strategy profiles surviving IDSDS are "sophisticated" since both players obtain the same payoff in each of these strategy profiles.

2. [**Bargaining between a labor union and a firm**] Consider the following bargaining game between a labor union and a firm. In the first stage, the labor union chooses the wage level, w, that all workers will receive. In the second stage, the firm responds to the wage w by choosing the number of workers it hires, $h \in [0, 1]$. The labor union's payoff function is $u_L(w, h) = w \cdot h$, thus being increasing in wages and in the number of workers hired. The firm's profit is:

$$\pi(w, h) = \left(h - \frac{h^2}{2}\right) - (w \cdot h)$$

Intuitively, revenue is increasing in the number of workers hired, h, but at a decreasing rate (i.e., $h - \frac{h^2}{2}$ is increasing but concave in h), reaching a maximum when the firm hires all workers, i.e., $h = 1$ where revenue becomes $1/2$.

(a) Applying backward induction, analyze the optimal strategy of the last mover (firm). Find the firm's best response, $h(w)$.

(b) Anticipating the best response function of the firm that you found in part (a), $h(w)$, determine the labor union's optimal wage in the first stage of the game, w^*.

- See page at the end of this answer key.

3. [**Contests with symmetric valuations and two players**] Let us examine equilibrium investment in a contest, where two players compete to earn a prize, and the probability of winning the prize is a function of a player's own investment relative to the aggregate investment of all the players. Contests are often used to model promotions within a firm (where every worker invests time and effort into being selected for a promotion), political campaigns (where candidates invest money and resources to capture a larger share of votes), and R&D races (where firms invest resources into discovering a new product, such as drug). We consider two players, players i and j, who assign the same value to the prize, $V \geq 1$. The probability of winning the prize is given by

$$p_i = \frac{x_i^r}{x_i^r + x_j^r}$$

where x_i denotes player i's investment and the parameter $r \geq 1$ represents the effectiveness of player i's investment, which is assumed to be the same for every player. For simplicity, we normalize the cost of every unit of investment to one dollar.

(a) Setup and solve the utility maximization program for the players.

- We begin with player i, and the case for player j is analyzed analogously. The expected value of the prize for player i is

$$p_i \times V + (1 - p_i) \times 0 = \frac{x_i^r}{x_i^r + x_j^r} V$$

where recall that, to contest for the prize, player i has to spend x_i. Therefore, he solves

$$\max_{x_i \geq 0} EU_i [x_i | V] = \left(\frac{x_i^r}{x_i^r + x_j^r} V - x_i \right)$$

Taking first order conditions with respect to x_i,

$$\frac{\partial EU_i [x_i | V]}{\partial x_i} = \frac{r x_i^{r-1} \left(x_i^r + x_j^r \right) - r x_i^r x_i^{r-1}}{(x_i^r + x_j^r)^2} V - 1$$

Therefore, setting the above FOC equal to zero, and rearranging, the optimal investment for player i satisfies

$$\frac{r x_i^{r-1} x_j^r}{(x_i^r + x_j^r)^2} V = 1 \tag{1}$$

Analogously, the optimal investment for player j satisfies

$$\frac{r x_j^{r-1} x_i^r}{(x_i^r + x_j^r)^2} V = 1 \tag{2}$$

Invoking symmetry, $x_i^* = x_j^* = x^*$, expression (1) becomes

$$\frac{r x^{r-1} x^r}{(x^r + x^r)^2} V = 1,$$

which simplifies to $\frac{r x^{2r-1}}{4x^{2r}} V = 1$, or $\frac{r}{4x} V = 1$, yielding an equilibrium investment of

$$x^* = \frac{rV}{4}.$$

(b) How does the equilibrium investment of player i change with V and r?

- Differentiating x^* with respect to the common valuation that both players assign to the prize, V, we find

$$\frac{\partial x^*}{\partial V} = \frac{r}{4}$$

which is positive because $r \geq 1$. In words, bidders invest more if they assign a higher valuation to the prize.

- Differentiating x^* with respect to the investment effectiveness parameter, r, we obtain
$$\frac{\partial x^*}{\partial r} = \frac{V}{4}$$
which is positive because $V \geq 0$. In words, bidders invest more if their investments are more effective.

(c) Define Rent Dissipation as
$$D \equiv V - \left(x_i^* + x_j^*\right)$$
which can be understood as how many resources the society is left with once players i and j are completed with their investments. Find the Rent Dissipation D. Is it increasing or decreasing in the value that players assign to the prize, V? Is it increasing or decreasing in the effectiveness parameter r?

- Substituting the equilibrium investment, x^*, into the above expression, rent dissipation is
$$D = V - (x^* + x^*) = V - 2 \times \frac{rV}{4} = \frac{(2-r)\,V}{2} \tag{3}$$
- Differentiating expression (3) with respect to V,
$$\frac{\partial D}{\partial V} = \frac{2-r}{2}$$
Since $1 \leq r \leq 2$, rent dissipation increases with the value that the players assign to the prize, V. Intuitively, the more valuable is the prize to the winner, the more aggressive they bid such they obtain less rent.
- Differentiating expression (3) with respect to r,
$$\frac{\partial D}{\partial r} = -\frac{V}{2}$$
implying that rent dissipation decreases with the effectiveness of investment, r. Intuitively, the more effective is the investment, the more rent can player i capture.