EconS 503 - Microeconomic Theory II
Final Exam - Answer Key

1. **Helping small firms reduces welfare?** Consider a Cournot duopoly with two firms facing inverse demand function

$$p(q_1, q_2) = 1 - q_1 - q_2$$

Every firm i's marginal cost is $c_i > 0$, and firms do not face fixed costs. In addition, assume that marginal costs of firm 1 satisfy $c_1 < 1$ while those of firm 2 satisfy $c_2 < \frac{1+c_1}{2}$, so firm 2 benefits from a cost advantage relative to firm 1.

(a) Find each firm's best response function.

- Each firm i solves

$$\max_{q_i \geq 0} \ \pi_i = (1 - q_i - q_j)\, q_i - c_i q_i$$

Taking first order conditions with respect to q_i, we obtain

$$\frac{\partial \pi_i}{\partial q_i} = 1 - 2q_i - q_j - c_i = 0$$

Solving for q_i yields firm i's best response function

$$q_i(q_j) = \frac{1-c_i}{2} - \frac{1}{2} q_j$$

where, as usual, $\frac{1-c_i}{2}$ represents the vertical intercept of the best response function, while $-\frac{1}{2}$ is its negative slope.

b. Find the Nash equilibrium of the Cournot game.

- Using firm i's and j's best response functions

$$q_i(q_j) = \frac{1-c_i}{2} - \frac{1}{2} q_j \quad \text{and} \quad q_j(q_i) = \frac{1-c_j}{2} - \frac{1}{2} q_i$$

Simultaneously solving for q_i and q_j yields the output pair

$$q_1^* = \frac{1 - 2c_1 + c_2}{3} \quad \text{and} \quad q_2^* = \frac{1 - 2c_2 + c_1}{3}.$$

(b) Using your answer in part (b), determine equilibrium price, profits, consumer surplus, and social welfare (where social welfare is defined as the sum of consumer surplus and the profits of both firms).

- *Prices.* Let us first find equilibrium price

$$\begin{aligned} p^* &= 1 - q_i^* - q_j^* \\ &= 1 - \frac{1 - 2c_1 + c_2}{3} - \frac{1 - 2c_2 + c_1}{3} \\ &= \frac{1 + c_1 + c_2}{3} \end{aligned}$$

- *Profits.* We then obtain equilibrium profits

$$\begin{aligned}\pi_i^* &= \overbrace{\left(1-q_i^*-q_j^*\right)}^{p^*}q_i^* - c_i q_i^* \\ &= \frac{1+c_1+c_2}{3}\frac{1-2c_1+c_2}{3} - c_i\frac{1-2c_1+c_2}{3} \\ &= \frac{(1-2c_i+c_j)^2}{9} \quad \text{for every firm } i.\end{aligned}$$

- *Consumer surplus.* The consumer surplus is given by

$$CS^* = \frac{1}{2}\overbrace{(1-p^*)}^{\text{height}}\overbrace{Q^*}^{\text{base}}$$

where $Q^* = q_i^* + q_j^*$ denotes aggregate output in equilibrium. Since $Q = 1-p$ by definition, consumer surplus simplifies to

$$\begin{aligned}CS^* &= \frac{1}{2}\left(Q^*\right)^2 \\ &= \frac{1}{2}\left(\frac{1-2c_1+c_2}{3}+\frac{1-2c_2+c_1}{3}\right)^2 \\ &= \frac{1}{2}\left(\frac{2-c_1-c_2}{3}\right)^2 \\ &= \frac{(2-c_1-c_2)^2}{18}\end{aligned}$$

- *Welfare.* Hence, social welfare is

$$\begin{aligned}W^* &= CS^* + \pi_1^* + \pi_2^* \\ &= \frac{(2-c_1-c_2)^2}{18} + \frac{(1-2c_1+c_2)^2}{9} + \frac{(1-2c_2+c_1)^2}{9} \\ &= \frac{11c_1^2 - 8c_2 + 11c_2^2 - 2c_1\left(4+7c_2\right)+8}{18}.\end{aligned}$$

d. Using the social welfare you found in part (c), determine under which conditions social welfare increases/decreases in c_2. Interpret.

- We can directly differentiate the social welfare W^* found above with respect to c_2, to obtain

$$\frac{\partial W^*}{\partial c_2} = \frac{11c_2 - 7c_1 - 4}{9}$$

Let us now solve $\frac{11c_2-7c_1-4}{9} = 0$ for c_2 in order to identify for which of values of c_2 the above derivative is positive. Solving for c_2 yields $c_2 = \frac{4+7c_1}{11}$. Hence, the derivative $\frac{\partial W^*}{\partial c_2}$ is positive as long as

$$c_2 > \frac{7c_1+4}{11}.$$

- *Compatibility with initial conditions.* We can finally check if the condition we just found, $c_2 > \frac{7c_1+4}{11}$, is compatible with the initial assumption of the exercise, $c_2 < \frac{1+c_1}{2}$. For these two inequalities to simultaneously hold, we need $\frac{1+c_1}{2} > \frac{7c_1+4}{11}$ for all values of c_1, which holds since the difference

$$\frac{1+c_1}{2} - \frac{7c_1+4}{11} = \frac{3(1-c_1)}{22}$$

is positive given that $1 > c_1$ by definition. Therefore, when c_2 is relatively low, i.e., $0 < c_2 < \frac{7c_1+4}{11}$, welfare W^* decreases in c_2 (alternatively, a decrease in c_2 increases social welfare). In contrast, when c_2 is relatively high, $\frac{1+c_1}{2} > c_2 > \frac{7c_1+4}{11}$, welfare W^* increases in c_2 (that is, a decrease in c_2 decreases social welfare). For instance, when firm 1's costs are $c_1 = \frac{1}{10}$, we obtain that derivative $\frac{\partial W^*}{\partial c_2}$ is positive as long as $c_2 > \frac{7c_1+4}{11} = \frac{7\frac{1}{10}+4}{11} = \frac{47}{110} \simeq 0.42$. That is, when $c_2 > 0.42$.

- **Intuition**:
  - When firm 2's costs c_2 are low, the market share of firm 2 is really high, and a marginal increase in c_2 decreases W^* ($\frac{\partial W^*}{\partial c_2} < 0$ with respect to c_2 when c_2 is low).
  - However, when c_2 is relatively high, the market share of firm 2 is small, and a marginal increase in c_2 now increases W^* ($\frac{\partial W^*}{\partial c_2} > 0$ with respect to c_2 when c_2 is high). In short, social welfare is non-monotonic in c_2.

  That is why Lahiri and Ono's (1988) article was titled "Helping minor firms reduces welfare" since a firm with a small market share that sees its cost, c_2, marginally decrease produces an overall reduction in social welfare.[1]

2. **Pure-exchange economy.** Consider a pure-exchange economy with two individuals, A and B, each with utility function $u^i(x^i, y^i)$ where $i = \{A, B\}$, whose initial endowments are $e^A = (10, 0)$ and $e^B = (0, 10)$, that is, individual A (B) owns all units of good x (y, respectively).

   (a) Assuming that utility functions are $u^i(x^i, y^i) = \min\{x^i, y^i\}$ for all individuals $i = \{A, B\}$, find the set of PEAs and the set of WEAs.

   - *PEAs.* Since the utility functions are not differentiable we cannot follow the property of $MRS^A_{x,y} = MRS^B_{x,y}$ across consumers. Figure 1 helps us identify the set of PEAs. Points away from the 45^o-line, satisfying $y^A = x^A$, such as N, cannot be pareto efficient since we can still find other points, such as M, where consumer 2 is make better off while consumer 1 reaches the same utility level as under N. Once we are at points on the 45^o-line, such as M, we cannot find other points making at least once consumer better off (and keep the other consumer at least as well off). Hence, the set of PEAs is

$$\{(x^A, y^A), (x^B, y^B) : y^A = x^A \text{ and } y^B = x^B\}$$

[1] Sajal Lahiri and Yoshiyasu Ono (1988) "Helping Minor Firms Reduces Welfare," *Economic Journal*, vol. 98, issue 393, pp. 1199-1202.

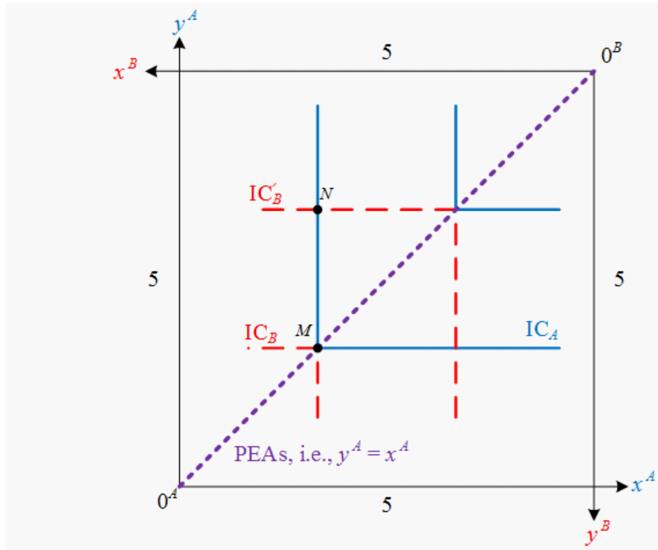

Figure 1. Edgeworth box and PEAs.

- *WEAs.* Using good 2 as the numeraire, i.e., $p_2 = 1$, the price ratio becomes $\frac{p_1}{p_2} = p_1$. The budget line of both consumers therefore has a slope $-p_1$ and crosses the point representing the initial endowment e in Figure 2 (where e lies at the lower right-hand corner).

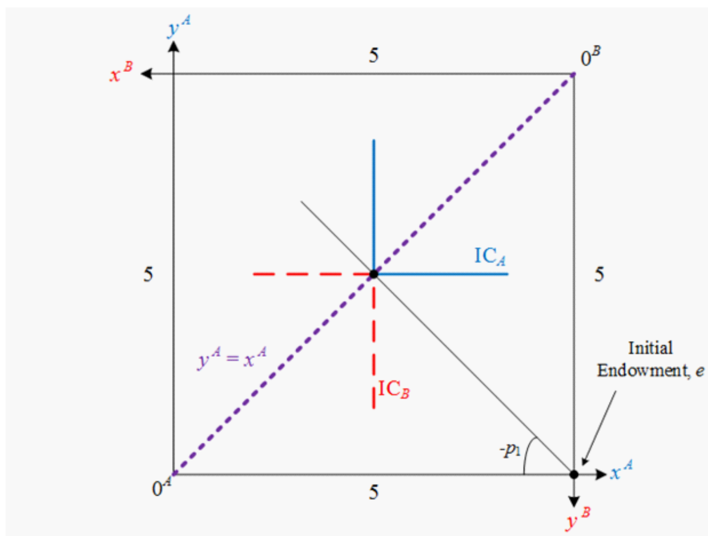

Figure 6.6. Edgeworth box and WEA.

Therefore, the WEA is given by the vector

$$\{(5, 5), (5, 5)\},$$

where every consumer enjoys 5 units of every good.

(b) Assuming utility functions of $u^A(x^A, y^A) = x^A y^A$ and $u^B(x^B, y^B) = \min\{x^B, y^B\}$, find the set of PEAs and WEAs.

- *PEAs.* By the same argument as in question (a), the set of PEAs satisfies $y^A = x^A$, as depicted in Figure 2. Point N cannot be efficient as we can still find other feasible points, such as M, where at least one consumer is made strictly better off (in this case consumer A). At points on the 45^o-line, however, we can no longer find alternatives that would constitute a Pareto improvement.

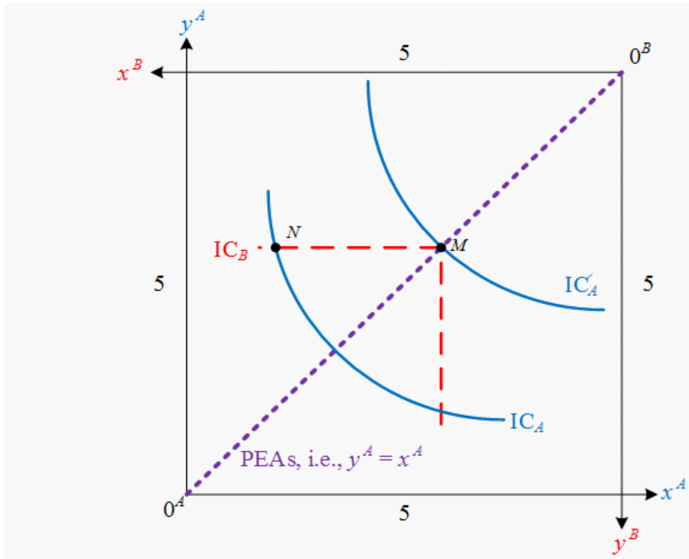

Figure 2. Edgeworth box and PEAs.

- *WEAs.* Using good y as the numeraire, $p_y = 1$, so that the price vector becomes $\mathbf{p} = (p_x, 1)$. Hence, *Consumer A's UMP is*

$$\max_{x^A, y^A} \quad x^A y^A$$
$$\text{subject to } p_x x^A + y^A = 10p_x$$

Taking first-order conditions

$$\begin{aligned} y^A - \lambda^A p_x &= 0 \\ x^A - \lambda^A &= 0 \\ p_x x^A + y^A &= 10p_x \end{aligned}$$

Combining the first two FOCs and rearranging, we have

$$p_x x^A = y^A$$

and substituting this equation into the third FOC yields

$$p_x x^A + p_x x^A = 10p_x \implies x^A = 5$$

and substituting this back into $p_x x^A = y^A$

$$y^A = 5p_x$$

Consumer B's UMP is not differentiable, but in equilibrium his Walrasian demands satisfy $x^B = y^B$. Substituting this into his budget constraint yields

$$p_x x^B + x^B = 10 \implies x^B = y^B = \frac{10}{p_x + 1}$$

Furthermore, the feasibility condition for good x entails

$$5 + \frac{10}{p_x + 1} = 10 + 0, \text{ or } p_x = 1$$

Therefore, the market of good x will clear at an equilibrium price of $p_x = 1$, i.e., $z_x(p_x, 1) = 0$ when $p_x = 1$. Since market y clears when market x does (by Walras' law), $z_y(p_x, 1)$ must also be zero when $p_x = 1$. Summarizing, the equilibrium price $p_x = 1$ yields a WEA

$$\{(5,5),(5,5)\}.$$

3. **DRM between a government official and an expert.** Consider the president of a country (P) facing a binary decision $k \in \{-1, 1\}$, e.g., whether to sign a bill ($k = 1$) or not ($k = -1$). Similarly as in cheap-talk games, his utility depends on whether his decision coincides with the state of nature (so he deviates as little as possible from the true state of nature, θ). In particular, considering that the state of nature is also binary, $\theta \in \{-1, 1\}$, the president's utility is

$$u_P(k, \theta) = \begin{cases} 0 & \text{if } k = \theta, \text{ and} \\ -1 & \text{otherwise} \end{cases}$$

For simplicity, assume that both states of nature are equally likely. Before choosing k, the president talks to an expert (E). Specifically, the expert privatley observes a noisy signal s of the true state of nature θ, where $s \sim N(\theta, 1)$, and given that signal s, the expert sends a message $m \in \mathbb{R}$ to the president. Hence, the president's choice can be described as a function of the message he receives from the expert, $k(m)$. The expert's utility is

$$u_E(k, \theta) = \begin{cases} 0 & \text{if } k = \theta, \\ -q & \text{if } \theta = 1 \text{ but } k = -1, \text{ and} \\ -(2-q) & \text{if } \theta = -1 \text{ but } k = 1 \end{cases}$$

where parameter $q \in (0, 2)$ can be interpreted as the expert's bias towards one type of error. For instance, when $q = 0.1$, the expert's utility is relatively high (low) when the president chooses $k = -1$ ($k = 1$) when the true state of nature was $\theta = 1$ ($\theta = -1$, respectively). The opposite argument applies when $q \to 2$.

Finally, note that the above description represents an indirect revelation mechanism, in which the expert sends a message $m \in \mathbb{R}$ to the president, and the president responds with allocation function $k(m) \in \{-1, 1\}$, entailing the above utilities for the president and the expert depending on the profile of (k, θ)-pairs.

(a) Consider a direct revelation mechanism in which the expert announces his privately observed signal s, and for each signal s, the efficient allocation function $k^*(s) \in \{-1, 1\}$ that maximizes the president's expected utility, $E_\theta[u_P(k, \theta)]$. That is, $k^*(s) = 1$ if and only if $E_\theta[u_P(1, \theta)|s] \geq E_\theta[u_P(-1, \theta)|s]$. Find under which conditions on the signal s, $k^*(s) = 1$. [*Hint:* This should be short.]

- Condition $E_\theta[u_P(1,\theta)|s] \geq E_\theta[u_P(-1,\theta)|s]$ entails

$$\begin{aligned} & [0 \times \Pr\{\theta = 1|s\}] + [(-1) \times \Pr\{\theta = -1|s\}] \\ \geq \;& [0 \times \Pr\{\theta = -1|s\}] + [(-1) \times \Pr\{\theta = 1|s\}] \end{aligned}$$

where the first term in each side of the inequality represents that the president chooses $k = \theta$, entailing a utility of 0, while the second terms describe the possibility of the president choosing $k \neq \theta$ and thus obtaining a utility of -1. We can simplify the above expression as follows

$$(-1)\Pr\{\theta = -1|s\} \geq (-1)\Pr\{\theta = 1|s\}$$

and even further, as follows,

$$\frac{1}{2}(-1)\Pr\{s|\theta = -1\} \geq \frac{1}{2}(-1)\Pr\{s|\theta = 1\}$$

which, multiplying both sides by -1 and rearranging, yields $\Pr\{s|\theta = -1\} < \Pr\{s|\theta = 1\}$. By symmetry, this condition is equivalent to $s > 0$. Hence, we can rewrite allocation rule $k^*(s)$ as

$$k^*(s) = \begin{cases} 1 & \text{if } s > 0 \text{, and} \\ -1 & \text{otherwise} \end{cases}$$

(b) Assume in this part of the exercise that the expert's bias is exactly $q = 1$. Show that the above allocation rule $k^*(s)$ is incentive compatible. [*Hint:* This is easy, you don't need to do any math.]

- When $q = 1$, the expert's utility becomes

$$u_E(k,\theta) = \begin{cases} 0 & \text{if } k = \theta, \\ -1 & \text{if } \theta = 1 \text{ but } k = -1 \text{, and} \\ -1 & \text{if } \theta = -1 \text{ but } k = 1 \end{cases}$$

and since he obtains -1 when $k \neq \theta$, we can rewrite his utility more compactly as

$$u_E(k,\theta) = \begin{cases} 0 & \text{if } k = \theta \text{, and} \\ -1 & \text{otherwise.} \end{cases}$$

thus coinciding with the utility function of the president. If the utility function of the sender and receiver exactly coincide, the sender has no incentives to misreport his signal, thus making the mechanism incentive compatible.

(c) For the remainder of the exercise, assume that the expert's bias is exactly $q = \frac{1}{2}$. Consider an indirect revelation mechanism in which the expert sends a message m to the president, and that the president responds using the following allocation rule

$$k(m) = \begin{cases} 1 & \text{if } m \geq K \text{, and} \\ -1 & \text{otherwise} \end{cases}$$

This is actually a simple *cutoff rule*: If the message m that the expert sends is sufficiently high (i.e., it is higher or equal to a cutoff $K \in \mathbb{R}$), the president responds signing the bill; but if the message is lower than cutoff K the president responds not signing the bill.

1. Show that this mechanism is incentive compatible if and only if the cutoff is exactly $K = \frac{\log 3}{2}$. [*Hint:* Recall that signals are normally distributed, implying a density function of $f(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right)$. Then, $\Pr\{s|\theta\} = f(s-\theta)$, which entails $\frac{f(s-\theta)}{f(s+\theta)} = \exp(2s)$.]
   - Similar to part (a) above, we can find under which conditions the expert prefers that the president selects $k = 1$ than $k = -1$. In particular,

     $$E_\theta[u_E(1,\theta)|s] \geq E_\theta[u_E(-1,\theta)|s]$$

     which entails

     $$\begin{aligned}&[0 \times \Pr\{\theta = 1|s\}] + \left[\left(-2 - \frac{1}{2}\right) \times \Pr\{\theta = -1|s\}\right] \\ \geq\ &[0 \times \Pr\{\theta = -1|s\}] + \left[\left(-\frac{1}{2}\right) \times \Pr\{\theta = 1|s\}\right]\end{aligned}$$

     which further reduces to

     $$\left(-\frac{3}{2}\right) \times \Pr\{\theta = -1|s\} \geq \left(-\frac{1}{2}\right) \times \Pr\{\theta = 1|s\}$$

     and to

     $$\left(-\frac{3}{2}\right) \times \frac{1}{2} \times \Pr\{s|\theta = -1\} \geq \left(-\frac{1}{2}\right) \times \frac{1}{2} \times \Pr\{s|\theta = 1\}$$

     which rearranging, yields

     $$3 < \frac{\Pr\{s|\theta = 1\}}{\Pr\{s|\theta = -1\}}$$

     In addition, since signal s is normally distributed, the probability $\Pr\{s|\theta\} = f(s-\theta)$, where $f(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right)$. Hence, we can rewrite the above inequality as follows

     $$3 < \frac{\Pr\{s|\theta = 1\}}{\Pr\{s|\theta = -1\}} = \frac{f\,(s-1)}{f(s+1)} = \exp(2s)$$

     or more compactly,

     $$3 < \exp(2s)$$

     Finally, solving for signal s (by applying logs on both sides of the inequality), we obtain

     $$\frac{\log 3}{2} \simeq 0.23 < s$$

2. Provide a short verbal explanation of your result.
   - In words, the above condition says that the expert prefers the same action as the president ($k = 1$ over $k = -1$) when the president's cutoff rule is to select $k = 1$ when the reported signal is $s > K = \frac{\log 3}{2} \simeq 0.23$. However, note that any cutoff rule with $K > 0$ entails that the allocation rule is biased towards not signing the bill, $k = -1$. Intuitively, the president needs a stronger signal to sign the bill.