EconS 503 - Advanced Microeconomics II
Handout on Mechanism Design

1. Public Good Provision

Imagine that you and your colleagues want to buy a coffee machine for your office. Suppose that some of you may be heavily addicted to coffee and are willing to pay more for the machine than the others. However, you do not know your colleagues' willingness to pay for the machine. The cost of the machine is C. We would like to find a decision rule in which (i) each individual reports a valuation (i.e., direct mechanism), and (ii) the coffee maker is purchased if and only if it is efficient to do so. Let us next analyze if it is possible to find a cost-sharing rule which gives incentive for everyone to report his valuation truthfully.
In particular, assume n individuals, each of them with private valuation $\theta_i \sim U[0,1]$. The allocation function is binary $y \in \{0,1\}$, i.e., the coffee machine is purchased or not. Let t_i be the transfer from individual i, implying a utility of

$$u_i(y, \theta_i, t_i) = y\theta_i - t_i$$

Let $i \in \{1, ..., n\}$ denote the individuals, and let $i = 0$ denote the original owner of the good.

(a) What is the efficient rule, $y^*(\theta_1, ..., \theta_n)$?

Answer:

The efficient rule is

$$y^*(\theta_1, ..., \theta_n) = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} \theta_j \geq C \\ 0 & \text{otherwise} \end{cases}$$

In words, the coffee machine is purchased if and only if the sum of all valuations exceeds its total cost.

(b) Consider the equal-share rule; when the public good is provided, the cost is equally divided by all n individuals.

(i). Before starting any computation, what would you expect - whether each individual would overstate or understate their valuation?

(ii). Confirm that the transfer rule is written by:

$$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

(iii). Let $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})$ be individual i's payoff when i reports $\tilde{\theta}_i$ instead of his true valuation θ_i, while the others truthfully report their valuations θ_{-i}. Show that

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-i})$$

(iv). Let $U_i\left(\tilde{\theta}_i|\theta_i\right)$ be individual i's expected payof when he reports $\tilde{\theta}_i$ instead of the true valuation θ_i. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-i})\right]$$

(v). Suppose that i's private valuation θ_i satisfies $\theta_i > \frac{C}{n}$. Assuming that the others are telling the truth, what is the best response for i? What if $\theta_i < \frac{C}{n}$? Is this mechanism strategy-proof? Is this mechanism Bayesian incentive compatible?

Answer:

(i). Because of free-rider incentives, each individual may have an incentive to understate his valuation. The equal-share payment rule, however, makes transfers independent of his report.

(ii). By the equal-share rule, each individual will pay $\frac{C}{n}$ if the project happens, and 0 otherwise. Hence, the transfer rule is

$$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

(iii). Using the definition of player i's utility function, we can plug in the above equal-share transfer rule to obtain

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= \theta_i y^*(\tilde{\theta}_i, \theta_{-i}) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \theta_i y^*(\tilde{\theta}_i, \theta_{-i}) - \frac{C}{n} y^*(\tilde{\theta}_i, \theta_{-i}) \\ &= \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-i}) \end{aligned}$$

(iv). Player i's expected payoff for misreporting $\tilde{\theta}_i \neq \theta_i$ is just the expected value of the utility found above, that is

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = E_{\theta_{-i}}\left[V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})\right] = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-i})\right]$$

(v). If player i's valuation θ_i satisfies $\theta_i > \frac{C}{n}$, $U_i\left(\tilde{\theta}_i|\theta_i\right)$ is maximized when $E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-i})\right]$ is maximized. Hence, individual i would report $\tilde{\theta}_i$ as large as possible, i.e., $\tilde{\theta}_i = 1$. In contrast, if θ_i satisfies $\theta_i < \frac{C}{n}$, individual i would report $\tilde{\theta}_i$ as small as possible, i.e., $\tilde{\theta}_i = 0$. The mechanism is neither strategy-proof nor Bayesian incentive compatible.

(c). Consider now the proportional payment rule:

$$t_i(\theta) = \frac{\theta_i C}{\sum_j \theta_j} y^*(\theta)$$

where every individual i pays a share of the total cost equal to the proportion that his reported valuation signifies out of the total reported valuations.

(i). Under this rule, what would you expect - whether each individual would overstate or understate the valuation?

(ii). Show that the utility of reporting $\tilde{\theta}_i$ is now

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-i})$$

(iii). For simplicity, suppose two individuals, $n = 2$ and a total cost of $C = 1$. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \tilde{\theta}_i\left(\theta_i - \log\left(\tilde{\theta}_i + 1\right)\right)$$

(iv). Is this mechanism strategy-proof? Is it Bayesian incentive compatible?

(v). Which way is everyone biased, overstate or understate? What is the intuition?

Answer:

(i). Now the payment is a function of the report. Notice that this cost-sharing rule is balanced-budget. Hence, you may expect that the agents have incentive to free-ride.
(ii). The payoff to each individual will be their actual valuation, less the amount the have to pay based on what they report if the project happens, and 0 otherwise. That is,

$$\begin{aligned}
V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= \theta_i y^*(\tilde{\theta}_i, \theta_{-i}) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\
&= \theta_i y^*(\tilde{\theta}_i, \theta_{-i}) - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j} y^*(\tilde{\theta}_i, \theta_{-i}) \\
&= \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-i})
\end{aligned}$$

(iii). Again, by definition, the expected utility of misreporting $\tilde{\theta}_i$ is

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = E_{\theta_{-i}}\left[\left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-i})\right]$$

Suppose now that $n = 2$ and $C = 1$. Then the above expression becomes

$$\begin{aligned}
U_i\left(\tilde{\theta}_i|\theta_i\right) &= E_{\theta_{-i}}\left[\left(\theta_i - \frac{\tilde{\theta}_i}{\tilde{\theta}_i + \theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-i})\right] \\
&= \int_{1-\tilde{\theta}_i}^{1} \left(\theta_i - \frac{\tilde{\theta}_i}{\tilde{\theta}_i + \theta_j}\right) d\theta_j \\
&= \left[\theta_i\theta_j - \tilde{\theta}_i \log(\tilde{\theta}_i + \theta_j)\right]_{1-\tilde{\theta}_i}^{1} \\
&= \theta_i - \tilde{\theta}_i \log(\tilde{\theta}_i + 1) - \left[\theta_i(1 - \tilde{\theta}_i) - \tilde{\theta}_i \log(\tilde{\theta}_i + (1 - \tilde{\theta}_i))\right] \\
&= \tilde{\theta}_i\left(\theta_i - \log(\tilde{\theta}_i + 1)\right)
\end{aligned}$$

(iv). It is straightforward to show that the expected utility of reporting $\tilde{\theta}_i$ is decreasing in player i's report $\tilde{\theta}_i$, since

$$\frac{\partial}{\partial \tilde{\theta}_i} U_i\left(\tilde{\theta}_i|\theta_i\right)\Big|_{\tilde{\theta}_i=\theta_i} = \frac{\theta_i^2}{1+\theta_i} - \log(\theta_1+1) < 0 \text{ for all } \theta_i \in (0,1]$$

implying that every player i has incentives to underreport his true valuation θ_i as much as possible, i.e., $\tilde{\theta}_i = 0$. Hence, this mechanism is neither strategy-proof nor Bayesian incentive compatible.

(v). The negative sign in part (iv) suggests that $U_i\left(\tilde{\theta}_i|\theta_i\right)$ is maximized at $\tilde{\theta}_i$ smaller than θ_i. Each individual has an incentive to understate the valuation.

(4). Consider now the VCG mechanism. Recall that the efficient rule $y^*(\theta)$ determines that the coffee machine is bought if and only if total valuations satisfy $\sum_i \theta_i \geq C$. Remember that we need to include the original owner of the public good; $i = 0$. Then, the total surplus when the valuation of individual i is considered in $\theta = (\theta_1, \theta_2, ..., \theta_n)$ is

$$\sum_{j\neq i} v_j\left(y^*(\theta), \theta_j\right) = \begin{cases} \sum_{j\neq i} \theta_j & \text{if } \sum_j \theta_j \geq C \\ C & \text{if } \sum_j \theta_j < C \end{cases}$$

while total surplus when the valuation of individual i is ignored, θ_{-i}, is

$$\sum_{j\neq i} v_j\left(y^*(\theta_{-i}), \theta_j\right) = \begin{cases} \sum_{j\neq i} \theta_j & \text{if } \sum_{j\neq i} \theta_j \geq C \\ C & \text{if } \sum_{j\neq i} \theta_j < C \end{cases}$$

The only difference in total surplus arises from the allocation rule which specifies that, when θ_i is considered, the good is purchased if and only if $\sum_j \theta_j \geq C$, whereas when θ_i is ignored, the good is bought if and only if $\sum_{j\neq i} \theta_j \geq C$. Hence, the VCG transfer is

$$\begin{aligned} t_i^*(\theta) &= -\left(\sum_{j\neq i} v_j(y^*(\theta), \theta_j) - \sum_{j\neq i} v_j(y^*(\theta_{-i}), \theta_j)\right) \\ &= \begin{cases} C - \sum_{j\neq i} \theta_j & \text{if } \sum_{j\neq i} \theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Intuitively, player i pays the difference between everyone else's valuations, $\sum_{j\neq i} \theta_j$, and the total cost of the good, C. Such a payment, however, only occurs when aggregate valuations exceed the total cost, $\sum_j \theta_j \geq C$, and thus the good is purchased, and when the valuations of all other players do not yet exceed the total cost of the good, $\sum_{j\neq i} \theta_j < C$, so the difference $C - \sum_{j\neq i} \theta_j$ is paid by player i in his transfer.

(i). Show that in this mechanism player i's utility from reporting a valuation $\tilde{\theta}_i \neq \theta_i$ is

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= v_i\left(y^*\left(\tilde{\theta}_i, \theta_{-i}\right), \theta_i\right) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \begin{cases} 0 & \text{if } \tilde{\theta}_i + \sum_{j\neq i} \theta_j < C \\ \sum_j \theta_j - C & \text{if } \sum_{j\neq i} \theta_j < C \leq \tilde{\theta}_i + \sum_{j\neq i} \theta_i \\ \theta_i & \text{if } C \leq \sum_{j\neq i} \theta_j \end{cases} \end{aligned}$$

(ii). Is this strategy-proof? Is this Bayesian incentive compatible?

(iii). For simplicity, suppose two individuals, $n = 2$, and a total cost of $C = 0.5$. Compute y^*, t_1^* and t_2^* for the following (θ_1, θ_2) pairs.

θ_1	θ_2
0.1	0.3
0.3	0.3
0.3	0.8
0.8	0.8

(iv). Show that the expected revenue from this mechanism is $E\left[t_1^*(\theta_1, \theta_2) + t_2^*(\theta_1, \theta_2)\right] = \frac{1}{6} \simeq 0.167$. Based on what you calculated in part (iii), is this problematic?

Answer:

(i). This is just the definition of the payoff function for the VCG.

(ii). In order to test if this direct revelation mechanism is strategy-proof,
1) Suppose that $C \leq \sum_{j \neq i} \theta_j$, i.e., the public good will be purchased regardless of individual i's reported valuation. Then $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = \theta_i$, which is independent of player i's reported valuation, $\tilde{\theta}_i$. Hence, telling the truth is player i's best response.
2) Now suppose that $\sum_{j \neq i} \theta_j < C \leq \sum_j \theta_j$, i.e., individual i's valuation is pivotal. Then by reporting a valutation $\tilde{\theta}_i$ such that $\tilde{\theta}_i \geq C - \sum_{j \neq i} \theta_j$, his utility becomes $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = \sum_j \theta_j - C \geq 0$. This includes the case of telling the truth; $\tilde{\theta}_i = \theta_i \geq C - \sum_{j \neq i} \theta_j$. If, instead, individual i lies by reporting $\tilde{\theta}_i < C - \sum_{j \neq i} \theta_j$, then his utility becomes $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = 0$ since the good is not purchased given that $\tilde{\theta}_i < C - \sum_{j \neq i} \theta_j$ entails $\tilde{\theta}_i + \sum_{i \neq j} \theta_j < C$. Hence, misreporting his valuation cannot be profitable.
3) Finally, suppose that $\sum_j \theta_j < C$, i.e., the public good will not be purchased regardless of individual i's valuation. Then, by honestly revealing his valuation, $\tilde{\theta}_i = \theta_i < C - \sum_{j \neq i} \theta_j$, his payoff is $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = 0$ since the good is not purchased. By lying, $\tilde{\theta}_i \geq C - \sum_{j \neq i} \theta_j$, his payoff is $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = \sum_j \theta_j - C < 0$. Telling a lie is then not profitable. Hence, truth-telling is the best strategy for i, regardless of the values of θ_{-i}. The VCG mechanism is thus strategy-proof, and also Bayesian incentive compatible.

(iii). For the case of $\theta_1 = 0.1$ and $\theta_2 = 0.3$, we have that VCG transfers become

$$t_i^*(\theta) = \begin{cases} -\sum_{j \neq i} \theta_j + C & \text{if } \sum_{j \neq i} \theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases}$$

implying that the transfer player 1 pays is

$$t_1(\theta) = \begin{cases} -0.3 + 0.5 & \text{if } 0.3 < 0.5 \leq 0.4 \\ 0 & \text{otherwise} \end{cases}$$

and the transfer that player 2 pays is

$$t_2(\theta) = \begin{cases} -0.1 + 0.5 & \text{if } 0.1 < 0.5 \leq 0.4 \\ 0 & \text{otherwise} \end{cases}$$

As we can see, the upper inequality does not hold, and thus the good is not purchased, $y^*(\theta) = 0$, and transfers are zero, $t_1^*(\theta) = t_2^*(\theta) = 0$. Following the same steps, the results for valuation pairs $(0.3, 0.3)$, $(0.3, 0.8)$, and $(0.8, 0.8)$ are presented in the following table

θ_1	θ_2	$y^*(\theta)$	$t_1^*(\theta)$	$t_2^*(\theta)$
0.1	0.3	0	0	0
0.3	0.3	1	0.2	0.2
0.3	0.8	1	0	0.2
0.8	0.8	1	0	0

(iv). If $\theta_2 \geq C$, then player 1 doesn't need to pay anything $t_1^* = 0$. If $\theta_2 < C$, then player 1's transfer is $t_1^* = -\theta_2 + C$ if and only if $\theta_1 + \theta_2 \geq C$. Hence, player 1's expected transfer is

$$\begin{aligned} E_\theta\left[t_1^*(\theta_1, \theta_2)\right] &= \int_{\{(\theta_1,\theta_2)|\theta_1+\theta_2 \geq C\}} (-\theta_2 + C) d\theta_1 d\theta_2 \\ &= \int_0^C \int_{-\theta_2+C}^1 (-\theta_2 + C) d\theta_1 d\theta_2 \end{aligned}$$

where the bounds of the inner integral emerge from the inequality that $-\theta_2 + C \leq \theta_1 \leq 1$, and the outer integral is that $\theta_2 < C$ in order for player 1 to be pivotal.

Simplifying the above expression, the expected transfer of player 1 becomes

$$\begin{aligned} & \int_0^C \left[(-\theta_2 + C) - (-\theta_2 + C)^2\right] d\theta_2 \\ = \;& C\left(C - C^2\right) + (2C - 1)\frac{C^2}{2} - \frac{C^3}{3} \\ = \;& \frac{C^2}{2} - \frac{C^3}{3} \end{aligned}$$

Evaluating at $C = 0.5$, $E_\theta\left[t_1^*(\theta_1, \theta_2)\right] = \frac{1}{12}$; and by symmetry, $E_\theta\left[t_2^*(\theta_1, \theta_2)\right] = \frac{1}{12}$, entailing that the expected revenue of the original owner of the public good becomes

$$E_\theta\left[t_1^*(\theta_1, \theta_2) + t_2^*(\theta_1, \theta_2)\right] = \frac{1}{6} \simeq 0.167$$

This is problematic, because the expected revenue, 0.167, is smaller than the total cost, 0.5, implying a budget deficit. The VCG mechanism has two nice properties: efficiency and incentive compatibility. However, balanced budget condition and participation constraint are not necessarily satisfied.

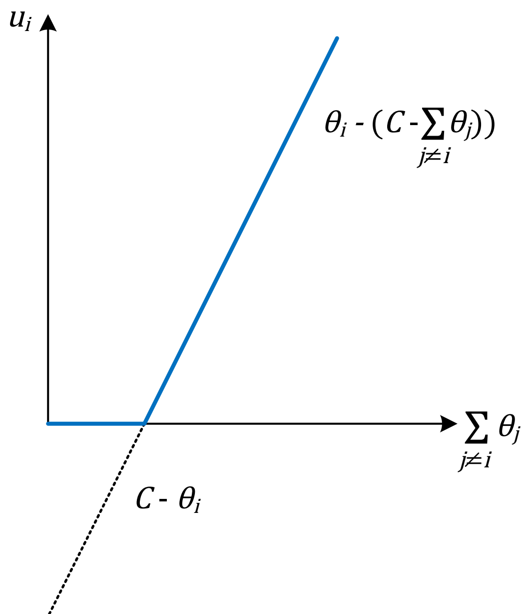
2. Implementation of Efficient Public Good Provision by Charging Pivotal Agents

Suppose that agent i's value for a good being auctioned, θ_i, is a random variable with support $[0, \beta_i]$. Each agent submits a bid $\tilde{\theta}_i$. The public good (which costs C to produce) is produced if total bids are larger than the production cost, $\sum_j \tilde{\theta}_j \geq C$. If this condition is not satisfied the agents pay nothing. If $C - \sum_{j \neq i} \tilde{\theta}_j - \beta_i \leq 0 < C - \sum_{j \neq i} \tilde{\theta}_j$ the public good is produced if and only if agent i's value is sufficiently high. Such an agent is said to be "pivotal." Define

$$t(\tilde{\theta}_{-i}) = \max \left\{ 0, C - \sum_{j \neq i} \tilde{\theta}_j \right\}$$

If agent i is pivotal and has submitted a bid above this transfer he pays t_i. Otherwise agent i pays nothing.

(a) Show that if agent i bids his value, his payoff is a function of $\sum_{j \neq i} \theta_j$ as depicted below.

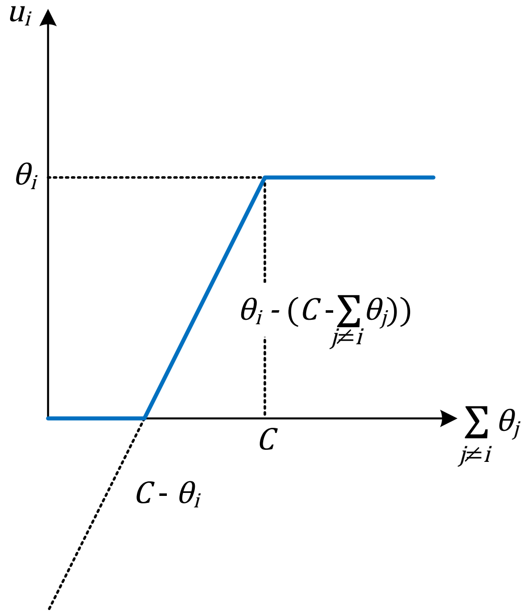


Answer:

If agent i bids his value, θ_i and, assuming that all other agents bid their true valuation, $\theta_i + \sum_{j \neq i} \theta_j < C$ (that is, $\sum_{j \neq i} \theta_j < C - \theta_i$) then the item is not produced and $t = 0$. Therefore the agent's payoff is zero. If $\sum_{j \neq i} \theta_j \geq C - \theta_i$ the good is produced and the agent pays

$$t(\theta_{-i}) = \max\{0, C - \sum_{j \neq i} \theta_j\}$$

Agent i's payoff is depicted below



where, starting from the origin, for low values of $\sum_{j\neq i}\theta_j$, the good is not purchased. At the point where $\sum_{j\neq i}\theta_j = C - \theta_i$, the good is purchased, and a kink in the payoff graph emerges, as agent i is now pivotal and must pay a transfer of $C - \sum_{j\neq i}\theta_j$, making his total payoff

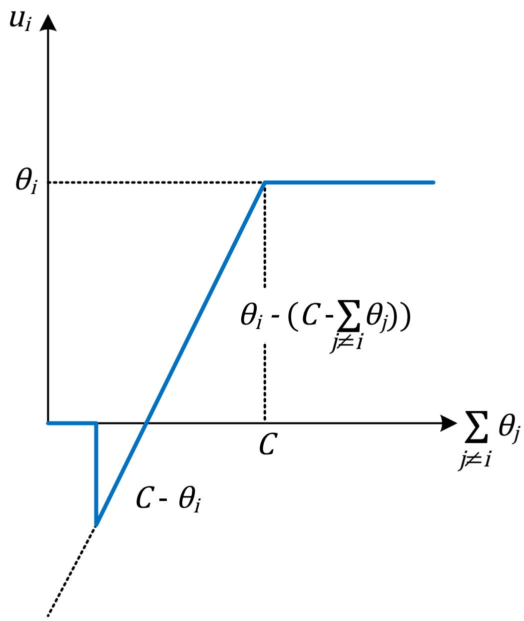
$$
\begin{aligned}
V_i(\theta_i|\theta_i, \theta_{-i}) &= v_i\left(y^*\left(\theta_i, \theta_{-i}\right), \theta_i\right) - t_i^*\left(\theta_i, \theta_{-i}\right) \\
&= \theta_i - (C - \sum_{j\neq i}\theta_j) \geq 0
\end{aligned}
$$

At the point where $\sum_{j\neq i}\theta_j = C$, the good is purchased without agent i's valuation, and thus agent i is no longer pivotal. His transfer become 0 and he receives his full valuation, θ_i, as his payoff.

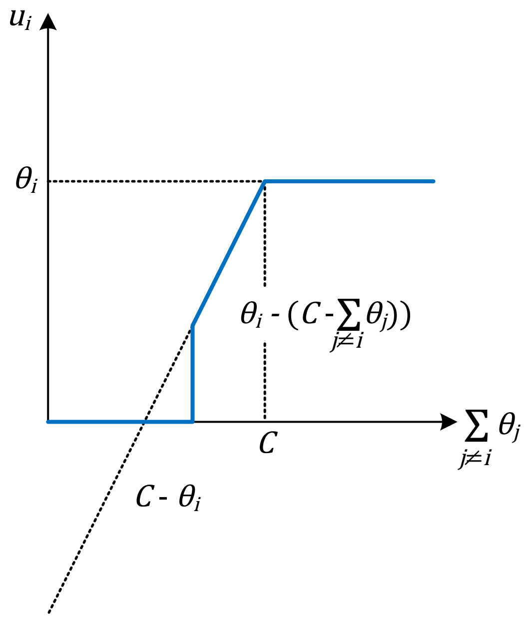
(b) Draw the payoff graph (i) with $\tilde{\theta}_i < \theta_i$ and (ii) with $\tilde{\theta}_i > \theta_i$.

Answer:

If agent i announces $\tilde{\theta}_i > \theta_i$ the public good is produced if total bids exceed the item's cost, $\tilde{\theta}_i + \sum_{j \neq i} \theta_j \geq C$; that is, $\sum_{j \neq i} \theta_j \geq C - \tilde{\theta}_i$. Thus the new payoff function is as shown below.



Note that agent i's payoff is negative if $\sum_{j \neq i} \theta_j \in [C - \tilde{\theta}_i, C - \theta_i)$ and is otherwise unaffected. This is due to agent i's inflated valuation causing the good to be purchased too soon, which also makes agent i pivotal for a lower value of $\sum_{j \neq i} \theta_j$. Thus agent i's expected payoff is strictly lower if he announces $\tilde{\theta}_i > \theta_i$. An almost identical argument shows that his expected payment is also strictly lower if he announces $\tilde{\theta}_i < \theta_i$ as depicted below.



In this case, when $\sum_{j \neq i} \theta_j \in [C - \theta_i, C - \tilde{\theta}_i)$, The good is not purchased when agent i would receive a positive payoff from it being purchased and being a pivotal agent. Thus, for that range, agent i's payoff is strictly lower if he announces $\tilde{\theta}_i < \theta_i$.

(c) Explain why it is a dominant strategy for agent i to bid his value.

Answer:

This argument follows the same from as in part (b). Let's expand it a bit.
Case 1: $\theta_i < \theta_i + \sum_{j \neq i} \theta_j < C$. In this case, individual i is not pivotal, and the public good would not be purchased. This leads individual i to receive a payoff of 0. If indidivual i were to report a valuation $\tilde{\theta}_i < \theta_i$, there would be no change. Likewise, if individual i reported a valuation $\theta_i < \tilde{\theta}_i < C - \sum_{j \neq i} \theta_j$, there would still be no change. If, however, individual i reported a valuation sufficiently high enough such that $\tilde{\theta}_i \geq C - \sum_{j \neq i} \theta_j$, then the public good would be purchased, and individual i would be charged for his pivotal role

$$\begin{aligned} t(\theta_{-i}) &= \max\{0, C - \sum_{j \neq i} \theta_j\} \\ &= C - \sum_{j \neq i} \theta_j > 0 \end{aligned}$$

Thus, individual i's payoff would be $\theta_i - \left(C - \sum_{j \neq i} \theta_j\right) = \sum_j \theta_j - C < 0$, and he has a weakly dominant strategy to remain truthful.
Case 2: $\sum_{j \neq i} \theta_j < C \leq \theta_i + \sum_{j \neq i} \theta_j$. In this case, individual i is pivotal, and the public good will be purchased. This causes individual i to receive a payoff of $\sum_j \theta_j - C > 0$ for being pivotal. If indidivual i were to report a valuation $\tilde{\theta}_i > \theta_i$, there would be no change. Likewise, if individual i reported a valuation $\theta_i > \tilde{\theta}_i > C - \sum_{j \neq i} \theta_j$, there would still be no change since his payoff is just a function of everyone else's valuations. If, however, individual i reported a valuation sufficiently low enough such that $\tilde{\theta}_i < C - \sum_{j \neq i} \theta_j$, then the public good would not be purchased, and individual i would receive a payoff of 0. Thus, individual i has a weakly dominant strategy to remain truthful.
Case 3: $C < \sum_{j \neq i} \theta_j \leq \theta_i + \sum_{j \neq i} \theta_j$. In this case, individual i is not pivotal, and the public good will be purchased leaving agent i with a payoff of 0. Changing his payoff in either direction will cause no change in the outcome, and thus, individual i has a weakly dominant strategy to tell the truth.

3. MWG 23.F.2

Consider a monopolist with costs $c > 0$ and multiple consumers with types $\theta > 0$. The consumers have utility functions $\theta v(x) - t$ where x is the amount of the good consumed and $v'(\cdot) > 0$ $v''(\cdot) < 0$. θ is distributed across the support $[\underline{\theta}, \bar{\theta}]$ with $\bar{\theta} > \underline{\theta} > 0$ distributed with a CDF $\Phi(\cdot)$ with positive density $\phi(\cdot) > 0$.

Consider the buyer, who we will denote as agent $i = 1$. His utility function is

$$u^1(\theta, x, t) = \theta v(x) - t$$

Hence, its first order derivative with respect to θ is $u^1_\theta(\theta, x, t) = v(x)$ and its second order derivative with respect to $x\theta$ is $u^1_{x\theta}(\theta, x, t) = v'(x) > 0$. Hence, we have that the single-crossing property is satisfied given that the marginal utility of additional units of x is increasing in the buyer's type θ.
Next, consider the seller, who we will denote as agent $i = 0$. His utility function is

$$u^0(\theta, x, t) = t - c \cdot x$$

Using the Revelation Principle we can focus only on Direct Revelation Mechanisms $f(\theta) = (x(\theta), t(\theta))$ that solves the seller's maximization problem:

$$\max_{x(\theta), t(\theta)} \quad E[t(\theta) - c \cdot x(\theta)]$$

subject to the SCF $f(\theta) = (x(\theta), t(\theta))$ being Bayesian Incentive Compatible (BIC) and Individually Rational (IR). Let's denote by $U(\theta) = \theta v(x(\theta)) - t(\theta)$ the expected utility of the buyer when truthfully revealing his type θ. Hence, we can solve for $t(\theta)$ to obtain $t(\theta) = \theta v(x(\theta)) - U(\theta)$, which we substitute in the above expression to obtain:

$$\max_{x(\theta), t(\theta)} \quad E[\underbrace{\theta v(x(\theta)) - U(\theta)}_{t(\theta)} - c \cdot x(\theta)]$$

subject to:

$$\left.\begin{array}{c} (1)\ x(\cdot) \text{ is nondecreasing in } \theta \\ (2)\ U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(x(s))ds, \quad \forall \theta \in [\underline{\theta}, \bar{\theta}] \end{array}\right\} BIC$$

$$(3)\ U(\theta) \geq 0 \quad \forall \theta \in [\underline{\theta}, \bar{\theta}] \} \ IR$$

In addition, if constraint (2) holds, then constraint (3) is satisfied if and only if $U(\underline{\theta}) \geq 0$. We can hence rewrite the above program replacing constraint (3) for $U(\underline{\theta}) \geq 0$, which we denote as (3′).

$$\max_{x(\theta), t(\theta)} \quad E[\theta v(x(\theta)) - U(\theta) - c \cdot x(\theta)]$$

subject to:

$$(1)\ x(\cdot) \text{ is nondecreasing in } \theta$$

$$(2)\ U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v(x(s))ds, \quad \forall \theta \in [\underline{\theta}, \bar{\theta}]$$

$$(3')\ U(\theta) \geq 0$$

Plugging constraint (2) in the objective function yields

$$\max_{x(\theta), t(\theta)} \quad \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta v(x(\theta)) \underbrace{- U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v(x(s))ds}_{\text{From (2)}} - c \cdot x(\theta) \right] \phi(\theta) d\theta$$

subject to:

$$(1)\ x(\cdot) \text{ is nondecreasing in } \theta$$

$$(3)\ U(\underline{\theta}) \geq 0 \quad \forall \theta \in [\underline{\theta}, \bar{\theta}]$$

Note that operating with the objective function we have

$$
\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}} \left[\theta v(x(\theta)) - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v(x(s))ds - c \cdot x(\theta)\right] \phi(\theta)d\theta \\
= &\int_{\underline{\theta}}^{\bar{\theta}} \theta v(x(\theta))\phi(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\theta} v(x(s))ds\phi(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} c \cdot x(\theta)\phi(\theta)d\theta - \underbrace{U(\underline{\theta})}_{\text{Constant}}
\end{aligned}
$$

Looking at the middle term

$$
\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\theta} v(x(s))ds\phi(\theta)d\theta
$$

we can apply integration by parts. Let

$$
\begin{aligned}
h(x) &= \int_{\underline{\theta}}^{\theta} v(x(s))ds \quad g'(x) = \phi(\theta)d\theta \\
h'(x) &= v(x(\theta))d\theta \quad g(x) = \Phi(\theta)
\end{aligned}
$$

Recall from integration by parts,

$$
\int h(x)g'(x) = h(x)g(x) - \int g(x)h'(x)
$$

Substituting,

$$
\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\theta} \underbrace{v(x(s))ds}_{h}\underbrace{\phi(\theta)d\theta}_{g'} &= \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} v(x(\theta))d\theta}_{h} \underbrace{\left(\Phi(\theta)|_{\underline{\theta}}^{\bar{\theta}}\right)}_{=1 \text{ by cdf definition}} - \int_{\underline{\theta}}^{\bar{\theta}} \underbrace{\Phi(\theta)}_{g}\underbrace{v(x(\theta))d\theta}_{h'} \\
&= \int_{\underline{\theta}}^{\bar{\theta}} v(x(\theta))\left[1 - \Phi(\theta)\right]d\theta
\end{aligned}
$$

Substituting back into the second term of the objective function,

$$
\int_{\underline{\theta}}^{\bar{\theta}} \theta v(x(\theta))\phi(\theta)d\theta - \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} v(x(\theta))\left[1 - \Phi(\theta)\right]d\theta}_{\text{2nd term}} - \int_{\underline{\theta}}^{\bar{\theta}} c \cdot x(\theta)\phi(\theta)d\theta - U(\underline{\theta})
$$

and rearranging, we obtain

$$
\int_{\underline{\theta}}^{\bar{\theta}} \left[v(x(\theta))\left[\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)}\right] - c \cdot x(\theta)\right]\phi(\theta)d\theta - U(\underline{\theta})
$$

At the solution, $U(\underline{\theta}) = 0$. Otherwise, the monopolist could extract surplus from the buyer with the lowest valuation. Hence, we can rewrite again the seller's optimization problem as

$$
\max_{x(\theta), t(\theta)} \quad \int_{\underline{\theta}}^{\bar{\theta}} \left[v(x(\theta))\left[\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)}\right] - c \cdot x(\theta)\right]\phi(\theta)d\theta
$$

subject to:

(1) $x(\cdot)$ is nondecreasing in θ

Finally, ignoring the constraint and assuming an interior solution, we obtain that the function $x(\theta)$ that solves the problem must satisfy the following first-order condition

$$v'(x(\theta))\left[\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right] - c = 0 \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}]$$

or, rearranging,

$$v'(x(\theta))\theta - v'(x(\theta))\frac{1-\Phi(\theta)}{\phi(\theta)} = c$$

Intuitively, the monopolist increases the output sold to a buyer with valuation θ until the point where the marginal cost of producing one more unit, c, coincides with the marginal valuation that the buyer assigns to this additional unit less the valuation loss of all buyers with types above θ.

Monotonicity of the solution, $x(\theta)$. We now need to check that the ignored constratint (1) is indeed satisfied at the optimum. That is, we seek to show that $x'(\theta) \geq 0$. Differentiating the resulting FOC with respect to θ, and using $J(\theta) \equiv \theta - \frac{1-\Phi(\theta)}{\phi(\theta)}$ to denote the virtual valuation of an consumer with type θ, yields

$$v''(x(\theta))x'(\theta)J(\theta) + v'(x(\theta))J'(\theta) - 0 = 0$$

Rearranging,

$$v''(x(\theta))x'(\theta)J(\theta) = -v'(x(\theta))J'(\theta)$$

and solving for $x'(\theta)$ entails

$$x'(\theta) = -\frac{v'(x(\theta))J'(\theta)}{v''(x(\theta))J(\theta)} = -\frac{(+)\times(+)}{(-)\times(+)} = +$$

since the $v(\cdot)$ function satisfies $v'(\cdot) > 0$ and $v''(\cdot) \leq 0$, and the virtual valuation function $J(\theta)$ satisfies $J(\theta) > 0$ given that θ is distributed across the support $[\underline{\theta}, \bar{\theta}]$ with $\bar{\theta} > \underline{\theta} > 0$, and $J'(\theta) > 0$ by assumption. Therefore, $x'(\theta) \geq 0$ implying that the amount of good consumed x increases in the consumer's type, θ, as required by constraint (1).

Sufficiency: Note that this first-order condition characterizes the solution to the seller's actual maximization problem, given that $v''(\cdot) < 0$ which implies that the second-order conditions are satisfied.
$U(\theta)$: The optimal value of $U(\theta)$ can be obtained from $x(\theta)$ and constraint (2).
$t(\theta)$: The optimal value of the transfer $t(\theta)$ can be obtained from $U(\theta)$ and the expression of $t(\theta) = \theta v(x(\theta)) - U(\theta)$.

Intuition:

- The consumer with the highest valuation $\bar{\theta}$ is set at the first-best level since $\Phi(\bar{\theta}) = 1$, and from the first-order condition we get

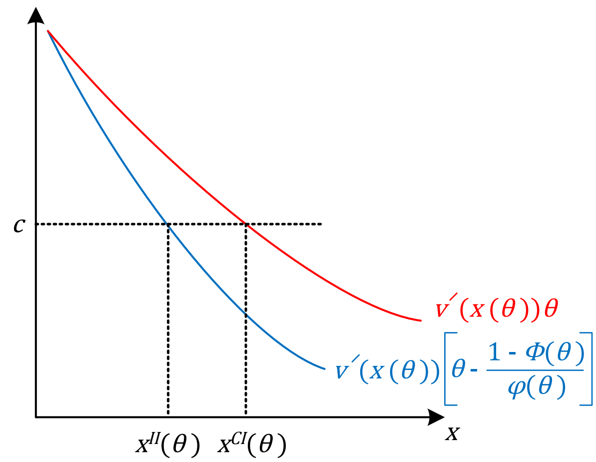
$$v'(x(\bar{\theta}))\left[\bar{\theta} - \frac{1-1}{\phi(\theta)}\right] - c = 0 \iff v'(x(\bar{\theta}))\bar{\theta} - c = 0.$$

  Thus coinciding with the FOC for this consumer under complete information. That is, no distortion at the top.

- All the other consumers $\theta \in [\underline{\theta}, \bar{\theta})$ get distorted since

$$v'(x(\theta))\left[\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right] < v'(x(\theta))\theta$$

given that $\frac{1-\Phi(\theta)}{\phi(\theta)} > 0$, $v' > 0$, and $\theta > 0$. As depicted in the figure below, the monopolist offers fewer units to these buyers under incomplete information of their valuations than under a complete information context. This is a common result in the literature on screening.



Example: Assume that $\theta \sim U[0,1]$, that $v(x) = \ln x$, and that $c = 1/4$, then the above FOC for an optimum becomes

$$\frac{1}{x(\theta)}\left[\theta - (1-\theta)\right] - \frac{1}{4} = 0 \quad \text{for all } \theta \in [0,1]$$

Solving for $x(\theta)$, we obtain an optimal outcome of $x(\theta) = 8\theta - 4$, which is clearly increasing in θ.