EconS 503 - Microeconomic Theory II
Homework #6 - Answer key

1. **Exercises from textbooks:**

   (a) Watson, Chapter 29: Exercises 7, 8 and 10 (see exercises at the end of this assignment).

   (b) MWG, Chapter 9: Exercise 9.C.7.

   (c) Bolton and Dewatripont, Exercise 6 (see page 650 since all exercises are at the end of the book).

   - See handout at the end of this answer key.

2. [**Entry deterrence with a sequence of potential entrants**] The following entry model is inspired on the original paper of Kreps and Wilson (*JET*, 1982). Consider an incumbent monopolist building a reputation as a tough competitor who does not allow entry without a fight. The entrant first decides whether to enter the market, and, if he does, the monopolist chooses whether to fight or acquiesce. If the entrant stays out, the monopolist obtains a profit of $a > 1$, and the entrant gets 0. If the entrant enters, the monopolist gets 0 from fighting and -1 from acquiescing if he is a "tough" monopolist, and -1 from fighting and 0 from acquiescing if he is a "normal" monopolist. The entrant obtains a profit of b if the monopolist acquiesces and $b - 1$ if he fights, where $0 < b < 1$. Suppose the entrant believes the monopolist to be tough (normal) with probability p ($1 - p$, respectively), while the monopolist observes his own type.

   (a) Depict a game tree representing this incomplete information game.

   (b) Solve for the PBE of this game.

   (c) Suppose the monopolist faces two entrants in sequence, and the second entrant observes the outcome of the first game (there is no discounting). Depict the game tree, and solve for the PBE. [*Hint:* you can use backward induction to reduce the game tree as much as possible before checking for the existence of separating or pooling PBEs. For simplicity, focus on the case in which prior beliefs satisfy $p \leq b$.]

   - See handout at the end of this answer key.

3. [**PBEs in bargaining**] A buyer and a seller are bargaining. The seller owns an object for which the buyer has a value v; the seller's value is zero. The buyer knows v but the seller does not. The seller's beliefs about v, which are common knowledge, are that $v = 30$ with probability $\frac{1}{2}$ and $v = 10$ with probability $\frac{1}{2}$. There are two periods of bargaining; there is no discounting (i.e., $\delta = 1$).

- In the first period, the seller makes an offer p_1 that represents a price that the buyer will need to pay to buy the object. The buyer can accept or reject the offer. If the buyer accepts, the offer is implemented and the game ends. If the buyer rejects, the game continues to the second period.
- In the second period, the seller (again) makes an offer p_2, which is the price the buyer will need to pay to buy the object. The buyer can accept or reject the offer. If the buyer accepts, the offer is implemented and the game ends. If the buyer rejects, then the seller keeps the object and the game ends.

If the buyer buys the object in the first period, then the payoffs are p_1 for the seller and $v - p_1$ for the buyer. Similarly, if the buyer buys the object in the second period, then the payoffs are p_2 for the seller and $v - p_2$ for the buyer. If the buyer does not buy the object, then the payoffs are zero for each player.

(a) Provide an extensive-form representation of this game.

(b) Find a Perfect Bayesian equilibrium in which the seller believes that any buyer that rejects a first-period offer is the type with valuation $v = 10$ with probability 1. (Justify your answer, and remember to fully specify the Perfect Bayesian equilibrium.)

- See handout at the end of this answer key.

4. **Cheap talk with three types.** Consider the cheap talk model with three types discussed in class (Investing recommendations game). Let us focus on the partially separating strategy profile where the Analyst (sender) recommends Buy both when the stock outperforms the market and when its neutral, but recommends Hold when the stock underperforms the market. In class, we made a simplifying assumption on off-the-equilibrium beliefs (after the Investor receives a Sell recommendation), denoted by γ_1, γ_2, and $1 - \gamma_1 - \gamma_2$.

(a) Without restricting off-the-equilibrium beliefs, find under which conditions the above partially separating strategy profile can be sustained as a PBE of this game.

(b) Consider now the pooling strategy profile where the Analyst recommends Buy regardless of the stock's type. Under which conditions can this strategy profile be supported as a PBE?

- See handout at the end of this answer key.

5. **Cheap talk with a continuum of types.** Consider the Crawford-Sobel cheap talk game with a continuum of types. In class we discussed the maximal number of partitions n that can be sustained as a PBE of the game.

(a) Find the ex-ante expected utility that the sender obtains in equilibrium. (By "ex-ante" we mean before observing the realization of parameter θ.)

- The *ex-ante* expected utility of the sender at a N-partition equilibrium is

$$EU_1 = E\left[-\left[p-(\theta+\delta)\right]^2\right]$$

since the policy in a given partition is given by $\frac{\theta_{k-1}+\theta_k}{2}$, we can express teh above expected utility as follows

$$EU_1 = -\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}[\underbrace{\frac{\theta_{k-1}+\theta_k}{2}}_{\text{Policy } p(m(\theta))} -x-\delta]^2 dx$$

where we integrate over all values of θ in that partition, and then sum over all N partitions. To facilitate our analysis, we expand the above expression as follows

$$\begin{aligned} EU_1 \quad = \quad & \underbrace{-\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}\left(\frac{\theta_{k-1}+\theta_k}{2}-x\right)^2 dx}_{\text{Term } A} \\ & \underbrace{+2\delta\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}\left(\frac{\theta_{k-1}+\theta_k}{2}-x\right)dx}_{\text{Tern } B} - \underbrace{\delta^2\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}dx}_{\text{Term } C} \end{aligned}$$

We next separately analyze each term.

- *Term A*. First, we can simplify term A as follows

$$\begin{aligned} -\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}\left(\frac{\theta_{k-1}+\theta_k}{2}-x\right)^2 dx &= \sum_{k=1}^{N}\left[\left(\frac{\theta_{k-1}+\theta_k}{2}-x\right)^3\right]_{\theta_{k-1}}^{\theta_k} \\ &= \frac{1}{3}\sum_{k=1}^{N}\left[\left(\frac{\theta_{k-1}+\theta_k}{2}-\theta_k\right)^3-\left(\frac{\theta_{k-1}+\theta_k}{2}-\theta_{k-1}\right)^3\right] \\ &= \frac{1}{3}\sum_{k=1}^{N}\left[-\left(\frac{\theta_k-\theta_{k-1}}{2}\right)^3-\left(\frac{\theta_k-\theta_{k-1}}{2}\right)^3\right] \\ &= -\frac{1}{12}\sum_{k=1}^{N}(\theta_k-\theta_{k-1})^3 \end{aligned}$$

- *Term B*. We can now simplify term B as follows

$$\begin{aligned} \sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}\left(\frac{\theta_{k-1}+\theta_k}{2}-x\right)dx &= \sum_{k=1}^{N}\frac{\theta_{k-1}+\theta_k}{2}\int_{\theta_{k-1}}^{\theta_k}dx-\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}xdx \\ &= \sum_{k=1}^{N}\frac{\theta_{k-1}+\theta_k}{2}(\theta_k-\theta_{k-1})-\sum_{k=1}^{N}\left(\frac{x^2}{2}\right)_{\theta_{k-1}}^{\theta_k} \\ &= \sum_{k=1}^{N}\frac{\theta_k^2-\theta_{k-1}^2}{2}-\sum_{k=1}^{N}\frac{\theta_k^2-\theta_{k-1}^2}{2} \\ &= 0 \end{aligned}$$

so that $2\delta \sum_{k=1}^{N} \int_{\theta_{k-1}}^{\theta_k} \left(\frac{\theta_{k-1}+\theta_k}{2} - x \right) dx = 0$.

- *Term C*. Finally, we simplify term C as follows

$$\sum_{k=1}^{N} \int_{\theta_{k-1}}^{\theta_k} dx = \sum_{k=1}^{N} [x]_{\theta_{k-1}}^{\theta_k} = \sum_{k=1}^{N} (\theta_k - \theta_{k-1}) = \theta_N - \theta_0 = 1$$

so that $\delta^2 \sum_{k=1}^{N} \int_{\theta_{k-1}}^{\theta_k} dx = \delta^2$.

- Combining our results from terms A-C, the expected utility of the sender (e.g., Special Interest Group) becomes

$$EU_1 = \underbrace{-\frac{1}{12} \sum_{k=1}^{N} (\theta_k - \theta_{k-1})^3}_{\text{Term } A} + \underbrace{0}_{\text{Term } B} + \underbrace{\delta^2}_{\text{Term } C}$$

- Recalling that, for the Special Interest Group to send an incentive compatible message $m(\theta_k) = \theta_k$, the length of the k^{th} interval must satisfy

$$\theta_k - \theta_{k-1} = 4\delta (k-1)$$

Substituting this expression into the above expected utility, we find that

$$EU_1 = -\frac{1}{12} \sum_{k=1}^{N} \left(\overbrace{4\delta (k-1)}^{\theta_k - \theta_{k-1}} \right)^3 - \delta^2$$
$$= -\frac{1}{12} \delta^3 \sum_{k=1}^{N} (4k-4)^3 - \delta^2$$
$$= -\frac{64}{12} \delta^3 \sum_{k=1}^{N} (k-1)^3 - \delta^2$$

At this point, note that $\sum_{k=1}^{N} (k-1)^3 = \sum_{k=0}^{N-1} k^3$, which helps us rewrite the above expected utility as

$$EU_1 = -\frac{16}{3} \delta^3 \sum_{k=0}^{N-1} k^3 - \delta^2.$$

Further recalling that the sum $\sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}$, the expected utility of the sender becomes

$$EU_1 = -\frac{16}{3} \delta^3 \overbrace{\frac{N^2 (N-1)^2}{4}}^{\sum_{k=1}^{N} k^3} - \delta^2$$
$$= -\frac{4}{3} \delta^3 N^2 (N-1)^2 - \delta^2$$

(b) Find the *ex-ante* expected utility that the receiver obtains in equilibrium.

- The *ex-ante* expected utility that the receiver obtains in a N-partition equilibrium is

$$EU_2 = E\left[-\left(p-\theta\right)\right]^2$$

since the policy in a given partition is given by $\frac{\theta_{k-1}+\theta_k}{2}$, we can express teh above expected utility as follows

$$EU_2 = -\sum_{k=1}^{N}\int_{\theta_{k-1}}^{\theta_k}\left[\frac{\theta_{k-1}+\theta_k}{2}-x\right]^2 dx$$

where we integrate over all values of θ in that partition, and then sum over all N partitions. This term coincides with term A above, which helps us simplify it in the same fashion, yielding

$$\begin{aligned} EU_2 &= -\frac{1}{12}\sum_{k=1}^{N}\left(\theta_k-\theta_{k-1}\right)^3 \\ &= -\frac{4}{3}\delta^3 N^2\left(N-1\right)^2 \end{aligned}$$

Therefore, we can conclude that the expected utilities of sender and receiver are related as follows $EU_1 = EU_2 - \delta^2$.

(c) How are the above *ex-ante* expected utilities affected by an increase in the preference divergence parameter δ?

- Differentiating the *ex-ante* expected utility of the sender with respect to δ, we find

$$\frac{\partial EU_1}{\partial\delta} = -4\delta^2 N^2\left(N-1\right)^2 - 2\delta < 0$$

so that the *ex-ante* expected utility of the sender decreases in δ.

- Differentiate the *ex-ante* expected utility of the receiver with respect to δ, we obtain

$$\frac{\partial EU_2}{\partial\delta} = -4\delta^2 N^2\left(N-1\right)^2 < 0$$

so that the *ex-ante* expected utility of the receiver decreases with δ. In summary, the ex-ante expected utilities of both players in a cheap talk game decrease as their preferences become more differentiated.

EconS 503 – Homework #6

Answer key

Exercise #1 – MWG 9C7

(See handout of Review Session #6)

Exercise #2

a)

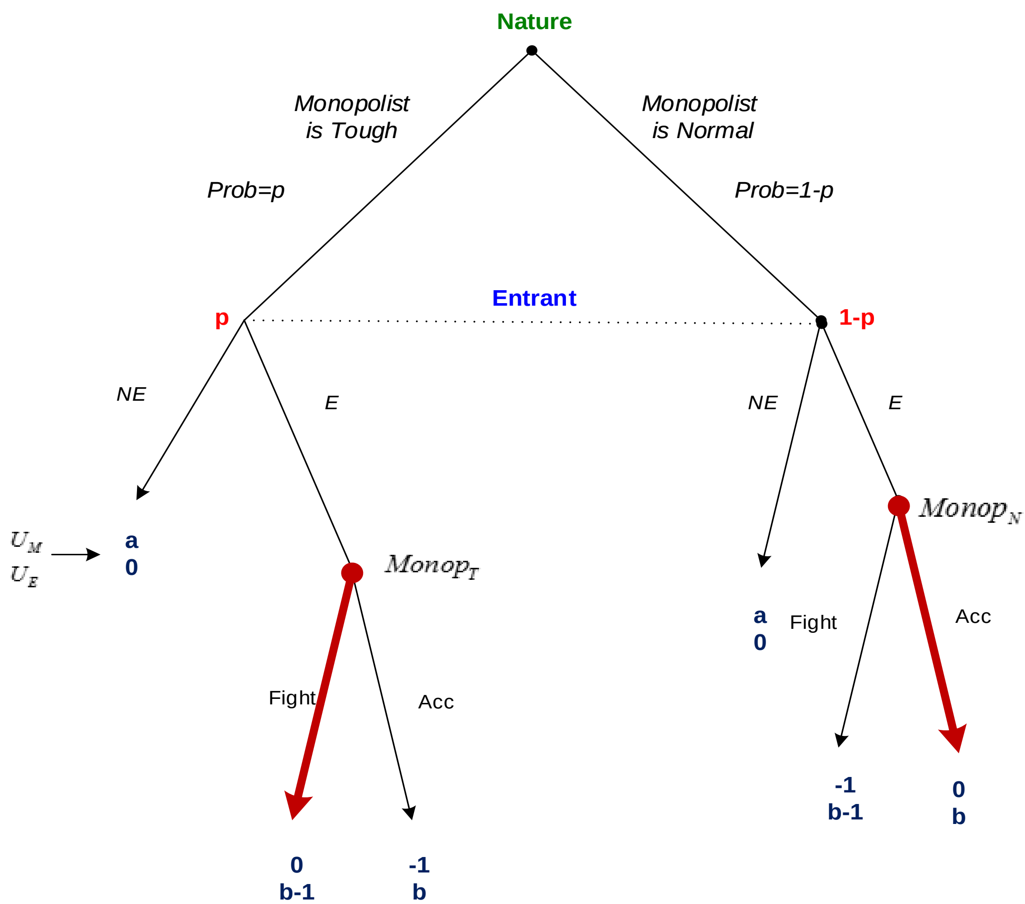


Figure 10.19. Entry deterrence with only one entrant

b) The tough monopolist fights with probability 1, since fight is a dominant strategy for him; while the normal monopolist accommodates with probability 1, since accommodation constitutes a dominant strategy for this type of incumbent. Indeed, at the node labeled with *Monop*$_T$ on the left-hand side of the game tree, the monopolist's payoff from fighting, 0, is larger than from accommodating, -1. In contrast, at the node labeled with $Monop_N$ for the normal monopolist, the monopolist's payoff from fighting, -1, is strictly lower than from accommodating, 0. Hence, the entrant's decision on whether or not to enter will be based on:

$$\mathrm{EU}_E(Enter|p) = p(b-1) + (1-p)b = b - p, \text{ and } \mathrm{EU}_E(Not\ Enter|p) = 0$$

Therefore, the entrant will decide to enter if and only if $b > p > 0$, or alternately, $b > p$. We can, hence, summarize the equilibrium as follows:

The entrant enters if $b > p$, but doesn't enter if $b \leq p$

The incumbent fights if tough, but accommodates if normal.

c) The following game tree depicts an entry game in which the incumbent faces two entrants in sequence

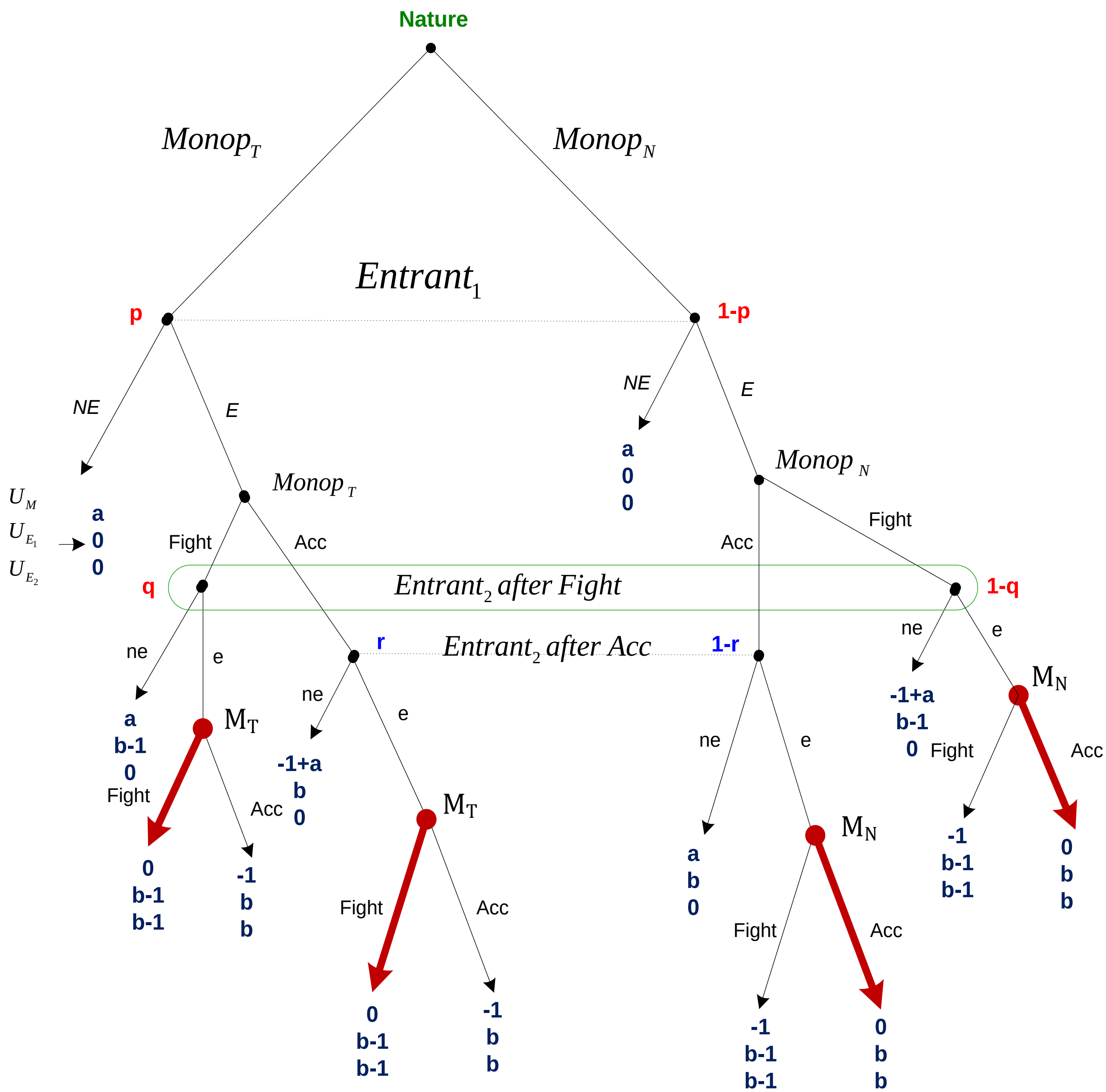


Figure 10.20. Entry deterrence with two entrants in sequence

Hence, applying backward induction on the proper subgames (those labeled with M_T and M_N in the last stages of the game tree) we can reduce the previous game tree to that in figure 10.21. For instance, after the second entrant chooses to enter despite observing a fight with the first entrant (left side of game tree), the tough monopolist chooses between fighting and accommodating in the node labeled M_T. In this setting, the tough monopolist prefers to fight, yielding a payoff of zero, rather than accommodate, which entails a payoff of -1. A similar analysis applies to the other node labeled with M_T (where the second entrant has entered after observing that the incumbent accommodated the first entrant). However, an opposite argument applies for the nodes marked with M_N in the right-hand side of the tree, where the normal monopolist prefers to accommodate the second entrant, regardless of whether he fought or accommodated the first entrant, since his payoff from accommodating (zero) is larger than from fighting (-1).

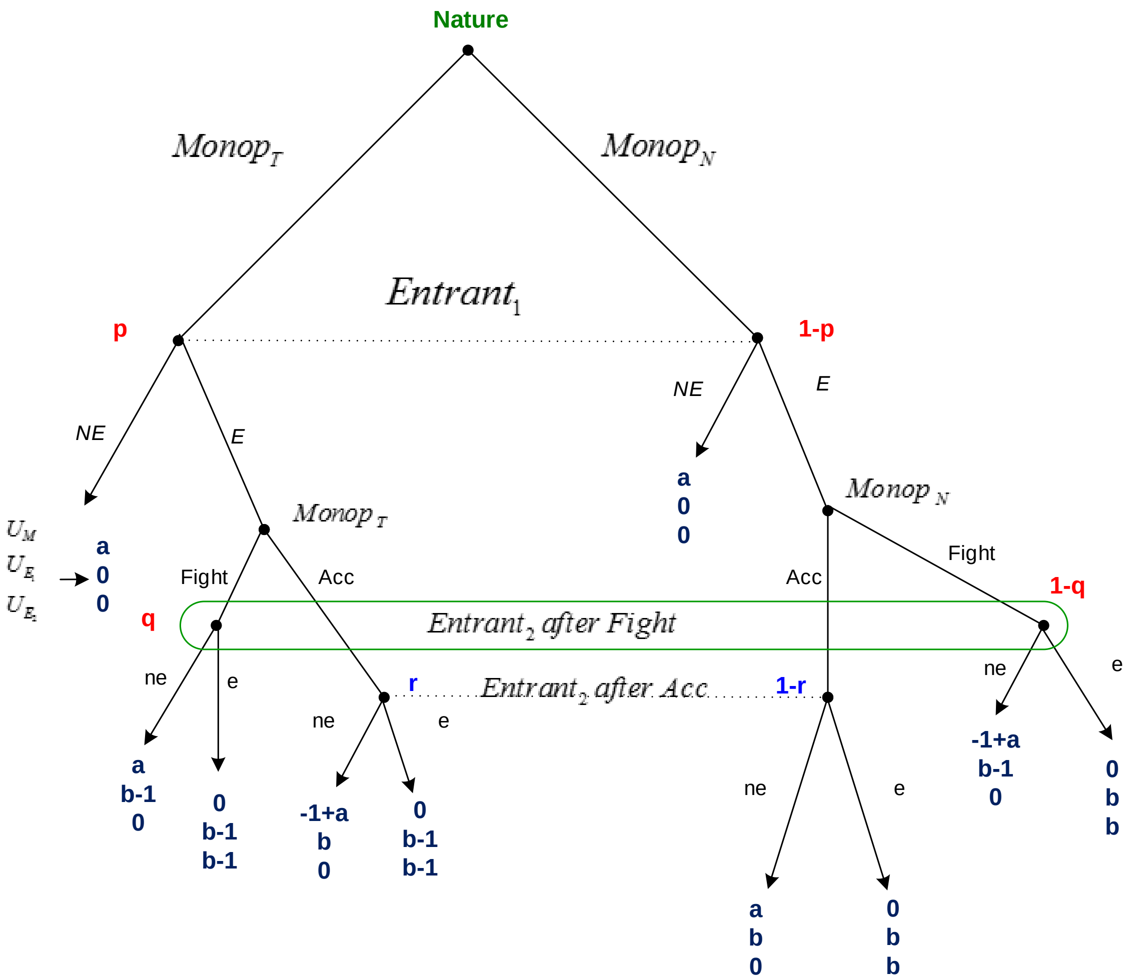


Figure 10.21. Reduced-form game

In addition, note that the first entrant behaves in exactly the same way as in exercise a): entering if and only if $b > p$. Hence, when $p \leq b$, the first entrant enters, as shown in exercise (a). Figure 10.22 shades this choice of the first entrant (green shaded branches).

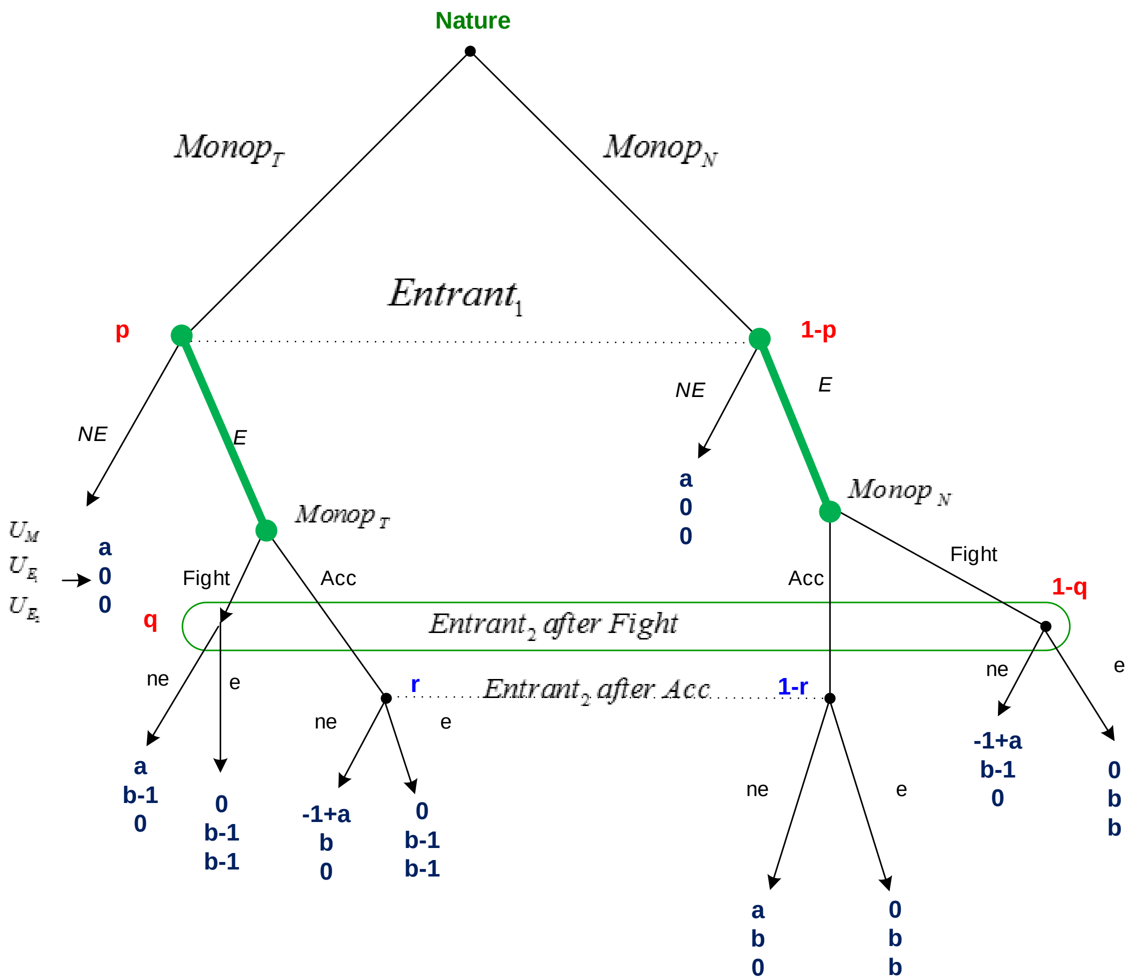


Figure 10.22. Entry of the first potential entrant

Therefore, upon entry, the first entrant gives rise to a beer-quiche type of signaling game, which can be more compactly represented as the game tree in figure 10.23. Intuitively, all elements before $Monop_T$ and $Monop_N$ can be predicted (i.e., the first entrant enters as long as $p \leq b$), while the subsequent stages characterize a signaling game between the monopolist (privately informed about its type) and the second entrant. In this context, the monopolist uses his decision to fight or accommodate the first entrant as a message to the second entrant, in order to convey or conceal his type.

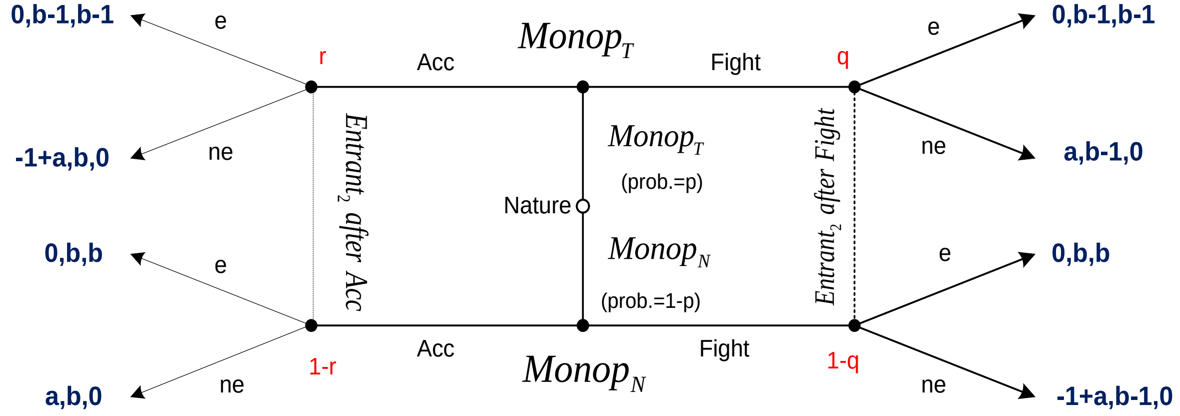


Figure 10.23. A further reduction of the game

(For every triplet of payoffs, the first corresponds to the monopolist, the second to the first entrant, and the third to the second entrant.) Let us now check if a pooling strategy profile in which both types of monopolists accommodate can be sustained as a PBE.

Pooling PBE with Acc. Figure 10.24 shades the braches corresponding to such a pooling strategy.

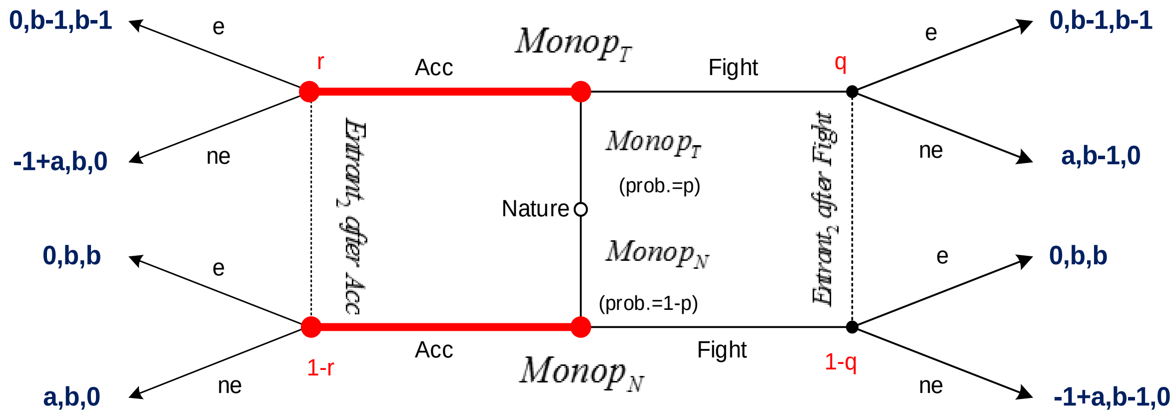


Figure 10.24. Pooling strategy profile - Acc

In this setting, posterior beliefs cannot be updated using Bayes' rule, which entails $r = p$. As in similar exercises, the observation of accommodation by the uninformed entrant does not allow him to further refine his beliefs about the monopolist's type. Hence, the second entrant responds entering (e) after observing that the monopolist accommodates (in equilibrium), since

$$p(b-1) + (1-p)b > p \cdot 0 + (1-p) \cdot 0 \leftrightarrow b > p$$

which holds in this case.

If, in contrast, the second entrant observes the off-the-equilibrium message of fight, then this player also enters if

$$q(b-1)+(1-q)b > q\cdot 0+(1-q)\cdot 0 \leftrightarrow b>q$$

Hence, the second entrant enters regardless of the incumbent's action if off-the-equilibrium beliefs, q, satisfy $q<b$, as depicted in figure 10.25 (see blue shaded branches). Otherwise, the entrant only enters after observing the equilibrium message of Acc.

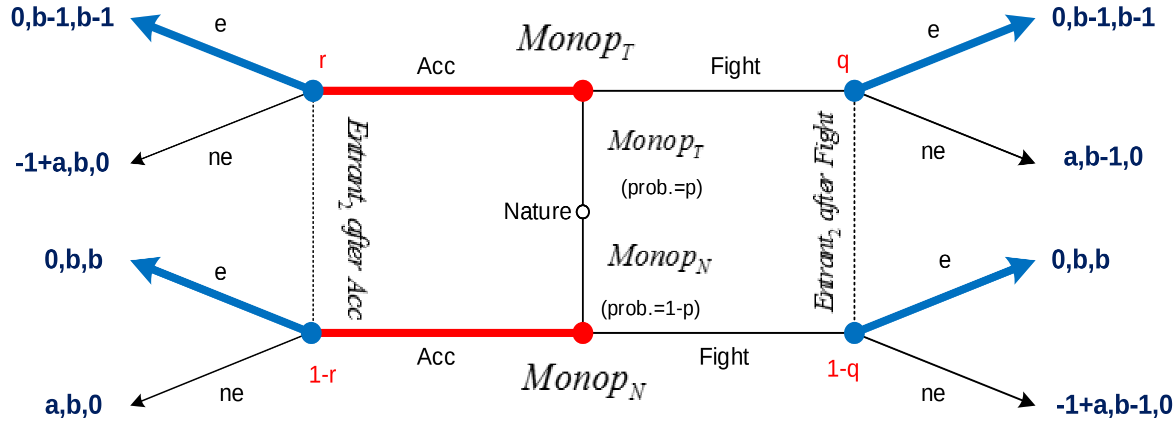


Figure 10.25. Pooling strategy profile – Acc (with responses)

The tough monopolist, M_T, is hence indifferent between Acc (as prescribed) which yields a payoff of 0, and deviate to Fight, which also yields a payoff of zero. A similar argument applies to the normal monopolist, M_N, in the lower part of the game tree. Hence, the pooling strategy profile where both types of incumbents accommodate can be sustained as a PBE.

A remark on the Intuitive Criterion. Let us next show that the above pooling equilibrium, despite constituting a PBE, violates the Cho and Kreps' (1987) Intuitive Criterion. In particular, the tough monopolist has incentives to deviate towards Fight if, by doing so, he is identified as a tough player, $q=1$, which induces the entrant to respond not entering. In this case, the tough monopolist obtains a payoff of a, which exceeds his equilibrium payoff of 0. In contrast, the normal monopolist doesn't have incentives to deviate since, even if his deviation to Fight deters entry, his payoff from doing so, -1+a, would still be lower than his equilibrium payoff of 0, given that -1+a<0 or a<1. Hence, only the tough monopolist has incentives to deviate, and the entrant's off-the-equilibrium beliefs can thus be restricted to q=1 upon observing that the monopolist fights. Intuitively, the entrant infers that the observation of Fight can only originate from the tough monopolist. In this case, the tough incumbent indeed prefers to select Fight, thus implying that the above pooling PBE violates Cho Kreps' (1983) Intuitive Criterion. □

Let us next check if this pooling strategy profile can be sustained when off-the-equilibrium beliefs satisfy, instead, $q\geq b$, thus inducing the entrant to respond not entering upon observing the off-the-equilibrium message of Fight, as illustrated in the game tree of figure 10.26 (see blue shaded branches in the right-hand side of the tree).

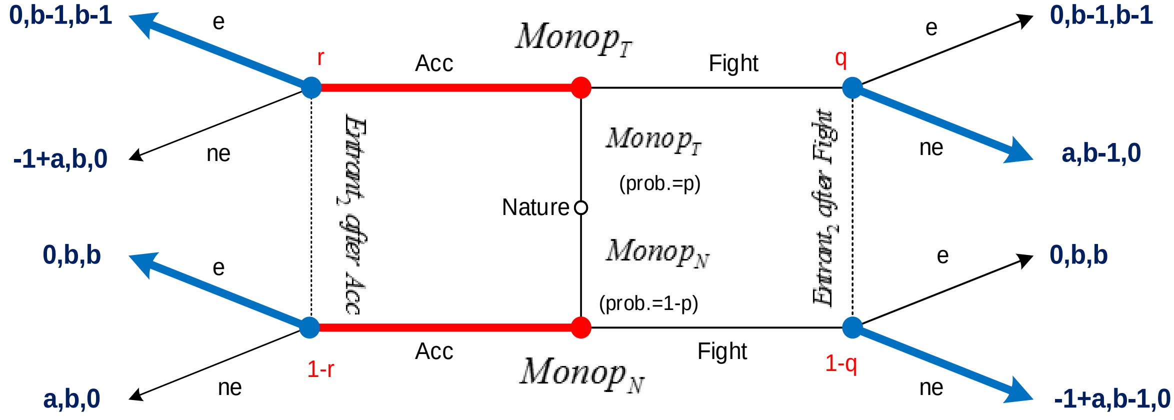


Figure 10.26. Pooling strategy profile – Acc (with responses)

In this case, the M_T has incentives to deviate from Acc, and thus the pooling strategy profile where both M_T and M_N select to Acc cannot be sustained as a PBE.

Pooling PBE with Fight. Let us now examine the opposite pooling strategy profile (Fight, Fight), in which both types of monopolists fight, as depicted in figure 10.27.

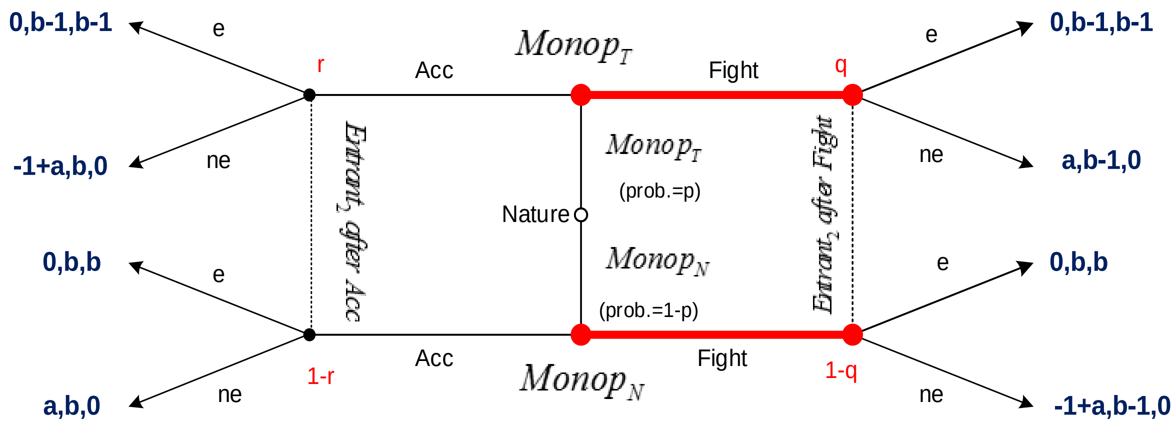


Figure 10.27. Pooling strategy profile - Fight

Hence, equilibrium beliefs after observing Fight cannot be updated, and thus satisfy q=p; while off-the-equilibrium beliefs are arbitrary $r \in [0,1]$ after observing the off-the-equilibrium message of accomodation. Given these beliefs, upon observing the equilibrium message of Fight, the entrant responds entering since

$$p(b-1) + p \cdot b > p \cdot 0 + (1-p) \cdot 0 \leftrightarrow b > p$$

which holds by definition. If, in contrast, the entrant observes the off-the-equilibrium message of Acc, then it responds entering if

$$r(b-1) + r \cdot b > p \cdot 0 + (1-r) \cdot 0 \leftrightarrow b > r$$

Hence, if b>r, the entrant enters both after observing Fight (in equilibrium) and Acc (off-the-equilibrium path). If, instead, $b \leq r$, then the entrant only responds entering after observing Fight, but is deterred otherwise. We next separately analyze each case. Figure 10.28 illustrates the entrant's responses when b>r.

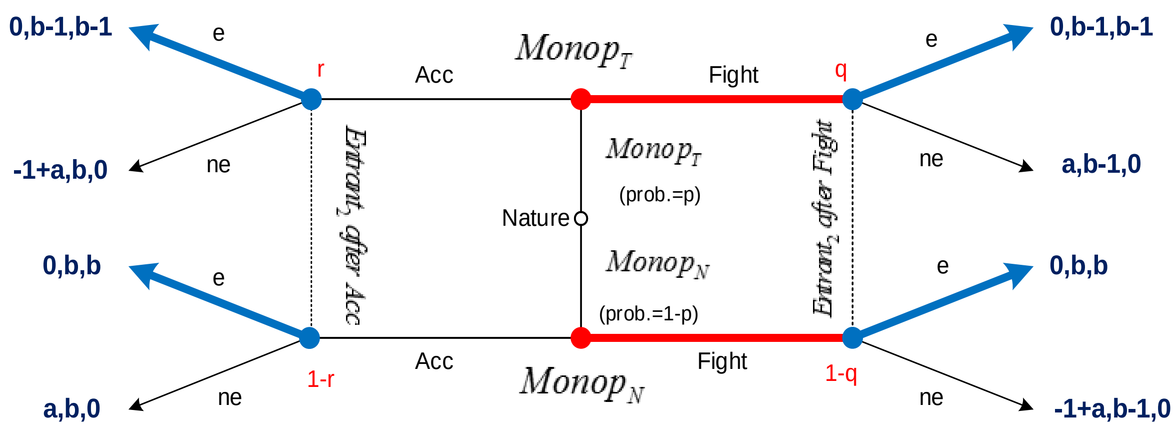


Figure 10.28. Pooling strategy profile – Fight (with responses)

In this context, the tough monopolist is indifferent between sticking to the equilibrium strategy of Fight, obtaining zero profits, or accommodating, which also yields zero profits. A similar argument applies to the normal monopolist in the lower part of the game tree. Hence, in this case the pooling strategy profile (Fight, Fight) can be supported as a PBE if off-the-equilibrium beliefs, r, satisfy r<b.

If, instead, $r \geq b$, then the entrant is deterred upon observing the off-the-equilibrium message of Acc, as figure 10.29 depicts.

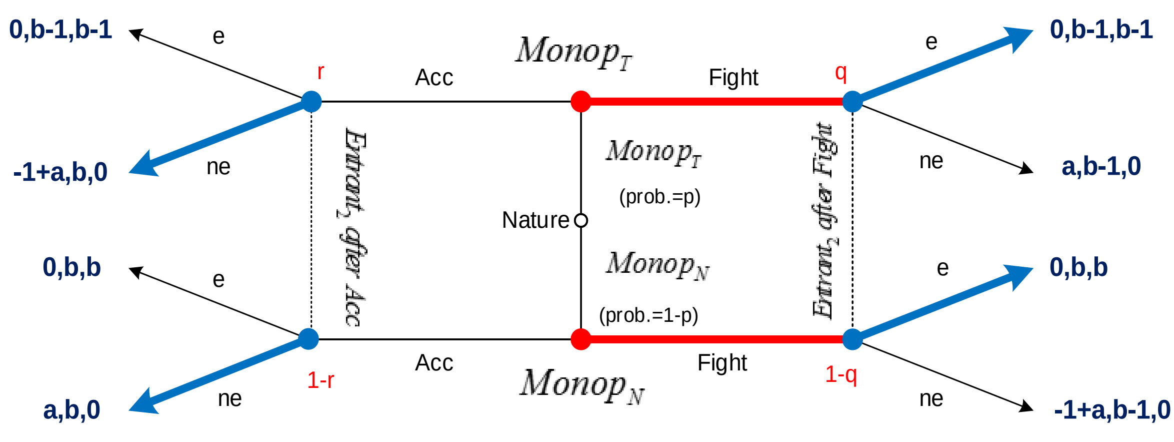


Figure 10.29 Pooling strategy profile – Fight (with responses)

The pooling strategy profile cannot be sustained in this context, since M_N has incentives to deviate towards Acc, obtaining a payoff of a, which exceeds its payoff of zero when he Fights. Hence, the pooling strategy profile (Fight, Fight) cannot be supported as a PBE when off-the-equilibrium beliefs satisfy $r \geq b$.

Separating PBE (Fight, Acc). Let us next examine if the separating strategy profile in which only the tough monopolist fights can be sustained as a PBE. Figure 10.30 depicts this strategy profile.

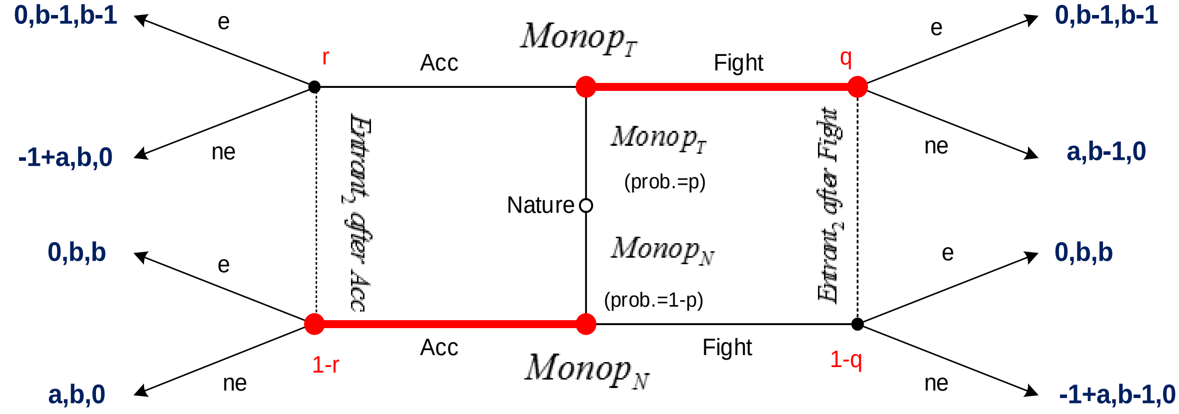


Figure 10.30. Separating strategy profile (Fight, Acc)

In this case, entrant's beliefs are updated to q=1 and r=0 using Bayes' rule, implying that, upon observing Fight, the entrant is deterred from the market since b-1<0, given that b<1 by definition. Upon observing Acc, the entrant is instead attracted to the market since b>0. Figure 10.31 illustrates the entrant's responses (see blue shaded arrows).

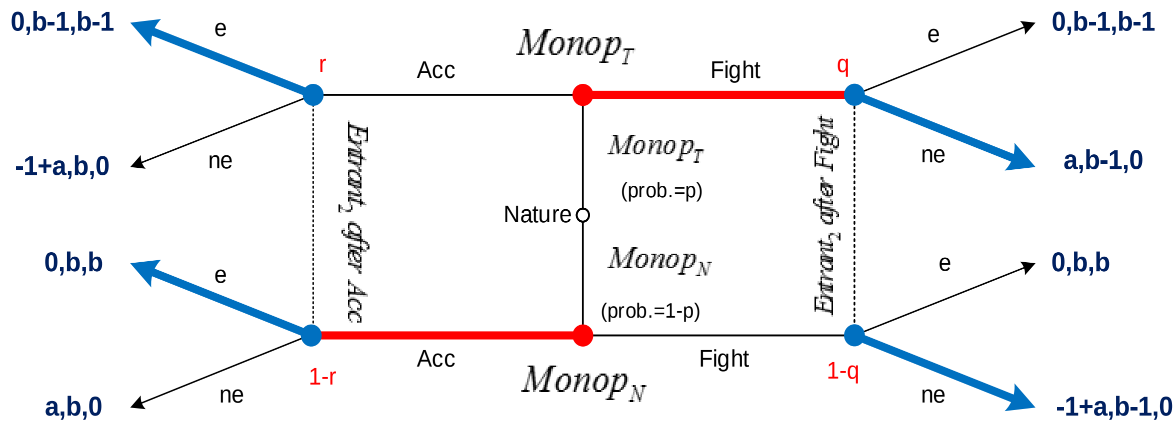


Figure 10.31. Separating strategy profile (Fight, Acc), with responses

In this setting, no type of monopolist has incentives to deviate: (1) the tough monopolist obtains a payoff of a by fighting (as prescribed) but only 0 from deviating towards Acc; and similarly (2) the normal monopolist obtains 0 by accommodating (as prescribed) but a negative payoff, -1+a, by deviating towards Fight, given that a<1 by definition. Hence, this separating strategy profile can be sustained as a PBE.

Separating PBE (Acc, Fight). Let us now check if the alternative separating strategy profile, in which only the normal monopolist fights, can be supported as a PBE (We know, this strategy sounds crazy, but we want to formally show that it cannot be sustained as a PBE.) Figure 10.32 illustrates this strategy profile.

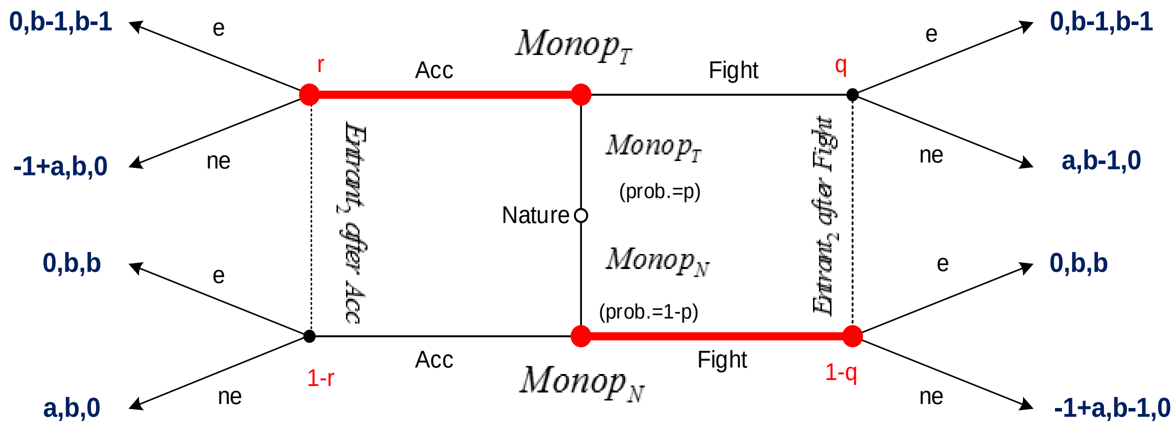


Figure 10.32. Separating strategy profile (Acc, Fight)

In this setting, the entrant's beliefs can be updated to r=1 and q=0, inducing the entrant to respond not entering after observing Acc, but entering after observing Fight, as depicted in figure 10.33 (see blue shaded branches).

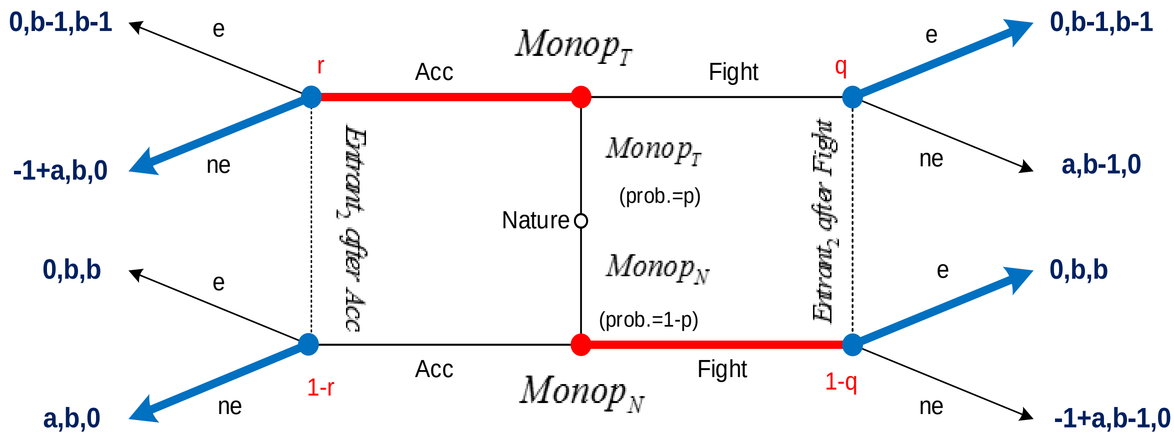


Figure 10.33. Separating strategy profile (Acc, Fight), with responses

Given these responses by the entrant, the tough monopolist has incentives to deviate from Acc, which yields a negative payoff of -1+a, to Fight, which yields a higher payoff of zero. Therefore, this separating strategy cannot be supported as a PBE.

Exercise #3

a) Standard figure.

b) At period 1, the buyer buys if $v - p_1 \geq 0$, or $v \geq p_1$. Hence, if $p_1 < 10$ the buyer buys regardless of his valuation; if p_1 lies between 10 and 30, he buys only when his valuation is v=30; and if p_1p1 is above 30 the buyer rejects regardless of his valuation. Hence, the seller expected profit is $\frac{1}{2}p_1$ since both types are equally likely. (In addition, if a first-period price of 30 is rejected the game proceeds to period 2, where the seller infers that the buyer's valuation must be lower than 30)

At period 2, let $\mu = Prob(v = 30|History\ at\ t = 2)$, and the buyer buys if his valuation v satisfies $v \geq p_2$

In addition, when beliefs satisfy

- $\mu > \frac{1}{2}$, the seller's optimal second-period price is $p_2 = \$30$,
- when $\mu < \frac{1}{2}$ the seller's optimal second-period price is $p_2 = \$10$, and
- when $\mu = \frac{1}{2}$ the seller is indifferent between setting a second-period price of $p_2 = \$10$ and $p_2 = \$30$.

We can now apply Bayes' rule to update the seller's beliefs about the seller's valuation upon observing that a price p_1 was rejected in the first period. In particular,

$$\mu(p_1) = \frac{\frac{1}{2}\alpha}{\frac{1}{2}\alpha + \left(1 - \frac{1}{2}\right)} = \frac{1}{2}$$

where α is the probability that the low-value buyer does not buy the product at a price p_1. Rearranging, we obtain

$$\frac{\alpha}{1 + \alpha} = \frac{1}{2}$$

And solving for probability α yields $\alpha = 1$, implying that in the second period the seller believes that the buyer who rejected price p_1 in the first-period game must be a high-value buyer with certainty.

Substituting for c and simplifying yields

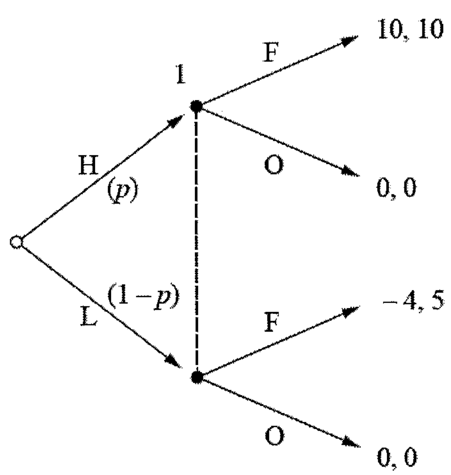
$$p_1 - p_1^2 \frac{4 - 3\delta}{(2-\delta)^2}.$$

Taking the derivative and solving the first-order condition for p_1 yields

$$p_1 = \frac{(2-\delta)^2}{8 - 6\delta}.$$

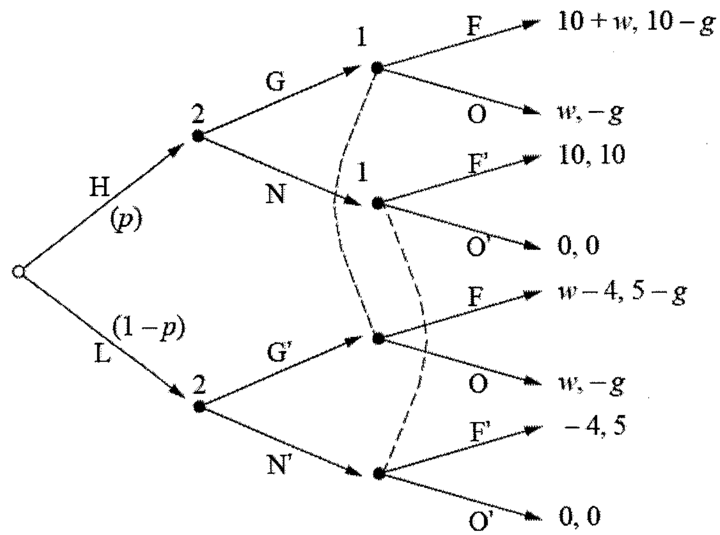
EXERCISE 7.

(a) The extensive form is:



In the Bayesian Nash equilibrium, player 1 forms a firm (F) if $10p - 4(1 - p) \geq 0$, which simplifies to $p \geq 2/7$. Player 1 does not form a firm (O) if $p < 2/7$.

(b) The extensive form is:



(c) Clearly, player 1 wants to choose F with the H type and O with the L type. Thus, there is a separating equilibrium if and only if the types of player 2 have the incentive to separate. This is the case if $10 - g \geq 0$ and $0 \geq 5 - g$, which simplifies to $g \in [5, 10]$.

(d) If $p \geq 2/7$, then there is a pooling equilibrium in which NN′ and F′ are played, player 1's belief conditional on no gift is p, player 1's belief conditional on a gift is arbitrary, and player 1's choice between F and O is optimal given this belief. If, in addition to $p \geq 2/7$, it is the case that $g \in [5, 10]$, then there is also a pooling equilibrium featuring GG′ and FO′. If $p \leq 2/7$, then there is a pooling equilibrium in which NN′ and OO′ are played (and player 1 puts a probability on H that is less than $2/7$ conditional on receiving a gift).

EXERCISE 8.

(a) A player is indifferent between O and F when he believes that the other player will choose O for sure. Thus, (O, O; O, O) is a Bayesian Nash equilibrium.

(b) If both types of the other player select Y, the H type prefers Y if $10p - 4(1-p) \geq 0$, which simplifies to $p \geq 2/7$. The L type weakly prefers Y, regardless of p. Thus, such an equilibrium exists if $p \geq 2/7$.

(c) If the other player behaves as specified, then the H type expects $-g + p(w + 10) + (1 - p)0$ from giving a gift. He expects pw from not giving a gift. Thus, he has the incentive to give a gift if $10p \geq g$. The L type expects $-g+p(9w+5)+(1-p)0$ if he gives a gift, whereas he expects pw if he does not give a gift. The L type prefers not to give if $g \geq 5p$. The equilibrium, therefore, exists if $g \in [5p, 10p]$.

EXERCISE 10.

(a) 1^H selects a_1^H to maximize $4a_1^H + 4a_2 - [a_1^H]^2$, which has a first-order condition of $4 - 2a_1^H \equiv 0$ implying $a_1^H = 2$.

Similarly, 1^L selects a_1^L to maximize $2a_1^L + 2a_2 - [a_1^L]^2$, which has a first-order condition of $2 - 2a_1^L \equiv 0$ implying $a_1^L = 1$.

Player 2 does not observe k and chooses a_2 to maximize $\frac{1}{2}[4a_1^H + 4a_2] + \frac{1}{2}[2a_1^H + 2a_2] - a_2^2$, which has a first-order condition of $2 + 1 - 2a_2 \equiv 0$ implying $a_2 = \frac{3}{2}$.

(b) There is an equilibrium in which both types of player 1 present evidence of their type. This requires that when no evidence is presented, player 2's belief is that $k = 4$. When player 1 shows her type to be H, both player 1 and 2 choose effort of 2, and when player 1 shows her type to be L, both players choose effort of 1. Following the out-of-equilibrium behavior of player 1 not disclosing evidence, player 2 chooses effort of $\frac{3}{2}$, and player 1 chooses effort of 2 when the state is H and 1 when the state is L.

There is also an equilibrium in which player 1 presents evidence in H and does not in L. Upon seeing no evidence presented, player 2 believes that $k = 4$. Player 1 chooses effort of 2 in H and 1 in L. Player 2 chooses effort of 2 when evidence of H is presented and chooses effort of 1 when either no evidence is presented or evidence of L is presented.

In both of these equilibria, player 2 knows the value of k from either direct evidence or from inferring that $k = 4$ due to player 1 not presenting evidence.

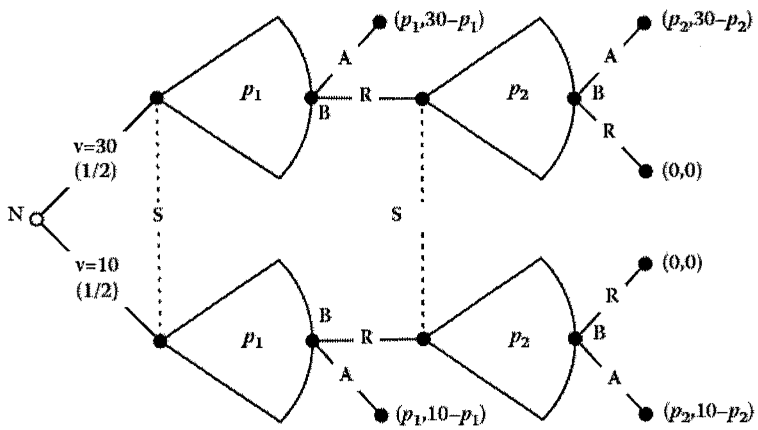
(c) After observing $k = 8$, player 1 would like for player 2 to know the value of k, but after observing $k = 4$, player 1 would like to not be able to convey the value of k.

We can also address this ex ante or prior to the realization of k as follows. When $k = 8$, player 1's payoff is $4[2 + \frac{3}{2}] - 4 = 10$, and when $k = 4$, player 1's payoff is $2[1 + \frac{3}{2}] - 1 = 4$. So player 1's expected payoff is 7. However, when k is known by player 2, player 1's payoffs are the following: when $k = 8$, $u_1 = 4[2 + 2] - 4 = 12$, and when $k = 4$, $u_1 = 2[1 + 1] - 1 = 3$. This yields an expected payoff for player 1 of $7.5 > 7$, so player 1 would prefer that player 2 know the value of k.

EXERCISE #3 [BARGAINING AND PBEs]

Answer:

(a)



(b) We work backwards from the end of the game. In period 2, the 10-buyer accepts if and only if ("iff") $p_2 \leq 10$, and the 30-type buyer accepts iff $p_2 \leq 30$. Given that the seller believes that the buyer is the 30-type with probability 0 in period 2, his sequentially rational response is $p_2 = 10$ so that the 10-buyer will accept.

Given that $p_2 = 10$, the 30-buyer will accept in period 1 iff $p_1 \leq 10$, since she could get a price of $p_2 = 10$ in period 2 by rejecting. For the second-period beliefs to be consistent, the 30-buyer must accept in period 1, so we have $p_1 \leq 10$. The 10-buyer also accepts in period 1 iff $p_1 \leq 10$, since she will not get a price lower than 10 in the second period. Note from the extensive form that the seller must believe that the two types of buyers are equally likely in period 1, and so his sequentially rational response to these beliefs (about both the buyer's type and the buyer's strategy) is to offer $p_1 = 10$. Then in equilibrium, both buyer types accept in period 1, allowing us to set period 2 beliefs arbitrarily.

Note that we can apply the same period 2 beliefs (that the buyer is the 30-type with probability 0) for all of the seller's period 2 information sets (there is one for each possible price in period 1). Since none of these are reached in equilibrium, this does not violate consistency. This completes the description of the equilibrium.

Chapter 3

Hidden Information, Signaling

3.1 Question 6

Consider a firm that can invest an amount I in a project generating high observable cash flow $C > 0$ with probability θ and 0 otherwise: $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H - \theta_L \equiv \Delta > 0$ and $\Pr[\theta = \theta_L] = \beta$. The firm needs to raise I from external investors who do not observe the value of θ. Assume that $\theta_L C - I > 0$. Everybody is risk neutral and there is no discounting.

1. Suppose that the firms can only promise to repay an amount R chosen by the firm (with $0 \leq R \leq C$) when cash flow is C and 0 otherwise. Can a good firm signal its type?

2. Suppose now that the firm also has the possibility of pledging some assets as collateral for the loan: Should a "default" occur (the firm being unable to repay R), an asset of value K to the firm is transferred to the creditor whose valuation is xK with $0 < x < 1$. The size of the collateral K is a choice variable. Give a necessary and sufficient condition for the "best" Perfect Bayesian Equilibrium to be separating. How does it depend on β and x? Explain.

3.1.1 No Collateral

Both firms would want to undergo the project since $\theta_L C > I$. A good firm cannot signal its type, since for a separating equilibrium to exist we need $R_H \neq R_L$. However, this cannot be an equilibrium. This can be seen clearly from the incentive compatibility condition for a firm of type i

$$\theta_i (C - R_i) \geq \theta_i (C - R_j).$$

Whenever $R_i \neq R_j$ at least one type of firm will want to deviate.

Intuitively speaking, since both firms receive C when the project is successful and 0 when it fails, and we only have one repayment instrument, the bad firm can perfectly mimic the good firm.

3.1.2 Collateral

Separating Equilibrium

The best separating equilibrium is clearly the one with the least amount of K so $K_L = 0$. The loss $(1-x)K$ is higher for a low-type firm since it has a higher probability of being in default.

A separating equilibrium can be supported by the following beliefs:

$$\begin{aligned} \Pr(\theta = \theta_L | K > K^*) &= 0 \\ \Pr(\theta = \theta_L | K \le K^*) &= 1 \end{aligned}$$

and we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}. \end{aligned}$$

This holds since otherwise the low type could offer a higher payment and still be better off–there is no benefit from a positive K.

Thus we shall have the following incentive compatibility and individual rationality constraints:

$$\begin{aligned} \theta_L R_L &= I && \text{(IRL)} \\ xK^H(1-\theta_H) + \theta_H R_H &= I && \text{(IRH)} \\ \theta_L(C - R_L) &\ge \theta_L(C - R_H) - (1-\theta_L)K_H && \text{(ICL)} \\ \theta_H(C - R_H) - (1-\theta_H)K_H &\ge \theta_H(C - R_L) && \text{(ICH)} \end{aligned}$$

Rewriting (ICL) we obtain

$$R_L - R_H \le \frac{1-\theta_L}{\theta_L} K_H.$$

Similarly, re-arranging (ICH) gives

$$R_L - R_H \ge \frac{1-\theta_H}{\theta_H} K_H.$$

Putting these expressions together we obtain

$$\frac{1-\theta_H}{\theta_H} K_H \le R_L - R_H \le \frac{1-\theta_L}{\theta_L} K_H.$$

This works even if x is very small. The intuition is that the high type benefits from a lower R more often and suffers from the loss of K less often, since

$\theta_H > \theta_L$. Thus the best separating equilibrium minimizes the use of (socially) wasteful collateral, that is K_H is set as low as possible. Hence, in equilibrium, only (ICL) is binding and (ICH) is slack. Thus, in what follows we can ignore (ICH). Solving the following equalities which we obtained from the constraints using the fact that (ICL) is binding in equilibrium and combining (IRL) and (ICL), we find

$$xK^H(1-\theta_H) + \theta_H R_H = I \tag{IRH}$$
$$K_H(1-\theta_L) + \theta_L R_H = I. \tag{IRL, ICL}$$

Rewriting these conditions yields

$$\begin{aligned} R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} x K_H \\ R_H &= \frac{I}{\theta_L} - \frac{1-\theta_L}{\theta_L} K_H, \end{aligned}$$

and after some algebraic manipulation we obtain

$$\begin{aligned} K_H &= \frac{\Delta I}{\theta_H\left(1-\theta_L\right) - x\theta_L\left(1-\theta_H\right)} \\ R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} \frac{x\Delta I}{\theta_H\left(1-\theta_L\right) - x\theta_L\left(1-\theta_H\right)}. \end{aligned}$$

From above, we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}, \end{aligned}$$

since this is the best, or the least cost equilibrium. K is costly and hence there is no reason to use it in the low state. Notice that this separating equilibrium always exists.

Pooling Equilibrium

We can compare this to the best pooling equilibrium, where

$$\begin{aligned} K^P &= 0 \\ R^P &= \frac{I}{\theta_H - \beta\Delta}. \end{aligned}$$

However, this pooling equilibrium may not exist. It will exist if and only if

$$\theta_H\left(C - R^P\right) \geq \theta_H\left(C - R\right) - \left(1-\theta_H\right)K$$

where

$$\begin{aligned} R &= \frac{I - \left(1-\theta_L\right)xK}{\theta_L} \\ I &= \theta_L R + (1-\theta_L)xK, \end{aligned}$$

which follows from the assumption that for any deviation from $K^P = 0$ the investors will believe that the firm is of low type. So the pooling equilibrium is sustainable if there are no deviations given these beliefs. This is the worst belief in the sense that if we cannot find a pooling equilibrium supported by these beliefs then no pooling equilibrium exists (there will always be a profitable deviation from it). Combining the equations, we obtain

$$\theta_H \left(C - \frac{I}{\theta_H - \beta\Delta} \right) \geq \theta_H \left(C - \frac{I - (1-\theta_L)\, xK}{\theta_L} \right) - (1-\theta_H)\, K,$$

which is equivalent to

$$x \leq (\theta_H I(\frac{1}{\theta_L} - \frac{1}{\beta\theta_L + (1-\beta)\theta_H}) + (1-\theta_H)K)\frac{\theta_L}{\theta_H(1-\theta_L)K}.$$

Thus, the smaller x or β the more likely is the existence of a pooling equilibria. This is intuitive. A smaller x means that the signal is more costly, and hence a profitable deviation from the least-cost pooling equilibrium that has no costly collateral, is very difficult. Similarly, with a smaller β the less likely it is that the firm is a bad type (so a smaller cross-subsidy is needed).

Comparison

One way to compare the different equilibria would be to compare ex-ante expected profits of the firm for the separating and pooling equilibrium (compare section 3.1.1). The expected profits for the separating equilibrium are given by

$$\begin{aligned} \pi^S &= (1-\beta)\, \pi_H^S + \beta\pi_L^S \\ &= (1-\beta)\, [\theta_H(C - R_H) - (1-\theta_H)K_H] + \beta\theta_L(C - R_L) \\ &= C\, [\theta_L + (1-\beta)\, \Delta] - I - (1-\beta)\, (1-\theta_H)(1-x)K_H. \end{aligned}$$

where K_H is as defined above.

In contrast, expected profits for the pooling equilibrium are

$$\begin{aligned} \pi^P &= (1-\beta)\, \pi_H^P + \beta\pi_L^P \\ &= (1-\beta)\, \theta_H(C - R^P) + \beta\theta_L(C - R^P) \\ &= [\theta_L + (1-\beta)\, \Delta]\, (C - R^P) \\ &= [\theta_L + (1-\beta)\, \Delta]\, C - I. \end{aligned}$$

Hence, we have

$$\pi^P > \pi^S,$$

whenever the pooling equilibrium exists. The pooling equilibrium leads to higher profits as it avoids the use of (wasteful) collateral. Thus, the best perfect Bayesian equilibrium (when defined in this way) is separating if and only if the pooling equilibrium does not exist. Another way would be look for one equilibria Pareto-dominating the other. Here we would see when both types prefer the pooling equilibrium. This happens when β is very small and thus the tiny gain from signalling (avoiding the infinitesimal cross subsidy) is smaller than the costly signal. See section 3.1.1 for details.

Cheap talk with three types. Consider the cheap talk model with three types discussed in class (Investing recommendations game). Let us focus on the partially separating strategy profile where the Analyst (sender) recommends Buy both when the stock outperforms the market and when it's neutral, but recommends Hold when the stock underperforms the market. In class, we made a simplifying assumption on off-the-equilibrium beliefs (after the Investor receives a Sell recommendation), denoted by γ_1, γ_2, and $1 - \gamma_1 - \gamma_2$.

(a) Without restricting off-the-equilibrium beliefs, find under which conditions the above partially separating strategy profile can be sustained as a PBE of this game.

(b) Consider now the pooling strategy profile where the Analyst recommends Buy regardless of the stock's type. Under which conditions can this strategy profile be supported as a PBE?

Answer:

Part (a).

As depicted in Figure 1, when the Investor receives on-the-equilibrium message to Buy, he assesses that the stock outperforms the market or is neutral with equal probability, such that he chooses to Buy that yields him an expected payoff of ½ (as depicted by the blue arrows, and also check that the expected payoff is ½ or –½ when he chooses to Hold or Sell, respectively). On the other hand, when the Investor receives on-the-equilibrium message to Hold, he assigns full probability to the stock underperforming the market, such that he chooses to Sell that yields him a payoff of 1 (as depicted by the green arrows, and also check that the payoff is –1 or 0 when he chooses to Buy or Hold, respectively).

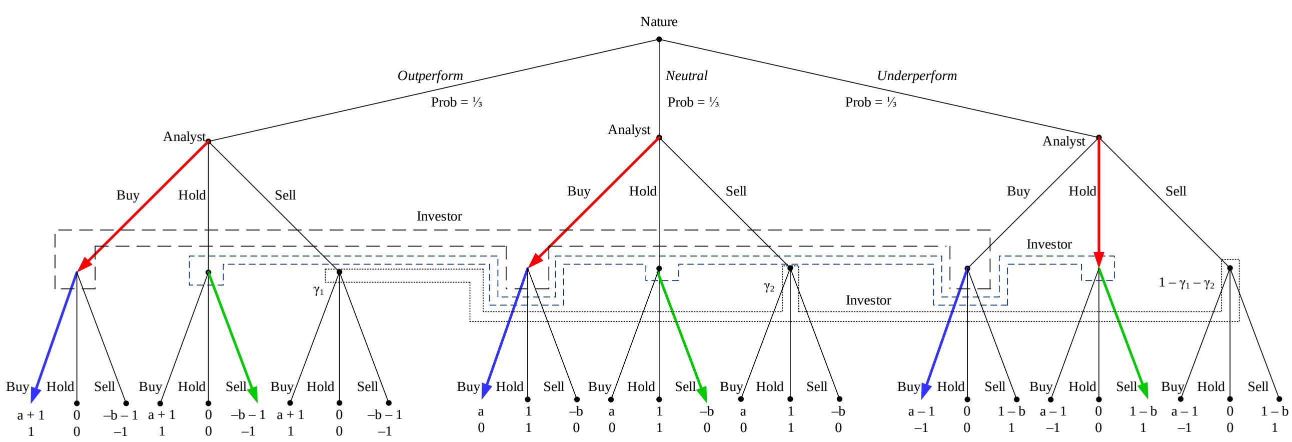


Figure 1: Analyst's recommendations to Buy or Hold and Investor's off-the-equilibrium beliefs on Sell

Lastly, the Analyst never recommends Buy, so that upon receiving an off-the-equilibrium message to Sell, the Investor assigns the following beliefs to the stock's performance:

$$\text{Prob(Outperform|Sell)} = \gamma_1$$

$$\text{Prob(Neutral|Sell)} = \gamma_2$$

$$\text{Prob(Underperform|Sell)} = 1 - \gamma_1 - \gamma_2$$

Having received an off-the-equilibrium message to Sell, the expected payoffs to the Investor are

$$E[\pi_2(\text{Sell|Buy})] = \gamma_1 \times 1 + \gamma_2 \times 0 + (1 - \gamma_1 - \gamma_2) \times (-1) = 2\gamma_1 + \gamma_2 - 1$$

$$E[\pi_2(\text{Sell|Hold})] = \gamma_1 \times 0 + \gamma_2 \times 1 + (1 - \gamma_1 - \gamma_2) \times 0 = \gamma_2$$

$$E[\pi_2(\text{Sell|Sell})] = \gamma_1 \times (-1) + \gamma_2 \times 0 + (1 - \gamma_1 - \gamma_2) \times 1 = -2\gamma_1 - \gamma_2 + 1$$

Then, the Investor prefers to Buy than Hold if $E[\pi_2(\text{Sell|Buy})] > E[\pi_2(\text{Sell|Hold})]$, that is,

$$2\gamma_1 + \gamma_2 - 1 > \gamma_2$$

$$\Rightarrow \gamma_1 > \frac{1}{2}$$

Next, the Investor prefers to Hold than Sell if $E[\pi_2(\text{Sell|Hold})] > E[\pi_2(\text{Sell|Sell})]$, that is,

$$\gamma_2 > -2\gamma_1 - \gamma_2 + 1$$

$$\Rightarrow \gamma_1 + \gamma_2 > \frac{1}{2}$$

Also, the Investor prefers to Buy than Sell if $E[\pi_2(\text{Sell}|\text{Buy})] > E[\pi_2(\text{Sell}|\text{Sell})]$, that is,

$$2\gamma_1 + \gamma_2 - 1 > -2\gamma_1 - \gamma_2 + 1$$
$$\Rightarrow 2\gamma_1 + \gamma_2 > 1$$

Lastly, check that $0 \leq 1 - \gamma_1 - \gamma_2 \leq 1$, which implies that $\gamma_1 + \gamma_2 \leq 1$.

Figure 2 depicts the Investor's best responses given the range of off-the-equilibrium beliefs. Intuitively, when the Investor believes that probability of the stock underperforming the market is high, that is, $\gamma_1 + \gamma_2 \leq \frac{1}{2}$, then the Investor would Sell the stock. Otherwise, if the Investor believes that that the probability of the stock overperforming the market is high, in particular, $\gamma_1 > \frac{1}{2}$, then the Investor would Buy the stock. In between, if the investor believes that the stock is more likely to be neutral, where $\gamma_1 + \gamma_2 > \frac{1}{2}$ and $\gamma_1 \leq \frac{1}{2}$, then the Investor would Hold the stock.

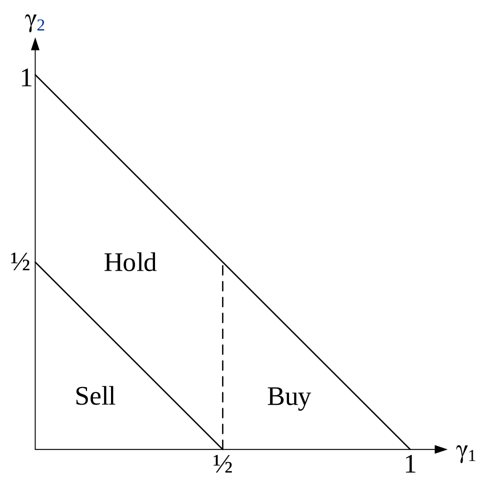


Figure 2: Investor's decisions given his off-the-equilibrium beliefs when recommended to Sell

Next, we consider the Analyst's recommendations.

- When the stock market outperforms, the Analyst's payoffs are

$$E[\pi_1(\text{Outperform}|\text{Buy})] = a + 1$$
$$E[\pi_1(\text{Outperform}|\text{Hold})] = -b - 1$$
$$E[\pi_1(\text{Outperform}|\text{Sell})] = \begin{cases} a+1 & if \quad \gamma_1 > \frac{1}{2} \\ 0 & if \quad \gamma_1 + \gamma_2 > \frac{1}{2} \; and \; \gamma_1 \leq \frac{1}{2} \\ -b-1 & if \quad \gamma_1 + \gamma_2 \leq \frac{1}{2} \end{cases}$$

  Therefore, for the Analyst to recommend Buy when the stock outperforms, we need

$$E[\pi_1(\text{Outperform}|\text{Buy})] \geq E[\pi_1(\text{Outperform}|\text{Hold})]$$
$$E[\pi_1(\text{Outperform}|\text{Buy})] \geq E[\pi_1(\text{Outperform}|\text{Sell})]$$

  which imply that

$$a + 1 \geq -b - 1$$
$$a + 1 \geq 0$$

  The first inequality simplifies to $a + b \geq -2$, which holds because $a, b \geq 0$ by definition. The second inequality simplifies to $a \geq -1$, which also holds for all non-negative values of a.

- When the stock market is neutral, the Analyst's payoffs are

$$E[\pi_1(\text{Neutral}|\text{Buy})] = a$$
$$E[\pi_1(\text{Neutral}|\text{Hold})] = -b$$

$$E[\pi_1(\text{Neutral}|\text{Sell})] = \begin{cases} a & if \quad \gamma_1 > \frac{1}{2} \\ 1 & if \quad \gamma_1 + \gamma_2 > \frac{1}{2} \; and \; \gamma_1 \leq \frac{1}{2} \\ -b & if \quad \gamma_1 + \gamma_2 \leq \frac{1}{2} \end{cases}$$

Therefore, for the Analyst to recommend Buy when the stock is neutral, we need

$$E[\pi_1(\text{Neutral}|\text{Buy})] \geq E[\pi_1(\text{Neutral}|\text{Hold})]$$

$$E[\pi_1(\text{Neutral}|\text{Buy})] \geq E[\pi_1(\text{Neutral}|\text{Sell})]$$

which imply that

$$a \geq -b$$

$$a \geq 1$$

The first inequality holds for all non-negative values of a and b. Whereas, when the Investor's off-the-equilibrium beliefs satisfy $\gamma_1 + \gamma_2 > \frac{1}{2}$ and $\gamma_1 \leq \frac{1}{2}$, we need $a \geq 1$ for the Analyst to recommend Buy (such that the Investor buys) than to recommend Sell (such that the Investor holds).

- When the stock market underperforms, the Analyst's payoffs are

$$E[\pi_1(\text{Underperform}|\text{Buy})] = a - 1$$

$$E[\pi_1(\text{Underperform}|\text{Hold})] = 1 - b$$

$$E[\pi_1(\text{Underperform}|\text{Sell})] = \begin{cases} a - 1 & if \quad \gamma_1 > \frac{1}{2} \\ 0 & if \quad \gamma_1 + \gamma_2 > \frac{1}{2} \; and \; \gamma_1 \leq \frac{1}{2} \\ 1 - b & if \quad \gamma_1 + \gamma_2 \leq \frac{1}{2} \end{cases}$$

Therefore, for the Analyst to recommend Hold when the stock underperforms, we need

$$E[\pi_1(\text{Underperform}|\text{Hold})] \geq E[\pi_1(\text{Underperform}|\text{Buy})]$$

$$E[\pi_1(\text{Underperform}|\text{Hold})] \geq E[\pi_1(\text{Underperform}|\text{Sell})]$$

which imply that

$$1 - b \geq a - 1$$

$$1 - b \geq 0$$

The first inequality becomes $a + b \leq 2$. Wheras, the second inequality becomes $b \leq 1$ when the Investor's off-the-equilibrium beliefs satisfy $\gamma_1 + \gamma_2 > \frac{1}{2}$ and $\gamma_1 \leq \frac{1}{2}$ for the Analyst to recommend Hold (such that the Investor sells) than to recommend Sell (such that the Investor holds).

- Combining the above, we need $a + b \leq 2$ for any off-the-equilibrium beliefs of the Investor. Specifically, when the Investor's off-the-equilibrium beliefs satisfy $\gamma_1 + \gamma_2 > \frac{1}{2}$ and $\gamma_1 \leq \frac{1}{2}$, we further need $a \geq 1$ and $b \leq 1$. Figure 3 highlights the ranges of a and b so that the Analyst's and Investor's strategies, {{Buy, Buy, Hold}, {Buy, Buy, Sell}}, can be supported as a semi-separating Perfect Bayesian equilibrium when the stock overperforms, is neutral, and underperforms respectively. Intuitively, the strategy profile can be sustained if the penalty (parameter *b*) is small relative to the bonus (*a*). A special case includes that in which the analyst does not suffer a penalty when the buyer sells (*b*=0) but receives a large bonus when the buyer buys stock (*a*>1); as depicted along the horizontal axis of Figure 3.

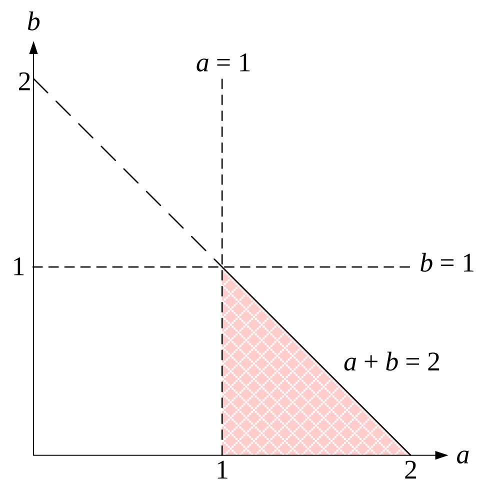


Figure 3: Ranges of a and b that support the semi-separating PBE

Part (b).

Having received a recommendation to Buy, the Investor cannot update his belief about the stock's performance, so that his belief coincides with nature's assignment of the stock's performance. Specifically,

$$\text{Prob}(\text{Outperform}|\text{Buy}) = \frac{1}{3}$$

$$\text{Prob}(\text{Neutral}|\text{Buy}) = \frac{1}{3}$$

$$\text{Prob}(\text{Underperform}|\text{Buy}) = \frac{1}{3}$$

yielding expected payoffs to the Investor as follows:

$$E[\pi_2(\text{Buy}|\text{Buy})] = \frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times (-1) = 0$$

$$E[\pi_2(\text{Buy}|\text{Hold})] = \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times 0 = \frac{1}{3}$$

$$E[\pi_2(\text{Buy}|\text{Sell})] = \frac{1}{3} \times (-1) + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$$

In Figure 4, when the Analyst always recommends to Buy (as depicted by the red arrows), the best response of the Investor is to Hold (as depicted by the blue arrows), which yields him an expected payoff of ⅓, than to Buy or Sell, both of which yield him an expected payoff of 0.

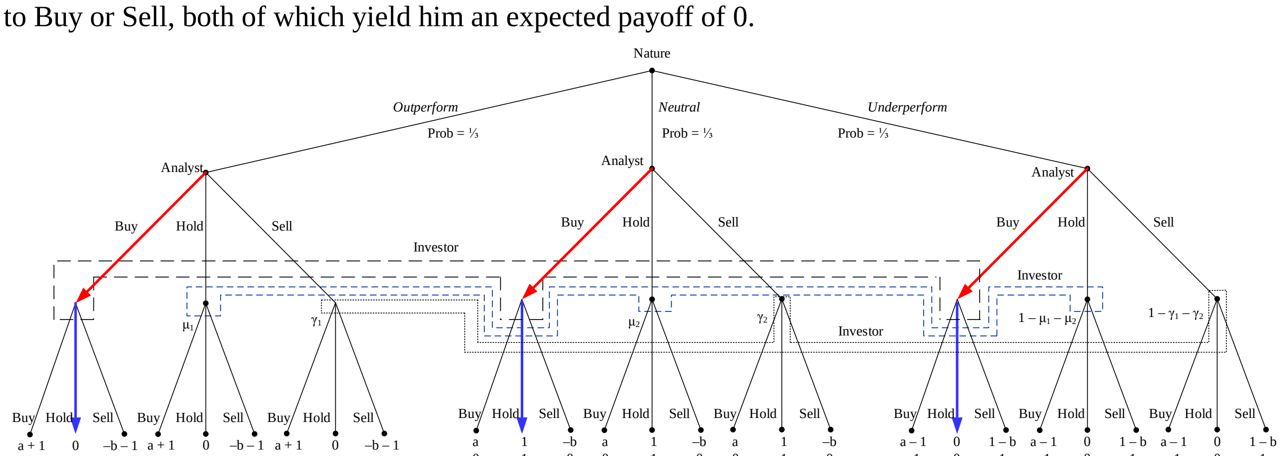


Figure 4: Analyst's recommendations to Buy and Investor's off-the-equilibrium beliefs on Hold and Sell

In this context, both messages of Hold and Sell are along off-the-equilibrium paths, so that the Investor assigns off-the-equilibrium beliefs to the stock's performance when Hold as follows:

$$\text{Prob}(\text{Outperform}|\text{Hold}) = \mu_1$$

$$\text{Prob}(\text{Neutral}|\text{Hold}) = \mu_2$$

$$\text{Prob}(\text{Underperform}|\text{Hold}) = 1 - \mu_1 - \mu_2$$

while the beliefs assignment to the stock's performance when Sell follows part (a).

Having received an off-the-equilibrium message to Hold, the expected payoffs to the Investor are

$$E[\pi_2(\text{Hold}|\text{Buy})] = \mu_1 \times 1 + \mu_2 \times 0 + (1 - \mu_1 - \mu_2) \times (-1) = 2\mu_1 + \mu_2 - 1$$

$$E[\pi_2(\text{Hold}|\text{Hold})] = \mu_1 \times 0 + \mu_2 \times 1 + (1 - \mu_1 - \mu_2) \times 0 = \mu_2$$

$$E[\pi_2(\text{Hold}|\text{Sell})] = \mu_1 \times (-1) + \mu_2 \times 0 + (1 - \mu_1 - \mu_2) \times 1 = -2\mu_1 - \mu_2 + 1$$

Then, the Investor prefers to Buy than Hold if $E[\pi_2(\text{Hold}|\text{Buy})] > E[\pi_2(\text{Hold}|\text{Hold})]$, that is,

$$2\mu_1 + \mu_2 - 1 > \mu_2$$

$$\Rightarrow \mu_1 > \frac{1}{2}$$

Next, the Investor prefers to Hold than Sell if $E[\pi_2(\text{Hold}|\text{Hold})] > E[\pi_2(\text{Hold}|\text{Sell})]$, that is,

$$\mu_2 > -2\mu_1 - \mu_2 + 1$$

$$\Rightarrow \mu_1 + \mu_2 > \frac{1}{2}$$

Also, the Investor prefers to Buy than Sell if $E[\pi_2(\text{Hold}|\text{Buy})] > E[\pi_2(\text{Hold}|\text{Sell})]$, that is,

$$2\mu_1 + \mu_2 - 1 > -2\mu_1 - \mu_2 + 1$$

$$\Rightarrow 2\mu_1 + \mu_2 > 1$$

Lastly, check that $0 \leq 1 - \mu_1 - \mu_2 \leq 1$, which implies that $\mu_1 + \mu_2 \leq 1$.

Figure 5 depicts the Investor's best responses given the range of off-the-equilibrium beliefs, which interpretation is the same as that for the off-the-equilibrium message to Sell (as in part (a) on Figure 2), except that the beliefs are now on the Analyst recommending him to Hold instead of to Sell.

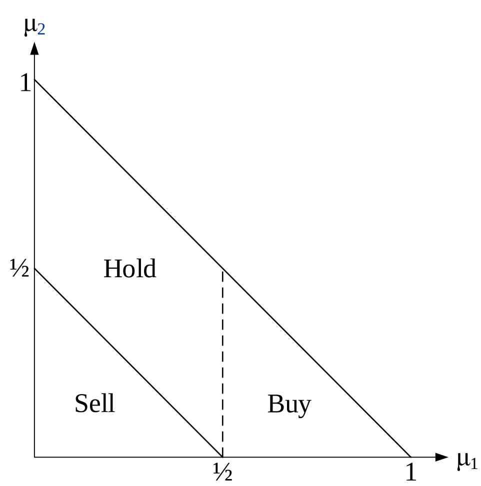


Figure 5: Investor's decisions given his off-the-equilibrium beliefs when recommended to Hold

Next, we consider the Analyst's recommendations.

- When the stock market outperforms, the Analyst's payoffs are

$$E[\pi_1(\text{Outperform}|\text{Buy})] = 0$$

$$E[\pi_1(\text{Outperform}|\text{Hold})] = \begin{cases} a+1 & if \quad \mu_1 > \frac{1}{2} \\ 0 & if \quad \mu_1 + \mu_2 > \frac{1}{2} \; and \; \mu_1 \leq \frac{1}{2} \\ -b-1 & if \quad \mu_1 + \mu_2 \leq \frac{1}{2} \end{cases}$$

$$E[\pi_1(\text{Outperform}|\text{Sell})] = \begin{cases} a+1 & if \quad \gamma_1 > \frac{1}{2} \\ 0 & if \quad \gamma_1 + \gamma_2 > \frac{1}{2} \; and \; \gamma_1 \leq \frac{1}{2} \\ -b-1 & if \quad \gamma_1 + \gamma_2 \leq \frac{1}{2} \end{cases}$$

Therefore, for the Analyst to recommend Buy when the stock outperforms, we need

$$\mathrm{E}[\pi_1(\text{Outperform}|\text{Hold})] \geq \mathrm{E}[\pi_1(\text{Outperform}|\text{Buy})]$$
$$\mathrm{E}[\pi_1(\text{Outperform}|\text{Hold})] \geq \mathrm{E}[\pi_1(\text{Outperform}|\text{Sell})]$$

which imply that

$$0 \geq a + 1$$
$$0 \geq -b - 1$$

Specifically, when the Investor's off-the-equilibrium beliefs satisfy $\mu_1 > \frac{1}{2}$ ($\gamma_1 > \frac{1}{2}$), he buys the stock upon receiving a recommendation to Hold (Sell). Therefore, the Analyst has an incentive to deviate from recommending Buy, which yields him a payoff of 0, to Hold (Sell), which yields him a payoff of $a + 1$ because the inequality $a \leq -1$ cannot be satisfied for all no non-negative values of a.

Whereas, the Investor's off-the-equilibrium beliefs satisfy $\mu_1 + \mu_2 \leq \frac{1}{2}$ ($\gamma_1 + \gamma_2 \leq \frac{1}{2}$), he sells the stock upon receiving a recommendation to Hold (Sell). Therefore, the Analyst has no incentive to deviate from recommending Buy, which yields him a payoff of 0, to Hold (Sell), which yields him a payoff of $-b - 1$, because the inequality $b \geq -1$ is satisfied for all non-negative values of b.

- When the stock market is neutral, the Analyst's payoffs are

$$\mathrm{E}[\pi_1(\text{Neutral}|\text{Buy})] = 1$$

$$\mathrm{E}[\pi_1(\text{Neutral}|\text{Hold})] = \begin{cases} a & if \quad \mu_1 > \dfrac{1}{2} \\ 1 & if \quad \mu_1 + \mu_2 > \dfrac{1}{2}\ and\ \mu_1 \leq \dfrac{1}{2} \\ -b & if \quad \mu_1 + \mu_2 \leq \dfrac{1}{2} \end{cases}$$

$$\mathrm{E}[\pi_1(\text{Neutral}|\text{Sell})] = \begin{cases} a & if \quad \gamma_1 > \dfrac{1}{2} \\ 1 & if \quad \gamma_1 + \gamma_2 > \dfrac{1}{2}\ and\ \gamma_1 \leq \dfrac{1}{2} \\ -b & if \quad \gamma_1 + \gamma_2 \leq \dfrac{1}{2} \end{cases}$$

Therefore, for the Analyst to recommend Buy when the stock is neutral, we need

$$\mathrm{E}[\pi_1(\text{Neutral}|\text{Hold})] \geq \mathrm{E}[\pi_1(\text{Neutral}|\text{Buy})]$$
$$\mathrm{E}[\pi_1(\text{Neutral}|\text{Hold})] \geq \mathrm{E}[\pi_1(\text{Neutral}|\text{Sell})]$$

which imply that

$$1 \geq a$$
$$1 \geq -b$$

Specifically, when the Investor's off-the-equilibrium beliefs satisfy $\mu_1 > \frac{1}{2}$ ($\gamma_1 > \frac{1}{2}$), he buys the stock upon receiving a recommendation to Hold (Sell). Therefore, the Analyst has no incentive to deviate from recommending Buy, which yields him a payoff of 1, to Hold (Sell), which yields him a payoff of a when the inequality $a \leq 1$ is satisfied.

Whereas, the Investor's off-the-equilibrium beliefs satisfy $\mu_1 + \mu_2 \leq \frac{1}{2}$ ($\gamma_1 + \gamma_2 \leq \frac{1}{2}$), he sells the stock upon receiving a recommendation to Hold (Sell). Therefore, the Analyst has no incentive to deviate from recommending Buy, which yields him a payoff of 1, to Hold (Sell), which yields him a payoff of $-b$, because the inequality $b \geq -1$ is satisfied for all non-negative values of b.

- When the stock market underperforms, the Analyst's payoffs are

$$\mathrm{E}[\pi_1(\text{Underperform}|\text{Buy})] = 0$$

$$E[\pi_1(\text{Underperform}|\text{Hold})] = \begin{cases} a-1 & if \quad \mu_1 > \frac{1}{2} \\ 0 & if \quad \mu_1+\mu_2 > \frac{1}{2}\ and\ \mu_1 \leq \frac{1}{2} \\ 1-b & if \quad \mu_1+\mu_2 \leq \frac{1}{2} \end{cases}$$

$$E[\pi_1(\text{Underperform}|\text{Sell})] = \begin{cases} a-1 & if \quad \gamma_1 > \frac{1}{2} \\ 0 & if \quad \gamma_1+\gamma_2 > \frac{1}{2}\ and\ \gamma_1 \leq \frac{1}{2} \\ 1-b & if \quad \gamma_1+\gamma_2 \leq \frac{1}{2} \end{cases}$$

Therefore, for the Analyst to recommend Hold when the stock underperforms, we need

$$E[\pi_1(\text{Underperform}|\text{Hold})] \geq E[\pi_1(\text{Underperform}|\text{Buy})]$$
$$E[\pi_1(\text{Underperform}|\text{Hold})] \geq E[\pi_1(\text{Underperform}|\text{Sell})]$$

which imply

$$0 \geq a-1$$
$$0 \geq 1-b$$

Specifically, when the Investor's off-the-equilibrium beliefs satisfy $\mu_1 > \frac{1}{2}$ ($\gamma_1 > \frac{1}{2}$), he buys the stock upon receiving a recommendation to Hold (Sell). Therefore, the Analyst has no incentive to deviate from recommending Buy, which yields him a payoff of 0, to Hold (Sell), which yields him a payoff of $a-1$ when the inequality $a \leq 1$ is satisfied.

Whereas, the Investor's off-the-equilibrium beliefs satisfy $\mu_1+\mu_2 \leq \frac{1}{2}$ ($\gamma_1+\gamma_2 \leq \frac{1}{2}$), he sells the stock upon receiving a recommendation to Hold (Sell). Therefore, the Analyst has no incentive to deviate from recommending Buy, which yields him a payoff of 0, to Hold (Sell), which yields him a payoff of $1-b$ whem the inequality $b \geq 1$ is satisfied.

- Combining the above, we see that when the Investor's off-the-equilibrium beliefs satisfy $\mu_1 > \frac{1}{2}$ ($\gamma_1 > \frac{1}{2}$), the Analyst will deviate from Buy to Hold (Sell) when the stock outperforms, and also when the stock is neutral or underperforms when $a > 1$. Whereas, when the Investor's off-the-equilibrium beliefs satisfy $\mu_1+\mu_2 \leq \frac{1}{2}$ ($\gamma_1+\gamma_2 \leq \frac{1}{2}$), the Analyst will not deviate from Buy to Hold (Sell) when $b \geq 1$.

  Further check that when the Investor's off-the-equilibrium beliefs satisfy $\mu_1+\mu_2 > \frac{1}{2}\ and\ \mu_1 \leq \frac{1}{2}$ ($\gamma_1+\gamma_2 > \frac{1}{2}\ and\ \gamma_1 \leq \frac{1}{2}$), the Investor will hold the stock so that the Analyst will not deviate from recommending Buy (and the Investor will also hold the stock) for any non-negative values of a and b.