ECONS 424 – STRATEGY AND GAME THEORY

MIDTERM EXAM #2 – *ANSWER KEY*

Exercise #1. Hawk-Dove game. Consider the following payoff matrix representing the Hawk-Dove game. Intuitively, Players 1 and 2 compete for a resource, each of them choosing to display an aggressive posture (hawk) or a passive attitude (dove). Assume that payoff $V > 0$ denotes the value that both players assign to the resource, and $C > 0$ is the cost of fighting, which only occurs if they are both aggressive by playing hawk in the top left-hand cell of the matrix.

		Player 2	
		Hawk	*Dove*
Player 1	*Hawk*	$\frac{V-C}{2}, \frac{V-C}{2}$	$V, 0$
	Dove	$0, V$	$\frac{V}{2}, \frac{V}{2}$

a) Show that if $C \leq V$, the game is strategically equivalent to a Prisoner's Dilemma game.

b) The Hawk-Dove game commonly assumes that the value of the resource is less than the cost of a fight, i.e., $C > V > 0$. Find the set of pure strategy Nash equilibria.

Answer:

Part (a)

- When Player 2 (in columns) chooses *Hawk* (in the left-hand column), Player 1 (in rows) receives a positive payoff of $\frac{V-C}{2}$ by paying *Hawk*, which is higher than his payoff of zero from playing *Dove*. Therefore, for Player 1 *Hawk* is a best response to Player 2 playing *Hawk*. Similarly, when Player 2 chooses *Dove* (in the right-hand column), Player 1 receives a payoff of V by playing *Hawk*, which is higher than his payoff from choosing *Dove*, $\frac{V}{2}$; entailing that *Hawk* is Player 1's best response to Player 2 choosing *Dove*. Therefore, Player 1 chooses *Hawk* as his best response to all of Player 2's strategies, implying that *Hawk* is a strictly dominant strategy for Player 1.

- Symmetrically, Player 2 chooses *Hawk* in response to Player 1 playing *Hawk* (in the top row) and to

Player 1 playing *Dove* (in the bottom row), entailing that playing *Hawk* is a strictly dominant strategy for Player 2 as well.

- Since $C \leq V$,by assumption, the unique pure strategy Nash equilibrium is {*Hawk*, *Hawk*}, although {*Dove*, *Dove*} Pareto dominates the NE strategy as both players can improve their payoffs from $\frac{V-C}{2}$ to $\frac{V}{2}$. Since every player is choosing a strictly dominant strategy in the pure-strategy Nash equilibrium of the game, despite being Pareto dominated by another strategy profile, this game is strategically equivalent to a Prisoner's Dilemma game.

Part (b)

- *Finding best responses.* When Player 2 plays *Hawk* (in the left-hand column), Player 1 receives a negative payoff of $\frac{V-C}{2}$ since $C > V > 0$. Player 1 then prefers to choose *Dove* (with a payoff of zero) in response to Player 2 playing *Hawk*. However, when Player 2 plays *Dove* (in the right-hand column), Player 1 receives a payoff of V by responding with *Hawk*, and half of this payoff, $\frac{V}{2}$, when responding with *Dove*. Then, Player 1 plays *Hawk* in response to Player 2 playing *Dove*.
- Symmetrically, Player 2 plays *Dove* (*Hawk*) in response to Player 1 playing *Hawk* (*Dove*) to maximize his payoff.
- *Finding pure strategy Nash equilibria.* For illustration purposes, we underline the best response payoffs in the matrix below.

Player 2

		Hawk	*Dove*
Player 1	*Hawk*	$\frac{V-C}{2}, \frac{V-C}{2}$	$\underline{V}, \underline{0}$
	Dove	$\underline{0}, \underline{V}$	$\frac{V}{2}, \frac{V}{2}$

We can therefore identify two cells where the payoffs of both players were underlined as best response payoffs: (*Hawk, Dove*) and (*Dove, Hawk*). These two strategy profiles are the two pure strategies Nash equilibria.

Our results then resemble those in the Chicken game, since every player seeks to miscoordinate by choosing the opposite strategy of his opponent.

Exercise #2. Cournot competition and cost disadvantages. Consider two firms competing in quantities (a la Cournot), facing inverse demand function $p(Q) = a - Q$, where Q denotes aggregate output, that is, $Q = q_1 + q_2$. Firm 1's marginal cost of production is $c_1 > 0$, while firm 2's marginal cost is $c_2 > 0$, where $c_2 \geq c_1$ indicating that firm 2 suffers a cost disadvantage relative to firm 1. Firms face no fixed costs. For compactness, let us represent firm 2's marginal costs relative to c_1, so that $c_2 = \alpha c_1$, where parameter $\alpha \geq 1$ indicates the cost disadvantage that firm 2 suffers relative to firm 1. Intuitively, when α approaches 1 the cost disadvantage diminishes, while when α is significantly higher than 1 firm 2's cost disadvantage is severe. This allows us to represent both firms' marginal costs in terms of c_1, without the need to use c_2.

a. Write down every firm's profit maximization problem and find its best response function.
b. Interpret how each firm's best response function is affected by a marginal increase in α. [*Hint*: Recall that an increase in parameter alpha indicates that firms' marginal costs are more similar.]
c. Find the Nash equilibrium of this Cournot game.
d. Identify the regions of (a, α)-pairs for which equilibrium output levels of firm 1 and firm 2 are positive.
e. How are equilibrium output levels affected by a marginal increase in a? And by a marginal increase in α?

Answer:

Part (a). Firm 1 chooses its output q_1 to maximize its profits as follows

$$\max_{q_1 \geq 0} \ \pi_1 = (a - q_1 - q_2)q_1 - c_1 q_1$$

Differentiating with respect to q_1, yields

$$\frac{\partial \pi_1}{\partial q_1} = a - 2q_1 - q_2 - c_1 = 0$$

Solving for q_1 in the above equation gives us firm 1's best response function

$$q_1(q_2) = \frac{a - c_1}{2} - \frac{1}{2} q_2$$

which originates at $\frac{a-c_1}{2}$ when firm 2 produces zero units of output, $q_2 = 0$, but decreases in q_2 at a rate of 1/2.

Similarly, for firm 2, its profit maximization problem is

$$\max_{q_2 \geq 0} \ \pi_2 = (a - q_1 - q_2)q_2 - (\alpha c_1)q_2$$

where we wrote firm 2's costs as a function of firm 1's, that is, $c_2 = \alpha c_1$. Differentiating with respect to q_2, we obtain

$$\frac{\partial \pi_2}{\partial q_2} = a - q_1 - 2q_2 - \alpha c_1 = 0$$

Solving for output q_2, we find firm 2's best response function

$$q_2(q_1) = \frac{a - \alpha c_1}{2} - \frac{1}{2} q_1$$

which is symmetric to that of firm 1.

Part (b).

Firm 1. An increase in parameter α does not affect firm 1's best response function. It will, however, affect its equilibrium output, as we show in the next part of the exercise.

Firm 2. An increase in parameter α (reflecting firm 2's cost disadvantage relative to firm 1) means that firm 2's best response function suffers a downward shift in its vertical intercept, without altering its slope. Intuitively, this indicates that, when firm 2's costs are similar to those of firm 1 (i.e., α close to 1, exhibiting a small cost disadvantage) firm 2 produces a large output, for a given output of its rival q_1. However, when firm 2's costs are significantly larger than those of firm 1 (i.e., α increases above 1, exhibiting a large cost disadvantage) firm 2 produces a smaller output, for a given output of its rival q_1.

Part (c). To find the Nash equilibrium of the Cournot game, we insert one best response function into another. Note that we cannot use symmetry here, since the firms have difference costs and will hence, choose different equilibrium output.

Inserting the best response function of firm 2 into that of firm 1, $q_1(q_2) = \frac{a-c_1}{2} - \frac{1}{2}q_2$, we obtain

$$q_1 = \frac{a - c_1}{2} - \frac{1}{2}\left(\frac{a - \alpha c_1}{2} - \frac{1}{2} q_1\right)$$

Rearranging and solving for output q_1, we find an equilibrium output of

$$q_1^* = \frac{1}{3}\big(a - c_1(2 - \alpha)\big)$$

Inserting this result into firm 2's best response function, $q_2(q_1) = \frac{a-\alpha c_1}{2} - \frac{1}{2}q_1$, we find

$$q_2 = \frac{a - \alpha c_1}{2} - \frac{1}{2}\left[\frac{1}{3}\big(a - c_1(2 - \alpha)\big)\right]$$

which upon solving for output q_2 yields an equilibrium output of

$$q_2^* = \frac{1}{3}\left(a - c_1(2\,\alpha - 1)\right)$$

Therefore, the Nash equilibrium output pair can be summarized as follows

$$\{q_1^*, q_2^*\} = \left\{\frac{1}{3}\left(a - c_1(2-\alpha)\right), \frac{1}{3}\left(a - c_1(2\,\alpha - 1)\right)\right\}$$

Part (d). Equilibrium output for firm 1 is positive, $q_1^* > 0$, if and only if $\frac{1}{3}\left(a - c_1(2-\alpha)\right) > 0$. After solving for parameter a, this condition yields

$$a > c_1(2-\alpha) \qquad (1)$$

Similarly, equilibrium output for firm 2 is positive, $q_2^* > 0$, if and only if $\frac{1}{3}\left(a - c_1(2\,\alpha - 1)\right) > 0$, which after solving for parameter a, yields

$$a > c_1(2\,\alpha - 1) \qquad (2)$$

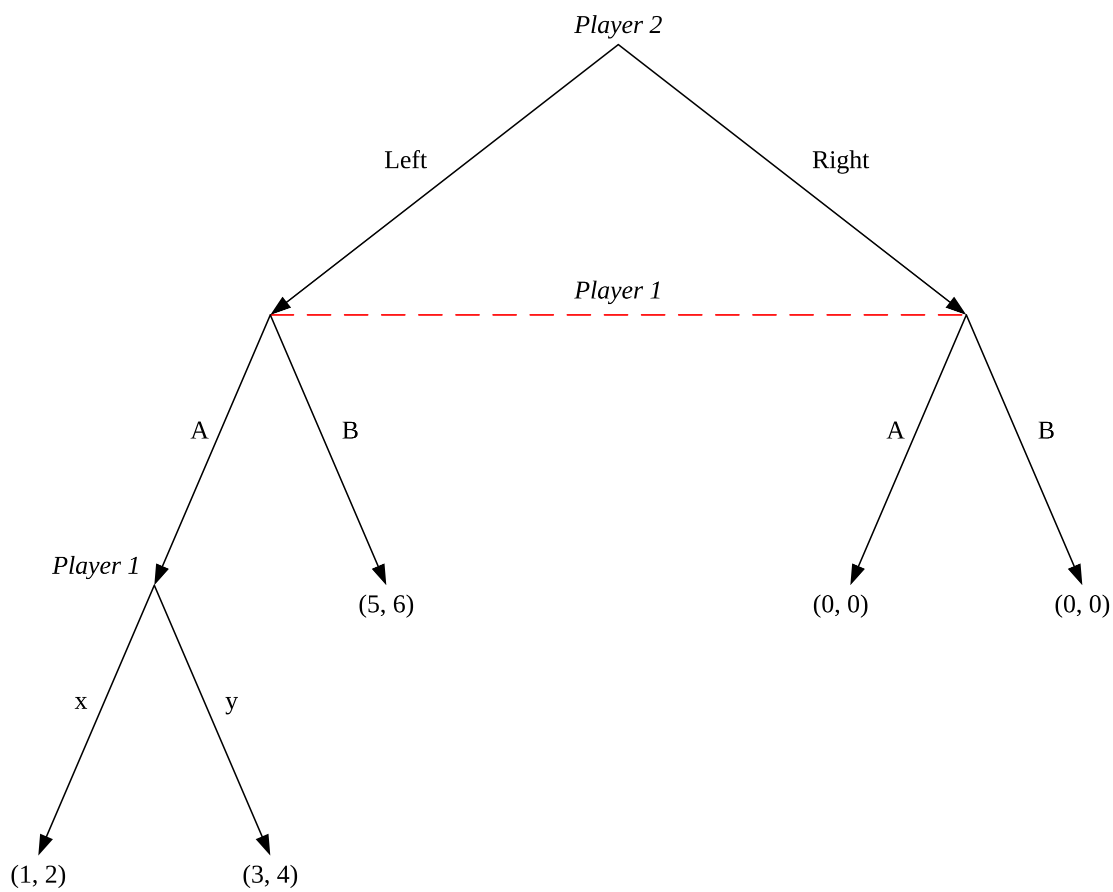
Since parameter α satisfies $\alpha \geq 1$ by assumption, we must have that $(2\,\alpha - 1) \geq (2-\alpha)$. Therefore, if condition (2) is satisfied, condition (1) also holds. In summary, both firms produce a positive equilibrium output if demand is sufficiently strong, that is, $a > c_1(2\,\alpha - 1)$.

To better understand this condition, note that cutoff $c_1(2\,\alpha - 1)$ originates at c_1 since $\alpha \geq 1$ by assumption; and increases in α , indicating that condition $a > c_1(2\,\alpha - 1)$ becomes more difficult to sustain as parameter α increases. Intuitively, an increase in α means that firm 2's cost disadvantage is more severe, implying that firm 2 only produces a positive output level if demand becomes sufficiently large, as captured by condition $a > c_1(2\,\alpha - 1)$.

Part (e). Firm 1's equilibrium output level $q_1^* = \frac{1}{3}\left(a - c_1(2-\alpha)\right)$ increases in both demand, as reflected by parameter a, and in the cost disadvantage that its rival (firm 2) suffers, as illustrated by parameter α.

Firm 2's equilibrium output level, $q_2^* = \frac{1}{3}\left(a - c_1(2\,\alpha - 1)\right)$ also increases in demand, as captured by parameter a, but it decreases in the cost disadvantage that it suffers relative to firm 1, captured by parameter α.

Exercise #3 – Sequential move game. Consider the following game tree, where player 2 chooses firstly whether to go Left (L) or Right (R). Then, without observing what player 2 chose, player 1 responds by playing A or B. Finally, player 1 is again called to move after playing A in the case in which player 2 initially chose Left. In this event player 1 can choose between action *x* and *y*.



a) How many strategies does player 1 have in this extensive form game?
b) Represent the game in its normal form payoff matrix.
c) Find the pure strategy Nash equilibria of the normal form game you represented in *b)*.
d) How many proper subgames can you identify? Draw them, and explain why these can be considered proper subgames.
e) Find the subgame perfect Nash equilibrium (SPNE) of this extensive form game.

Exercise #4. Stackelberg competition with efficiency changes. Consider the industry in the above setting but assume now that firm 1 is the industry leader while firm 2 is the follower; as in the Stackelberg model of quantity competition.

a) Assuming that firm i's cost function is $TC_i(q_i) = 4q_i$, find the equilibrium in this Stackelberg game.

b) Assuming that firm 2's cost function becomes $TC_2(q_2) = 2q_2$ after investing in the cost-saving technology, find the equilibrium in this Stackelberg game. How are your results from part (a) affected?

c) Repeat parts (a) and (b), but assuming that firm 2 is the industry leader while firm 1 is the follower.

Answer:

Part (a). By backwards induction, firm 2 takes the output of firm 1, q_1, as given and chooses its own output level, q_2, that solves the following profit maximization problem:

$$\max_{q_2 \geq 0} \ \pi_2(q_2) = p(Q)q_2 - TC_2(q_2)$$

$$= (6 - q_1 - q_2)q_2$$

Taking the first order condition with respect to q_2, and assuming an interior solution for q_2,

$$6 - q_1 - 2q_2 = 0 \ \text{ if } \ q_2 > 0$$

After rearranging, the best response function of firm 2 in response to firm 1 is given by

$$q_2(q_1) = 3 - \frac{q_1}{2}$$

Firm 1, anticipating that firm 2 will respond with best response function $q_2(q_1) = 3 - \frac{q_1}{2}$, chooses its own output level, q_1, that solves the following profit maximization problem:

$$\max_{q_1 \geq 0} \pi_1(q_1) = p(Q)q_1 - TC_1(q_1)$$

$$= \big(10 - q_1 - q_2(q_1)\big)q_1 - 4q_1$$

$$= \left(6 - q_1 - 3 + \frac{q_1}{2}\right)q_1$$

$$= \left(3 - \frac{q_1}{2}\right)q_1$$

Taking the first order condition with respect to q_1, and assuming an interior solution for q_1,

$$3 - q_1 = 0 \quad \text{if } \ q_1 > 0$$

such that the optimal output of firm 1 becomes

$$q_1^* = 3$$

Substituting the optimal output of firm 1 into firm 2's best response function, the optimal output of firm 2 becomes

$$q_2^* = 3 - \frac{3}{2} = 1\frac{1}{2}$$

Therefore, the leading firm 1 produces 3 units of output, while the following firm 2 produces 1.5 units of output, which is half of the production level of firm 1.

Part (b). By backwards induction, firm 2 takes the output of firm 1, q_1, as given and chooses its own output level, q_2, that solves the following profit maximization problem:

$$\max_{q_2 \geq 0} \pi_2 (q_2) = p(Q)q_2 - TC_2(q_2)$$

$$= (8 - q_1 - q_2)q_2$$

Taking the first order condition with respect to q_2, and assuming an interior solution for q_2,

$$8 - q_1 - 2q_2 = 0 \quad \text{if } q_2 > 0$$

After rearranging, the best response function of firm 2 in response to firm 1 is given by

$$q_2(q_1) = 4 - \frac{q_1}{2}$$

Firm 1, anticipating that firm 2 will respond with its best response function $q_2(q_1) = 4 - \frac{q_1}{2}$, chooses its own output level, q_1, that solves the following profit maximization problem:

$$\max_{q_1 \geq 0} \pi_1 (q_1) = p(Q)q_1 - TC_1(q_1)$$

$$= \big(10 - q_1 - q_2(q_1)\big)q_1 - 4q_1$$

$$= \left(6 - q_1 - 4 + \frac{q_1}{2}\right) q_1$$

$$= \left(2 - \frac{q_1}{2}\right) q_1$$

Taking the first order condition with respect to q_1, and assuming an interior solution for q_1,

$$2 - q_1 = 0 \text{ if } q_1 > 0$$

such that the optimal output of firm 1 becomes

$$q_1^* = 2$$

Substituting the optimal output of firm 1 into firm 2's best response function, the optimal output of firm 2 becomes

$$q_2^* = 4 - \frac{2}{2} = 3$$

When it is less expensive for firm 2 to produce the output, firm 2 doubles production from 1.5 to 3 units, and firm 1 reduces output to 2 units which now falls below that of firm 2.

Part (c). By backwards induction, firm 1 takes the output of firm 2, q_2, as given and chooses its own output level, q_1, that solves the following profit maximization problem:

$$\max_{q_1 \geq 0} \pi_1 (q_1) = p(Q)q_1 - TC_1(q_1)$$

$$= (6 - q_1 - q_2)q_1$$

Taking the first order condition with respect to q_1, and assuming an interior solution for q_1,

$$6 - 2q_1 - q_2 = 0 \text{ if } q_1 > 0$$

After rearranging, the best response function of firm 1 in response to firm 2 is given by

$$q_1(q_2) = 3 - \frac{q_2}{2}$$

Firm 2, anticipating that firm 1 will respond with best response function $q_1(q_2) = 3 - \frac{q_2}{2}$, chooses its own output level, q_2, that solves the following profit maximization problem:

$$\max_{q_2 \geq 0} \quad \pi_2(q_2) = p(Q)q_2 - TC_2(q_2)$$

$$= (10 - q_1(q_2) - q_2)q_2 - 2q_2$$

$$= \left(8 - 3 + \frac{q_2}{2} - q_2\right)q_2$$

$$= \left(5 - \frac{q_2}{2}\right)q_2$$

Taking the first order condition with respect to q_2, and assuming an interior solution for q_2,

$$5 - q_2 = 0 \text{ if } q_2 > 0$$

such that the optimal output of firm 2 becomes

$$q_2^* = 5$$

Substituting the optimal output of firm 2 into firm 1's best response function, the optimal output of firm 1 becomes

$$q_1^* = 3 - \frac{5}{2} = \frac{1}{2}$$

Therefore, in the case that the leading firm invests in cost-saving technology, firm 2 produces 5 units of output while the following firm 1 produces ½ units of output.

Exercise #5. Temporary punishments from deviation. Consider two firms competing in quantities (a la Cournot), facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. For simplicity, assume that firms face a common marginal cost of production $c = 10$.

a) *Unrepeated game.* Find the equilibrium output each firm produces when competing a la Cournot (that is, when they simultaneously and independently choose their output levels) in the unrepeated version of the game (that is, when firms interact only once). In addition, find the profits that each firm earns in equilibrium.

b) *Repeated game - Collusion.* Assume now that the CEOs from both companies meet to discuss a collusive agreement that would increase their profits. Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the output level that each firm should select to maximize joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

c) *Repeated game – Permanent punishment.* Consider a grim-trigger strategy in which every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. Otherwise, every firm deviates to the Cournot equilibrium thereafter (that is, every firm produces the Nash equilibrium of the unrepeated game found in part a forever). In words, this says that the punishment of deviating from the collusive agreement is *permanent*, since firms never return to the collusive outcome. For which discount factors this grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

d) *Repeated game – Temporary punishment.* Consider now a "modified" grim-trigger strategy. Like in the grim-trigger strategy of part (c), every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. However, if a deviation is detected by either firm, every firm deviates to the Cournot equilibrium during only 1 period, and then every firm returns to cooperation (producing the collusive output). Intuitively, this implies that the punishment of deviating from the collusive agreement is now *temporary* (rather than permanent) since it lasts only one period. For which discount factors this "modified" grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

e) Consider again the temporary punishment in part (d), but assume now that it lasts for two periods. How are your results from part (d) affected? Interpret.

f) Consider again the temporary punishment in part (d), but assume now that it lasts for three periods. How are your results from part (d) affected? Interpret.

Answer:

Part (a). Firm i's profit function is given by:

$$\pi_i(q_i, q_j) = [100 - (q_i + q_j)]q_i - 10q_i \qquad (7)$$

Differentiating with respect to output q_i yields,

$$100 - 2q_i - q_i - 10 = 0$$

Solving for q_i, we find firm i's best response function

$$q_i(q_j) = 45 - \frac{q_j}{2} \qquad (\text{BRF}_7)$$

Since this is a symmetric game, firm j's best response function is symmetric. Therefore, in a symmetric equilibrium both firms produce the same output level, $q_i = q_j$, which helps us rewrite the above best response function as follows

$$q_i = 45 - \frac{q_i}{2}$$

Solving this expression yields equilibrium output of $q_i^* = q_1^* = q_2^* = 30$. Substituting these results into profits $\pi_1(q_1, q_2)$ and $\pi_2(q_1, q_2)$, we obtain that

$$\pi_i(q_i^*, q_j^*) = \pi_i(30{,}30) = [100 - (30 + 30)]30 - 10 \times 30 = \$900$$

where firm's equilibrium profits coincide, that is, $\pi_i(q_i^*, q_j^*) = \pi_1(q_1^*, q_2^*) = \pi_2(q_1^*, q_2^*)$.
In summary, equilibrium profit is $900 for each firm and combined profits are $900+$900=$1800.

Part (b). In this case, firms maximize the sum $\pi_1(q_1, q_2) + \pi_2(q_1, q_2)$, as follows

$$\{[100 - (q_i + q_j)]q_i - 10q_i\} + \{[100 - (q_i + q_j)]q_j - 10q_j\} \qquad (8)$$

Differentiating with respect to output q_i, yields

$$100 - 2q_i - q_j - 10 - q_j = 0$$

Solving for q_i, we obtain

$$q_i(q_j) = 45 - q_j$$

and similarly when we differentiate with respect to q_j. In a symmetric equilibrium both firms produce the same output level, $q_i = q_j$, which helps us rewrite the above expression as follows

$$q_i = 45 - q_i$$

Solving for output q_i we obtain equilibrium output $q_i^* = q_1^* = q_2^* = \frac{45}{2} = 22.5$ units. This yields each firm a profit of

$$\pi_i(q_i^*, q_j^*) = \pi_i(22.5{,}22.5)$$
$$= \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\} + \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\}$$
$$= \$1012.5$$

Thus, each firm makes a profit of $1,012.5, which yields a joint profit of $2,025.

Part (c). For this part of the exercise, let us first list the payoffs that the firm can obtain from each of its output decisions:

- Cooperation yields a payoff of $1,012.5 for firm i.
- Defecting while firm j cooperates $(q_j = 22.5)$, by choosing the Cournot output $(q_i = 30)$ yields a profit of $1,125 for firm *i*. However, such defection is punished with Cournot competition in all subsequent periods, which yields a profit of only $900.

Therefore, firm *i* cooperates as long as

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + \cdots \geq 1125 + 900\delta + 900\delta^2 + \cdots$$

which can be simplified to

$$1012.5(1 + \delta + \delta^2 + \cdots) \geq 1125 + 900\delta(1 + \delta + \delta^2 + \cdots)$$

and expressed more compactly as

$$\frac{1012.5}{1-\delta} \geq 1125 + \frac{900\delta}{1-\delta}$$

Multiplying both sides of the inequality by $1-\delta$, yields

$$1012.5 \geq 1125(1-\delta) + 900$$

which rearranging and solving for discount factor δ entails

$$\delta \geq 0.5$$

Thus, for cooperation to be sustainable in an infinitely repeated game with permanent punishments, firms' discount factor δ has to be at least 0.5. In words, firms must put a sufficient weight on future payoffs.

Part (d). The set up is analogous to that in part (c) of the exercise, but we now write that cooperation is possible if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \ldots. \geq 1125 + 900\delta + 1012.5\delta^2 \ldots.$$

Importantly, note that payoffs after the punishment period (in this case, a one period punishment of Cournot competition with \$900 profits) returns to cooperation, explaining that all payoffs in the third term of the left-hand and right-hand side of the inequality coincide. We can therefore cancel them out, simplifying the above inequality to

$$1012.5 + 1012.5\delta \geq 1125 + 900\delta$$

Rearranging yields

$$112.5\delta \geq 112.5$$

$$\Rightarrow \delta \geq 1$$

ultimately simplifying to $\delta \geq 1$. That is, when deviations are only punished during one period cooperation can be sustained if firms assign the same value to current than future payoffs, which does not generally occur.

Part (e). Following a similar approach as in part (d), we write that cooperation can be sustained if and only if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 \ldots. \geq 1125 + 900\delta + +900\delta^2 + 1012.5\delta^3 \ldots.$$

The stream of payoffs after the punishment period (in this case, a two-period punishment of Cournot competition with \$900 profits) returns to cooperation. Hence, the payoffs in both the left- and right-hand side of the inequality after the punishment period coincide and cancel out. The above inequality then becomes

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \geq 1125 + 900\delta + 900\delta^2$$

which, after rearranging, yields

$$\delta^2 + \delta - 1 \geq 0$$

ultimately simplifying to $\delta \geq 0.62$. In words, when deviations are punished for two periods, cooperation can be sustained if firms' discount factor δ is at least 0.62.

Part (f). Following a similar approach as in part (e), we write that cooperation can be sustained if and only if

$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 + 1012.5\delta^4 \ldots .$$
$$\geq 1125 + 900\delta + 900\delta^2 + 900\delta^3 + 1012.5\delta^4 \ldots .$$

The stream of payoffs after the punishment period (in this case, a three-period punishment of Cournot competition with $900 profits) returns to cooperation. Hence, the payoffs in both the left- and right-hand side of the inequality after the punishment period coincide and cancel out. The above inequality then becomes

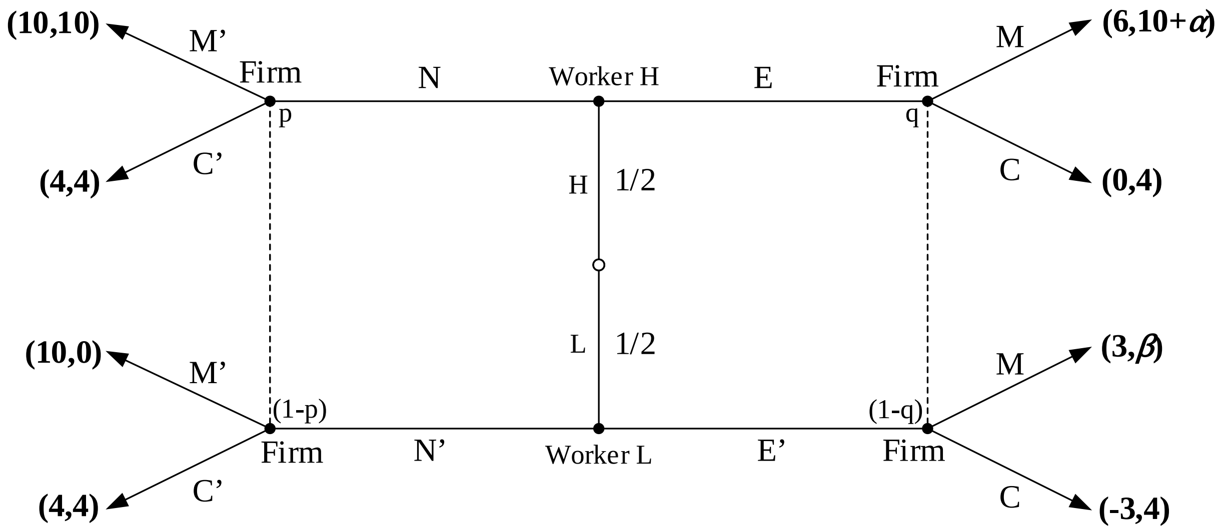
$$1012.5 + 1012.5\delta + 1012.5\delta^2 + 1012.5\delta^3 \geq 1125 + 900\delta + 900\delta^2 + 900\delta^3$$

which, after rearranging, yields

$$\delta^3 + \delta^2 + \delta - 1 \geq 0$$

ultimately simplifying to $\delta \geq 0.54$. Therefore, when deviations are punished for two periods, cooperation can be sustained if firms' discount factor δ is at least 0.54.

Exercise #6. Job-market signaling with productivity-enhancing education Consider a variation of the job-market signaling game we discussed in class (see point #18 in the EconS 424 website). Still assume two types of worker (high or low productivity), two messages he can send (education or no education), and two responses by the firm (hiring the worker as a manager or as a cashier). However, let us now allow for education to have a productivity-enhancing effect. In particular, when the worker acquires education (graphically represented in the right-hand side of the game tree below), the firm's payoff from hiring him as a manager is $10 + \alpha$ when the worker is of high productivity, and β when the worker is of low productivity. In words, parameter α (β) represents the productivity-enhancing effect of education for the high (low, respectively) productivity worker respectively, where we assume that $\alpha \geq \beta \geq 0$. For simplicity, we assume that education does not increase the firm's payoff when the worker is hired as a cashier, providing a payoff of 4 regardless of the worker's productivity. All other payoffs for both the firm and the worker are unaffected relative to the figure we saw in class.



Job-market signaling game with productivity-enhancing education

a) Consider the separating strategy profile EN'. Under which conditions of parameters α and β can this strategy profile be supported as a PBE?

Figure 9.12' depicts the separating strategy profile, in which only the high productivity worker acquires education, as indicated in the thick arrows E and N'.

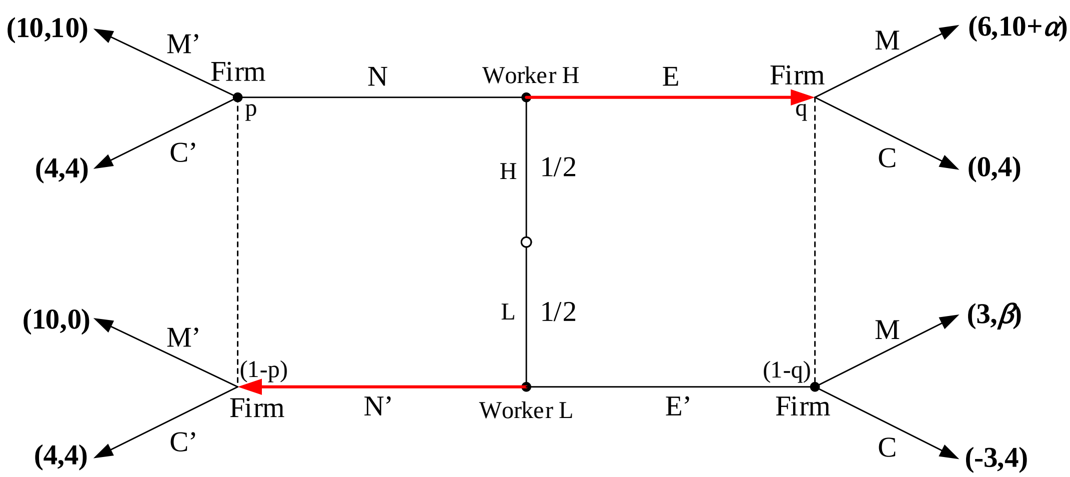


Figure 9.12'. Separating strategy profile EN'.

1. Responder's beliefs:

Firm's beliefs about the worker's type can be updated using Bayes' rule, as follows:

$$q = p(H|E) = \frac{p(E|\,H)p(H)}{p(E|\,H)p(H) + p(E|\,L)p(L)} = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + (1 - \frac{1}{2}) \times 0} = 1$$

Intuitively, upon observing that the worker acquires education, the firm infers that the worker must be of high productivity, i.e., q = 1, since only this type of worker acquires education in this separating strategy profile; while no education conveys the opposite information, i.e., p = 0, thus implying that the worker is not of high productivity but instead of low productivity. Graphically, the firm restricts its attention to the upper right-hand corner, i.e., q = 1, and to the lower left-hand corner, i.e., p = 0.

2. The firm's optimal response given its updated beliefs:

   After observing "Education" the firm responds with M. Graphically, the firm is convinced to be located in the upper right-hand corner of the game tree since q = 1. In this corner, the best response of the firm is M, which provides a payoff of 10 + α, rather than C, which only yields a payoff of 4.

   After observing "No education" the firm responds with C'. In particular, in this case the firm is convinced to be located in the lower left-hand corner of the game tree given that p = 0. In such a corner, the firm's best response is C', providing a payoff of 4, rather than M', which yields a zero payoff. Figure 9.13' illustrates these optimal responses for the firm.

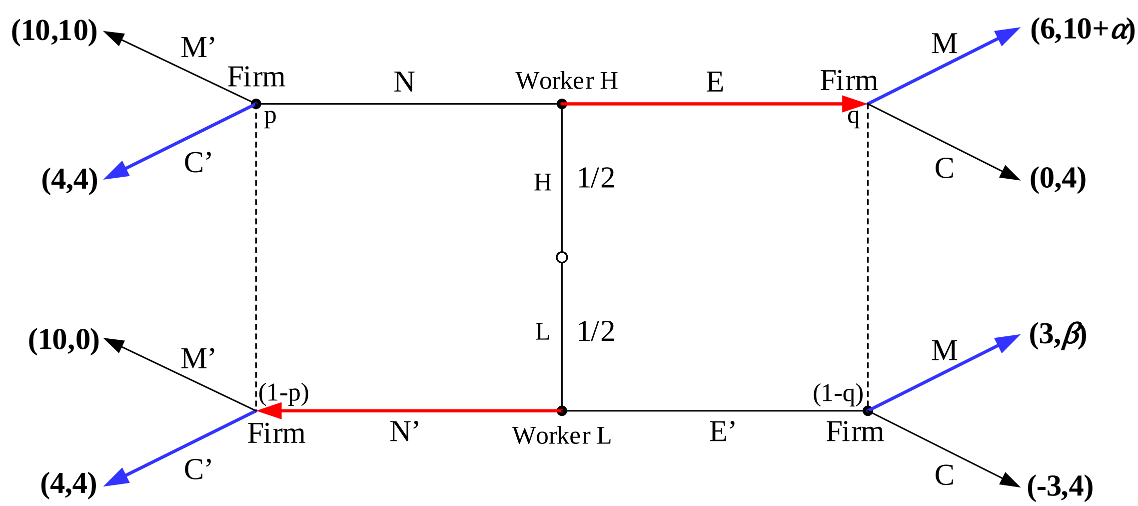


Figure 9.13'. Optimal responses of the firm in strategy profile EN'.

3. Given the previous steps 1 and 2, let us now find the worker's optimal actions:

   When he is a high-productivity worker, he acquires education (as prescribed in this strategy profile) since his payoff from E (6) is higher than that from N (4); as indicated in the upper part of figure 9.13'. When he is a low-productivity worker, he does not deviate from "No education", i.e., N', since his payoff from "No education", 4, is larger than that from "Education", 3; as indicated in the lower part of the game tree. Intuitively, even if acquiring no education reveals his low productivity to the firm, and ultimately leads him to be hired as a cashier, his payoff is larger than what he would receive when acquiring education (even if such education helped him to be hired as a manager). In short, the low-productivity worker finds it too costly to acquire education. Then, the separating strategy profile [N'E, C'M] can be supported as a PBE of this signaling game, where firm's beliefs are $q = 1$ and $p = 0$ for all values of $\alpha \geq \beta \geq 0$.

b) Consider now the pooling strategy profile NN'. Under which conditions of parameters α and β can this strategy profile be supported as a PBE?

Figure 9.20' depicts the pooling strategy profile in which both types of worker choose "No education" by shading branches N and N' (see thick arrows).

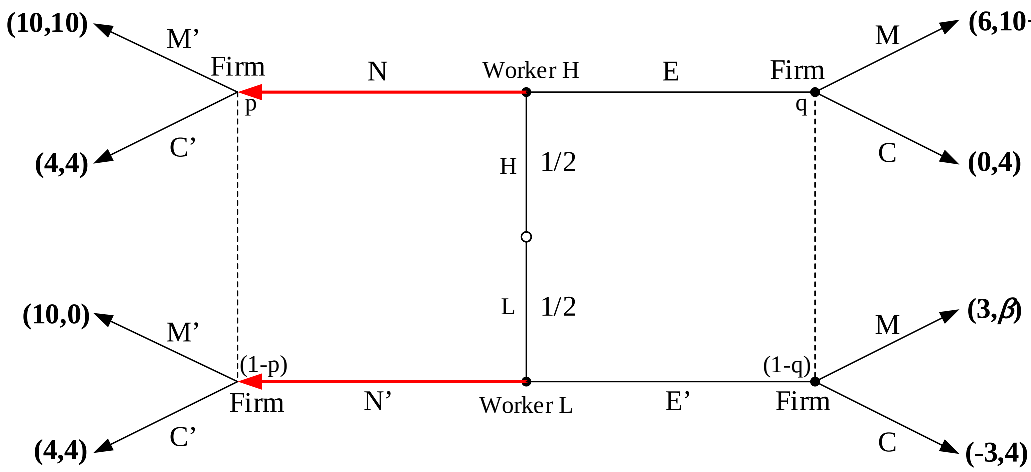


Figure 9.20'. Pooling strategy profile NN'.

1. Responder's beliefs:

   Analogously as the previous pooling strategy profile, the firm's equilibrium beliefs (after observing "No education") coincide with the prior probability of a high type, $p = \frac{1}{2}$; while its off-the-equilibrium beliefs (after observing "Education") are left unrestricted, i.e., $q \in [0,1]$.

2. Firms' optimal response given its updated beliefs:

   Given the previous beliefs, after observing "No education" (in equilibrium): if the firm hires the worker as a manager (M'), it obtains an expected payoff of

$$EU_F(M') = \frac{1}{2} \times 10 + \frac{1}{2} \times 0 = 5$$

   and if it hires him as a cashier (C'), its expected payoff only becomes

$$EU_F(C') = \frac{1}{2} \times 4 + \frac{1}{2} \times 4 = 4,$$

leading the firm to hire the worker as a manager (M') after observing the equilibrium message of "No education".

After observing "Education" (off-the-equilibrium): the expected payoff the firm obtains from hiring the worker as a manager (M) or cashier (C) are, respectively,

$$EU_F(M) = q \times (10 + \alpha) + (1 - q) \times \beta = (10 + \alpha - \beta)q + \beta$$
$$EU_F(C) = q \times 4 + (1 - q) \times 4 = 4$$

Hence, the firm responds hiring the worker as a manager (M) after observing the off-the-equilibrium message of Education if and only if $(10 + \alpha - \beta)q + \beta > 4$, or $q > \frac{4-\beta}{10+\alpha-\beta}$. Otherwise, the firm hires the worker as a cashier (C).

3. Given the previous steps 1 and 2, let us find the worker's optimal actions. In this case, we will also need to split our analysis into two cases (one in which $q \leq \frac{4-\beta}{10+\alpha-\beta}$ and thus the firm respond by hiring him as a cashier upon observing the off-the-equilibrium message of "Education", and the case in which $q \geq \frac{4-\beta}{10+\alpha-\beta}$, in which the firm hires the worker as a manager):

Case 1: When $q \leq \frac{4-\beta}{10+\alpha-\beta}$, the firm responds by hiring the worker as a cashier (C) upon observing the off-the-equilibrium message of "Education", as depicted in the right-hand shaded branches of figure 9.21'.

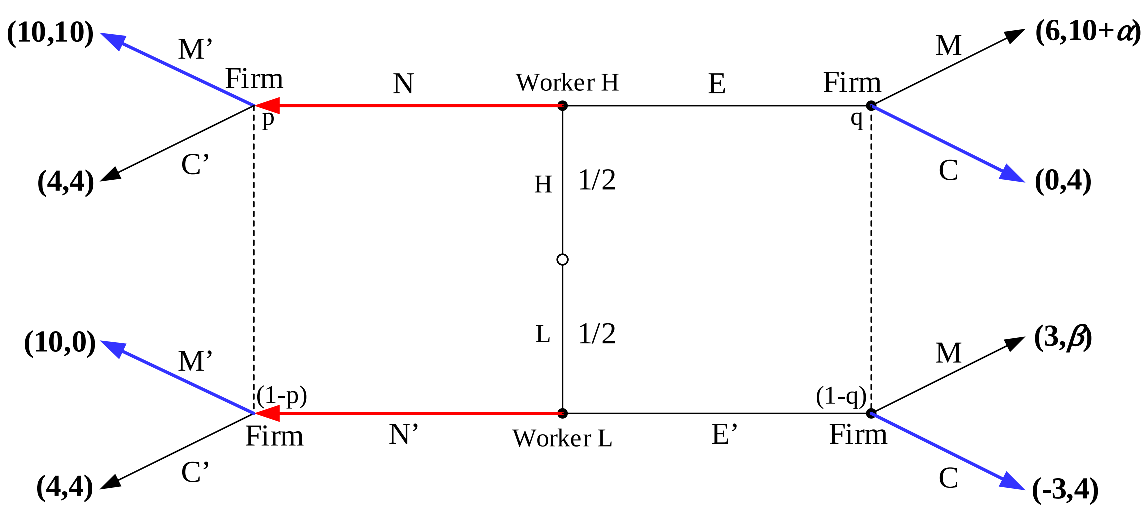
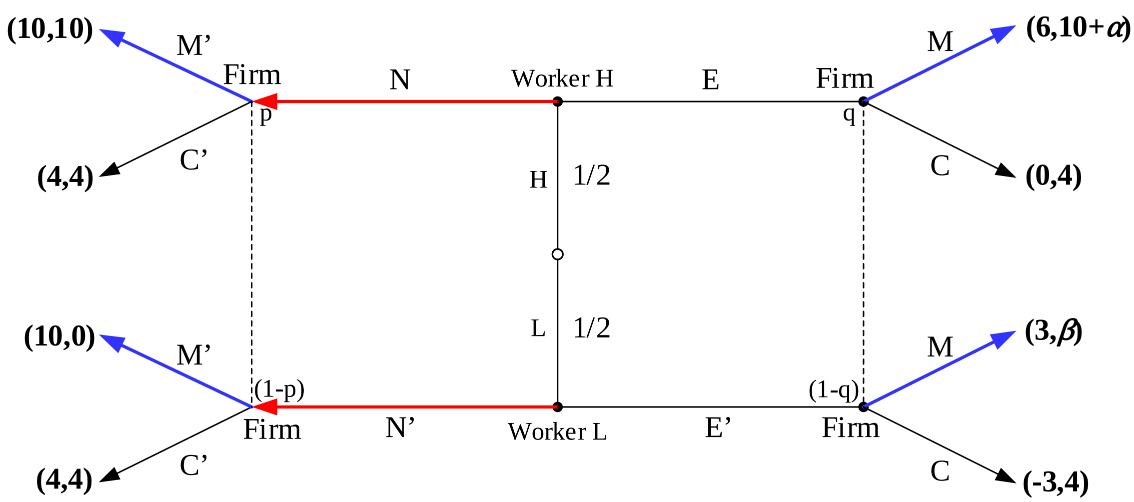


Figure 9.21'. Optimal responses in strategy profile NN' – Case 1.

If the worker is a high-productivity type, he does not deviate from "No education" since his payoff from N, 10, exceeds the zero payoff he would obtain by deviating to E; as indicated in the shaded branches upper part of the game tree. Similarly, if he is a low-productivity worker, he does not deviate from "No education" since his (positive) payoff from N' (10) is larger than the negative payoff he would obtain by deviating to E', -3; as indicated in the lower part of the game tree.

Therefore, the pooling strategy profile in with no type of worker acquires education, [NN', M'C], can be supported when off-the-equilibrium beliefs satisfy $q \leq \frac{4-\beta}{10+\alpha-\beta}$ for all values of $\alpha \geq \beta \geq 0$.

Case 2: When $q > \frac{4-\beta}{10+\alpha-\beta}$ the firm responds by hiring the worker as a manager (M) when he acquires education, as depicted in figure 9.22'.



If the worker is a high-productivity type, he plays "No education" (as prescribed in this strategy profile) since his payoff from doing so, 10, exceeds that from deviating to E, 6; as indicated in the shaded branches of upper part of the game tree. Similarly, if he is a low-productivity type, he plays "No education" (as prescribed) since his payoff from this strategy, 10, is higher than from deviating towards E', 3; as indicated in the lower part of the game tree.

Hence, the pooling strategy profile in which no worker acquires education, [NN', M'M], can be supported as a PBE when off-the-equilibrium beliefs satisfy $q > \frac{4-\beta}{10+\alpha-\beta}$, for all values of $\alpha \geq \beta \geq 0$ (that is, when $\beta > 4$, all off-the-equilibrium beliefs $q \in [0,1]$ support the poling strategy profile NN' as a PBE).

c) How are your results affected by an increase in parameter α? And an increase in parameter β?

First, increasing parameter α reduces probability cutoff $\bar{q}$, that is, $\frac{\partial \bar{q}}{\partial \alpha} = -\frac{4-\beta}{(10+\alpha-\beta)^2} < 0$.

Intuitively, as education improves the productivity of the high-productivity worker, the expected profit from hiring an educated worker as a manager increases, leading the firm to become more willing to respond hiring an educated worker as a manager.

A similar argument applies to an increase in parameter β since $\frac{\partial \bar{q}}{\partial \beta} = -\frac{6+\alpha}{(10+\alpha-\beta)^2} < 0$. In words, this result indicates that, as education improves the productivity of the low-productivity worker, the expected profit from hiring an educated worker as a manager increases, and the firm responds hiring him in this position for a larger range of priors (i.e., a larger range of probability q).

Finally, note that when both parameters are zero, $\alpha = \beta = 0$, probability cutoff $\bar{q}$ becomes $\bar{q} = \frac{4-0}{10+0-0} = \frac{2}{5}$, and we return to the same payoffs as in the exercise we discussed in class, and the same set of PBEs.

BONUS EXERCISE. Public good game with incomplete information. Consider a public good game between two players, A and B. The utility function of every player *i* is

$$u_i(g_i) = (w_i - g_i)^{1/2}\left[m\left(g_i + g_j\right)\right]^{1/2}$$

where the first term, $w_i - g_i$, represents the utility that the player receives from money (i.e., the amount of his wealth not contributed to the public good which he can dedicate to private uses). The second term indicates the utility he obtains from aggregate contributions to the public good, $g_i + g_j$, where $m \geq 0$ denotes the return from aggregate contributions. Since public goods are non-rival in consumption, player *i*'s utility from this good originates from both his own donations, g_i, and in those of player *j*, g_j.

Consider that the wealth of player A, $w_A > 0$, is common knowledge among the players; but that of player B, w_B, is privately observed by player B but not observed by player A. However, player A knows that player B's wealth level w_B can be high (w_B^H) or low (w_B^L) with equal probabilities, where $w_B^H > w_B^L > 0$. Let us next find the Bayesian Nash Equilibrium (BNE) of this public good game where player A is uninformed about the exact realization of parameter w_B.

a) Starting with the privately informed player B, set up his utility maximization problem, and separately find his best response function, first, when $w_B = w_B^H$ and, second, when $w_B = w_B^L$.
b) Find the best response function of the uninformed player A.
c) Using your results in parts (a) and (b), find the equilibrium contributions to the public good by each

player. For simplicity, you can assume that player A's wealth is $w_A = \$14$, and those of player B are $w_B^H = \$20$ when his wealth is high, and $w_B^L = \$10$ when his wealth is low. [*Hint*: You will find one equilibrium contribution for player A, but two equilibrium contributions for player B as his contribution is dependent on his wealth level w_B]

Answer:

Part (a). *Player B with high wealth.* When $w_B = w_B^H$, player B chooses his contribution to the public good $g_B^H \geq 0$ to solve the following utility maximization problem:

$$\max_{g_B^H \geq 0} \quad u_B(g_B^H) = (w_B^H - g_B^H)^{1/2}[m(g_A + g_B^H)]^{1/2}$$

where the first term represents his utility from private uses (that is, the amount of wealth not contributed to the public good), while the second term reflects his utility from total donations to the public good from him and player A.

Differentiating with respect to g_B^H, yields

$$\frac{\partial u_B(g_B^H)}{\partial g_B^H} = -\frac{\sqrt{m}}{2}\sqrt{\frac{g_A + g_B^H}{w_B^H - g_B^H}} + \frac{\sqrt{m}}{2}\sqrt{\frac{w_B^H - g_B^H}{g_A + g_B^H}}$$

Assuming interior solutions $(g_B^H > 0)$, we set the above first order condition equal to zero, which helps us rearrange the above expression as follows

$$g_A + g_B^H = w_B^H - g_B^H$$

Solving for player B's contribution to the public good, g_B^H, yields the following best response function,

$$g_B^H(g_A) = \frac{w_B^H}{2} - \frac{1}{2}g_A$$

which originates at half of his wealth, $\frac{w_B^H}{2}$, and decreases in player A's donation by 1/2. In words, when player A does not contribute to the public good ($g_A = 0$), player B donates half of his wealth, $\frac{w_B^H}{2}$, but for every dollar that player A contributes to the public good, player B reduces his donation by 50 cents.

Player B with low wealth. When $w_B = w_B^L$, player B chooses his contribution to the public good $g_B^L \geq 0$ to solve the following utility maximization problem

$$\max_{g_B^L \geq 0} \quad u_B(g_B^L) = (w_B^L - g_B^L)^{1/2}[m(g_A + g_B^L)]^{1/2}$$

Differentiating with respect to g_B^L, yields

$$\frac{\partial u_B(g_B^L)}{\partial g_B^L} = -\frac{\sqrt{m}}{2}\sqrt{\frac{g_A + g_B^L}{w_B^L - g_B^L}} + \frac{\sqrt{m}}{2}\sqrt{\frac{w_B^L - g_B^L}{g_A + g_B^L}}$$

Assuming interior solutions $(g_B^L > 0)$, we set the above first order condition equal to zero, which helps us simplify the expression as follows

$$g_A + g_B^L = w_B^L - g_B^L$$

After solving for player B's contribution to the public good, g_B^L, we find the following best response function

$$g_B^L(g_A) = \frac{w_B^L}{2} - \frac{1}{2}g_A$$

which originates at half of his wealth, $\frac{w_B^L}{2}$, and decreases in player A's contribution to the public good by 1/2.

Part (b). Player A chooses his contribution to the public good $g_A \geq 0$ to solve the following expected utility maximization problem,

$$\max_{g_A \geq 0} \; EU_A(g_A) = \frac{1}{2}(w_A - g_A)^{1/2}[m(g_A + g_B^L)]^{1/2} + \frac{1}{2}(w_A - g_A)^{1/2}[m(g_A + g_B^H)]^{1/2}$$

since he does not observe player B's wealth level, he does not know whether player B contributes g_B^L (which occurs when his wealth is low) or g_B^H (which happens when his wealth is high).
Differentiating with respect to g_A, yields

$$\frac{\partial u_A(g_A)}{\partial g_A} = \frac{\sqrt{m}}{4}\left[\sqrt{\frac{w_A - g_A}{g_A + g_B^L}} + \sqrt{\frac{w_A - g_A}{g_A + g_B^H}} - \sqrt{\frac{g_A + g_B^L}{w_A - g_A}} - \sqrt{\frac{g_A + g_B^H}{w_A - g_A}}\right]$$

Assuming interior solutions $(g_A > 0)$, we set the above first order condition equal to zero. Rearranging, we find that player A's best response function, $g_A(g_B^L, g_B^H)$, is implicitly defined by the following expression

$$\sqrt{\frac{g_A + g_B^L}{w_A - g_A}} + \sqrt{\frac{g_A + g_B^H}{w_A - g_A}} = \sqrt{\frac{w_A - g_A}{g_A + g_B^L}} + \sqrt{\frac{w_A - g_A}{g_A + g_B^H}}$$

Part (c). At a Bayesian Nash equilibrium (BNE), the contribution of each player constitutes a best response to one another (mutual best response). Hence, we can substitute the best responses of the two types of player B into the best response of player A, to obtain

$$\sqrt{\frac{g_A + \frac{w_B^L - g_A}{2}}{w_A - g_A}} + \sqrt{\frac{g_A + \frac{w_B^H - g_A}{2}}{w_A - g_A}} = \sqrt{\frac{w_A - g_A}{g_A + \frac{w_B^L - g_A}{2}}} + \sqrt{\frac{w_A - g_A}{g_A + \frac{w_B^H - g_A}{2}}}$$

While the expression is rather large, you can notice that it now depends on player A's contribution to the public good, g_A, alone. We can then start to simplify the above expression, obtaining

$$\sqrt{g_A + w_B^L} + \sqrt{g_A + w_B^H} = 2(w_A - g_A)\left(\frac{1}{\sqrt{g_A + w_B^L}} + \frac{1}{\sqrt{g_A + w_B^H}}\right)$$

which is further simplified to

$$\sqrt{(g_A + w_B^L)(g_A + w_B^H)} = 2(w_A - g_A)$$

Squaring both sides yields

$$(g_A + w_B^L)(g_A + w_B^H) = 4(w_A - g_A)^2$$
$$\Rightarrow 3g_A^2 - (8w_A + w_B^L + w_B^H)g_A + 4w_A^2 - w_B^L w_B^H = 0$$

Substituting wealth levels $w_A = \$14$, $w_B^H = \$20$, and $w_B^L = \$10$ into the above quadratic equation, we obtain

$$3g_A^2 - 142g_A + 584 = 0$$

The quadratic equation returns two roots, namely $g_A = 4.55$ or $g_A = 42.78$. However, since player A's contribution to public good cannot exceed his wealth, $w_A = \$14$, we take $g_A^* = 4.55$ as the only feasible solution. Therefore, the contribution made by player 2 when his wealth is high or low are

$$g_B^{H*} = \frac{20 - 4.55}{2} = 7.73$$

and

$$g_B^{L*} = \frac{14 - 4.55}{2} = 4.73$$

As a result, the Bayesian Nash equilibrium (BNE) of this public good game is the triplet of contributions

$$\{g_A^*, g_B^{H*}, g_B^{L*}\} = \{4.55, 7.73, 4.73\}$$