EconS 503 - Advanced Microeconomics II
Handout on Hidden Information and Signaling

Macho-Stadler Ch. 5 Ex. 8-9

A seller and a buyer enter into a relationship to transact a good. The good may or may not break. If it does break, its monetary value to the buyer is b_1, while if it breaks, it is only worth b_2 to the buyer, where $b_2 < b_1$. The probability that the good will break depends on its quality. If it is of good quality, the probability of breakage is q^G, while if it is of bad quality the breakage probability is $q^B > q^G$. Assume that the seller is risk-neutral and that the buyer is risk-averse. The seller proposes a contract that includes the price p at which the good is to be sold and a guarantee g, which is the amount that the seller must pay the buyer should the good break. The utility of the buyer is

$$\begin{aligned} u(b_1 - p) &\quad \text{if the good does not break, and} \\ u(b_2 - p + g) &\quad \text{if it does,} \end{aligned}$$

where $u' > 0$ and $u'' < 0$. The buyer will buy so long as his expected utility is greater than or equal to $u(0)$.

a) Calculate the optimal contract for each quality for the case of symmetric information on quality.

Answer:

Setting up the seller's maximization problem,

$$\max_{p^i, g^i} \; q^i(p^i - g^i) + (1 - q^i)(p^i) \qquad i = \{B, G\}$$

subject to the participation constraint of the buyer

$$q^i u(b_2 - p^i + g^i) + (1 - q^i)u(b_1 - p^i) \geq u(0)$$

The seller considers his expected profit because the good can break (with probability q^i) or not (with probability $1 - q^i$), where the superscript i denotes the good's type $i = \{B, G\}$. We can simplify the objective function to obtain,

$$\max_{p^i, g^i} \; p^i - q^i g^i \quad i = \{B, G\}$$
$$\text{subject to } \; q^i u(b_2 - p^i + g^i) + (1 - q^i)u(b_1 - p^i) \geq u\left(0\right)$$

Taking first-order conditions,

$$\begin{aligned} \partial p^i : &\quad 1 + \lambda\left[-q^i u'(b_2 - p^i + g^i) - (1 - q^i)u'(b_1 - p^i)\right] = 0 \\ \partial g^i : &\quad -q^i + \lambda\left[q^i u'(b_2 - p^i + g^i)\right] = 0 \end{aligned}$$

We can rearrange the second first-order condition, and solve for λ, to obtain

$$\lambda = \frac{1}{u'(b_2 - p^i + g^i)} > 0$$

since $u' > 0$ by definition, $\lambda > 0$ implying that the buyer's participation constraint is binding. Furthermore, we can substitute this result into the first first-order condition, yielding

$$1 = \frac{1}{u'(b_2 - p^i + g^i)} \left[q^i u'(b_2 - p^i + g^i) + (1 - q^i)u'(b_1 - p^i)\right]$$

and rearranging, we have

$$u'(b_2 - p^i + g^i) = u'(b_1 - p^i)$$

Taking the inverse of both sides,

$$b_2 - p^i + g^i = b_1 - p$$
$$\implies g^i = b_1 - b_2$$

We have now found one of our choice variables, g^i. The other variable, p^i can be found by substituting the above result into the participation constraint,

$$q^i u(b_2 - p^i + \underbrace{b_1 - b_2}_{g^i}) + (1 - q^i)u(b_1 - p^i) = u(0)$$

simplifying, we have

$$u(b_1 - p^i) = u(0)$$

and solving for p^i,

$$p^i = b_1 - u^{-1}(u(0))$$

Thus, our optimal contract is

$$(p^i, g^i) = (b_1 - u^{-1}(u(0)), b_1 - b_2) \quad i = \{B, G\}$$

b) Calculate the optimal contracts if the seller knows the quality but the buyer doesn't. Does the contract signal quality? Explain why, or why not.

Answer:

Since the optimal symmetric information contract does not depend on the type-dependent probability q^i, it will continue to be optimal under asymmetric information. No signal needs to be sent. Intuitively, the seller does not price as a function of his private information, and thus, there are no incentives to charge a higher than prescribed price.

For parts (c)-(f), assume now that the buyer is risk-neutral, with utility function $u(x) = x$, while the seller is risk-averse, with von Neumann-Morgenstern utility function $B(\cdot)$, where $B' > 0$ and $B'' < 0$. Note that this is the exact opposite as in previous parts, where the buyer was risk-averse and the seller was risk-neutral.

c) Calculate the optimal contract if information is symmetric.

Answer:

Setting up the seller's maximization problem,

$$\max_{p^i,g^i} \quad q^i B(p^i - g^i) + (1-q^i)B(p^i) \qquad i = \{B, G\}$$

subject to the participation constraint of the buyer,

$$q^i(b_2 - p^i + g^i) + (1-q^i)(b_1 - p^i) \geq 0$$

We can simplify the buyer's participation constraint to obtain,

$$\max_{p^i,g^i} \quad q^i B(p^i - g^i) + (1-q^i)B(p^i) \qquad i = \{B, G\}$$

$$\text{subject to } b_1 - q^i(b_1 - b_2 - g^i) - p^i \geq 0$$

Taking first-order conditions,

$$\begin{aligned}
\partial p^i : &\quad q^i B'(p^i - g^i) + (1-q^i)B'(p^i) - \lambda = 0 \\
\partial g^i : &\quad -q^i B'(p^i - g^i) + \lambda q^i = 0
\end{aligned}$$

We can rearrange the second first-order condition to obtain

$$\lambda = B'(p^i - g^i) > 0$$

This implies that the buyer's participation constraint is binding. Furthermore, we can substitute this result into the first first-order condition, yielding

$$q^i B'(p^i - g^i) + (1-q^i)B'(p^i) = B'(p^i - g^i)$$

Simplifying, we have

$$B'(p^i - g^i) = B'(p^i)$$

Inverting, we obtain our solution

$$\begin{aligned}
p^i - g^i &= p^i \\
&\implies g^i = 0
\end{aligned}$$

and thus, the optimal contract never includes a guarantee. Substituting these results into our participation constraint and simplifying, we obtain an optimal price of

$$p^i = b_1 - q^i(b_1 - b_2)$$

Thus, our optimal contract is

$$(p^i, g^i) = (b_1 - q^i(b_1 - b_2), 0) \quad i = \{B, G\}$$

Intuitively, when the seller is risk-neutral and the buyer risk-averse (parts (a)-(b)), the seller insures the buyer against the risk of breakdowns (i.e., offering a guarantee $g^i > 0$ for all types of goods $i = \{B, G\}$). In contrast, if the seller is risk-averse and the buyer is risk-neutral (as in this part of the exercise), the seller would like to be insured against income swings, so the buyer is now the agent bearing the risk, as no guarantees are offered for either type of good, $g^i = 0$.

d) Are the symmetric information contracts adequate when only the seller knows the quality of the good? Explain why, or why not.

Answer:

No. Both types of seller want to set a price of p^G, but the buyer is only willing to pay p^G if he is sure to get a good quality product. Recall the participation constraint for the low-quality good,

$$b_1 - q^B(b_1 - b_2) - p^B = 0$$

Since $q^B > q^G$, we have that $p^B < p^G$. Hence,

$$b_1 - q^B(b_1 - b_2) - p^G < 0$$

and the participation constraint is violated.

e) Provide an intuitive description of what a seller could do to signal that the good he is selling has low breakdown probability. Is it useful to offer a guarantee?

Answer:

A guarantee of $g > 0$ could be offered, since the guarantee is more costly to the seller of a product when the probability of breakage is high. This reasoning comes from the shape of the seller's utility curve, as a higher p is already on a flatter portion of the curve, and a small increase in g will have less of an effect on a higher price than a lower.

f) When a seller offers a guarantee, he needs to increase the price of the good in order to compensate the possible indemnity should the good break. Show analytically that the price rise required for an increase in guarantee on a low-quality good (such that the seller's utility is constant) is greater than that required for a high-quality good. Given this observation, consider the existence of a separating equilibrium and describe its characteristics.

Answer:

The expected utility of a seller is

$$EU(p^i, g^i) = q^i B(p^i - g^i) + (1 - q^i)B(p^i)$$

Differentiating, we get that in order for EU to remain constant, we must have

$$dEU = \left[q^i B'(p^i - g^i) + (1 - q)B'(p^i)\right] dp^i - q^i B'(p^i - g^i)dg^i = 0$$

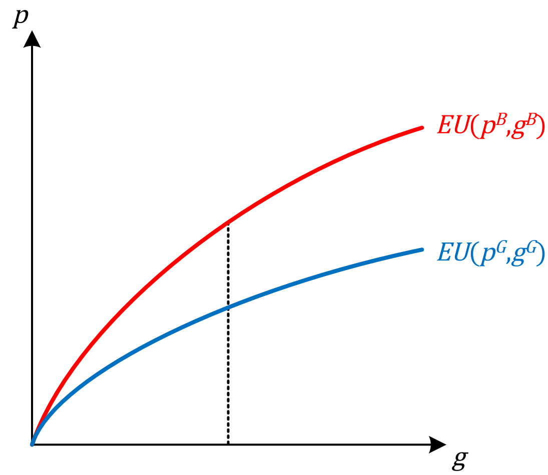
(This total differentiation with respect to p^i and g^i, keeping the expected utility constant $dEU = 0$, is equivalent to the total differentiation we did in consumer theory to find the MRS of an indifference curve, where we allowed for simultaneous changes in goods x and y, but $dU = 0$.) Rearranging the above expression yields

$$\left.\frac{dp^i}{dg^i}\right|_{EU=\text{Constant}} = \frac{q^i B'(p^i - g^i)}{q^i B'(p^i - g^i) + (1 - q)B'(p^i)} = \frac{1}{1 + \frac{1-q^i}{q^i}\frac{B'(p^i)}{B'(p^i - g^i)}}$$

which represents the slope of the seller's indifference curve. Since $q^B > q^G$, we have that $\frac{1-q^B}{q^B} < \frac{1-q^G}{q^G}$, thus implying

$$\left.\frac{dp^B}{dg^B}\right|_{EU=\text{Constant}} > \left.\frac{dp^G}{dg^G}\right|_{EU=\text{Constant}}$$

as depicted in the next figure.



This implies that a seller of low quality should increase his price more than a seller of high quality in order to compensate a greater guarantee. Hence, we find a separating equilibrium in that:

1. The seller with breakage probability q^B sells using his efficient contract

$$g^B = 0 \quad \text{and} \quad p^B = b_1 - q^B(b_1 - b_2)$$

   (This is a characteristic of separating equilibria: The type of principal, which no one is interested in mimicking, offers the same contract as under symmetric information).

2. The seller with breakage probability q^G includes $g^G > 0$. He will set the highest price such that the other type of seller is not interested in changing his own contract (that is, that it is an effective signal of a good-quality product) and that the buyer is interested in accepting (knowing that he is receiving good quality).

Ex. 15.12 (NS)

Signaling with entry accomodation

This question will explore signaling when entry deterrence is impossible, so the signaling firm accommodates its rival's entry. Assume deterrence is impossible because the two firms do not pay a sunk cost to enter or remain in the market. In particular, firm i's demand is given by

$$q_i = a_i - p_i + \frac{p_j}{2},$$

where a_i is product i's attribute (say, quality). Production is costless. Firm 1's attribute can be one of two values: either $a_1 = 1$, in which case we say firm 1 is the low type, or $a_1 = 2$, in which case we say it is the high type. Assume there is no discounting across periods for simplicity.

a) Compute the Nash equilibrium of the game of complete information in which firm 1 is the high type and firm 2 knows that firm 1 is the high type (assume for all parts that $a_2 = 1$).

Answer:

Setting up firm 1's profit maximization problem,

$$\max_{p_1^H \geq 0} p_1^H(2 - p_1^H + \frac{p_2}{2})$$

with first-order condition

$$2 - 2p_1^H + \frac{p_2}{2} = 0$$

Solving for p_1^H, the high type's best-response function is

$$p_1^H(p_2) = 1 + \frac{p_2}{4}$$

Setting up firm 2's profit maximization problem,

$$\max_{p_2 \geq 0} p_2(1 - p_2 + \frac{p_1^H}{2})$$

with first-order condition

$$1 - 2p_2 + \frac{p_1^H}{2} = 0$$

Solving for p_2, firm 2's best response function is

$$p_2(p_1^H) = \frac{1}{2} + \frac{p_1^H}{4}.$$

Solving both best response functions simultaneously yields

$$p_1^{H^*} = \frac{6}{5} = 1.2 \quad \text{and} \quad p_2^* = \frac{4}{5} = 0.8.$$

with corresponding profits of

$$\pi_1^{H^*} = \left(\frac{6}{5}\right)^2 = 1.44 \quad \text{and} \quad \pi_2^* = \left(\frac{4}{5}\right)^2 = 0.64$$

b) Compute the Nash equilibrium of the game in which firm 1 is the low type and firm 2 knows that firm 1 is the low type.

Answer:

This is similar to part (a) in that now, the firms are symmetric, and identical to firm 2. Setting up the profit maximization problem

$$\max_{p_i \geq 0} \; p_i(1 - p_i + \frac{p_j}{2})$$

with first-order condition

$$1 - 2p_i + \frac{p_j}{2} = 0$$

and solving for p_i, we have firm i's best response function

$$p_i(p_j) = \frac{1}{2} + \frac{p_j}{4}$$

Due to symmetry, we know that $p_1^L = p_2 = p$, and our best response function becomes

$$p = \frac{1}{2} + \frac{p}{4} \implies p = \frac{2}{3}$$

Thus, the equilibrium price is

$$p_1^{L^*} = p_2^* = \frac{2}{3} = 0.667$$

with corresponding profits of

$$\pi_1^{L^*} = \pi_2^* = \left(\frac{2}{3}\right)^2 = 0.444$$

c) Solve for the Bayesian-Nash equilibrium of the game of incomplete information in which firm 1 can be either type with equal probability. Firm 1 knows its type, but firm 2 only knows the probabilities.

Answer:

We can pull the best response functions for firm 1 directly from parts (a) and (b). They are

$$p_1^H(p_2) = 1 + \frac{p_2}{4}$$
$$p_1^L(p_2) = \frac{1}{2} + \frac{p_2}{4}$$

To derive firm 2's best response function, we need to set up his expected profit maximization problem,

$$\max_{p_2 \geq 0} \; \frac{1}{2}\left[p_2\left(1 - p_2 + \frac{p_1^H}{2}\right)\right] + \frac{1}{2}\left[p_2\left(1 - p_2 + \frac{p_1^L}{2}\right)\right]$$

Simplifying,

$$\max_{p_2 \geq 0} \; p_2\left(1 - p_2 + \frac{p_1^H}{4} + \frac{p_1^L}{4}\right)$$

Taking first-order conditions yields

$$1 - 2p_2 + \frac{p_1^H}{4} + \frac{p_1^L}{4} = 0$$

and solving for p_2 gives firm 2's best response as a function of p_1^H and p_1^L,

$$p_2(p_1^H, p_1^L) = \frac{1}{2} + \frac{p_1^H}{8} + \frac{p_1^L}{8}$$

This, along with firm 1's best response functions, is a system of 3 equations and 3 unknowns. Solving simultaneously yields

$$p_1^{L*} = \frac{41}{60} = 0.683, \quad p_1^{H*} = \frac{71}{60} = 1.183, \quad \text{and} \quad p_2^* = \frac{44}{60} = 0.733.$$

with corresponding profits of

$$\pi_1^{L*} = \left(\frac{41}{60}\right)^2 = 0.467, \quad \pi_1^{H*} = \left(\frac{71}{60}\right)^2 = 1.400, \quad \text{and} \quad \pi_2^* = \left(\frac{44}{60}\right)^2 = 0.538$$

d) Which of firm 1's types gains from incomplete information? Which type would prefer complete information (and thus would have an incentive to signal its type if possible)? Does firm 2 earn more profit on average under complete or under incomplete information?

Answer:

Firm 2 earns an expected payoff of about $0.542 (= (0.64 + 0.444)/2)$ under complete information and 0.538 under incomplete information, and thus would prefer complete information.

e) Consider a signaling variant of the model that has two periods.

- Firm 1 and 2 choose prices in the first period, when 2 has incomplete information about 1's type. Firm 2 observes firm 1's price in this period and uses the information to update its beliefs about 1's type. Then firms engage in another period of price competition.

Show that there is a separating equilibrium in which each type of firm 1 charges the same prices as computed in part (d). You may assume that, if firm 1 chooses an out-of-equilibrium price in the first period, then firm 2 believes that firm 1 is the low type with probability 1.

- *Hint*: To prove the existence of a separating equilibrium, show that the loss to the low type from trying to pool in the first period exceeds the second-period gain from having convinced firm 2 that it is the high type. Use your answers from parts (a)-(d) where possible to aid in your solution.

Answer:

We need to check that the low type would prefer its equilibrium profit (of 0.467) to the profit from mimicking the high type's price in the first period and then having firm 2 believe it has high costs (thus deterring entry). In particular, the low type earns:

- 0.467 from pricing low in the first period, and 0.444 in the second period (when entry ensues). Thus, its overall profits from pricing low are 0.911.
- 0.217 from pricing high in the first period (recall that pricing strategy entails a deviation from complete information strategies). In the second period, after such a high price, firm 2 believes that firm 1 is of high type, stays out of the market, and firm 1 earns $\left(\frac{7}{10}\right)^2 = 0.49$. Therefore, the overall profits of firm 1 are $0.217 + 0.49 = 0.707$.

Therefore, the overall profits from pricing low are larger than those of pricing high, implying that the low-type firm has no incentives to deviate from low pricing. This result can also be understood by noticing that the first-period loss from pooling, $0.467 - 0.217 = 0.25$, exceeds the second-period gain from pooling, $0.49 - 0.444 = 0.046$, entailing no incentive to deviate.