EconS 503 - Advanced Microeconomics II
Adverse Selection - II

1. Screening between two types of workers

Consider a setting where a principal (firm) seeks to hire an agent (worker), but cannot observe the worker's cost of effort, which ultimately affects the amount of effort that the worker exerts and thus the firm's profits. Hence, the manager would like to know the worker's cost of effort in order to design his salary.
In particular, the firm's profit function is

$$\pi(e, w) = x(e) - w$$

where function $x(e)$ represents the benefit that the manager obtains when the worker supplies e units of effort, which is increasing and concave in effort, $x'(e) \geq 0$ and $x''(e) \leq 0$. In addition, the worker's utility function is:

$$v(w, e|\theta) = u(w) - c(e, \theta)$$

where $u(w)$ denotes his utility from salary w, $u'(w) > 0$, $u''(w) < 0$, and $c(e, \theta)$ represents the worker's cost of exerting e units (e.g. hours) of effort when his type is θ. For simplicity, we assume that the worker can only be of two types, θ_L and θ_H, where $\theta_L < \theta_H$. Intuitively, a high-type worker faces a higher total and marginal cost of effort, i.e., $c'(e, \theta_L) < c'(e, \theta_H)$ for every $e > 0$.

For comparison purposes, we first analyze a complete information setting in which principal can observe the agent's type, and evaluate the optimal contract of salary and exerted effort, (w_i, e_i) where $i = \{H, L\}$ denotes the worker's type. In the following subsection, we examine a more realistic setting where the principal cannot observe the agent's type, but knows that the relative frequency of low-type workers in the population is $p \in (0, 1)$ while that of the high-type is $1 - p$. In such a context, the principal will need to design not a single contract (w_i, e_i) as under complete information, since that could lead some workers to lie about their true type in order to benefit from the contract meant for the other type of worker. Instead we will show that principal will design a menu of contracts, (w_L, e_L) for the low types and (w_H, e_H) for the high types, that induces each type of workers to choose the contract meant for him (self-selection of contracts).

1.1. Complete information

If the principal (firm) knew the agent is type $i = \{H, L\}$, it would solve

$$\max_{w_i, e_i} x(e_i) - w_i$$
$$\text{subject to } u(w_i) - c(e_i, \theta_i) \geq 0$$

When the PC constraint guarantees that the worker willingly accepts the contract since the firm can reduce w_i until PC holds with equality, PC must bind, implying

$$u(w_i) = c(e_i, \theta_i),$$

or inverting $u(\cdot)$ on both sides, $w_i = u^{-1}[c(e_i, \theta_i)]$. This result helps us simplify the above problem to an unconstrained maximization problem with only one choice variable variable (the effort level), as follows

$$\max_{e_i} \; x(e_i) - \underbrace{u^{-1}[c(e_i, \theta_i)]}_{w_i} \tag{1}$$

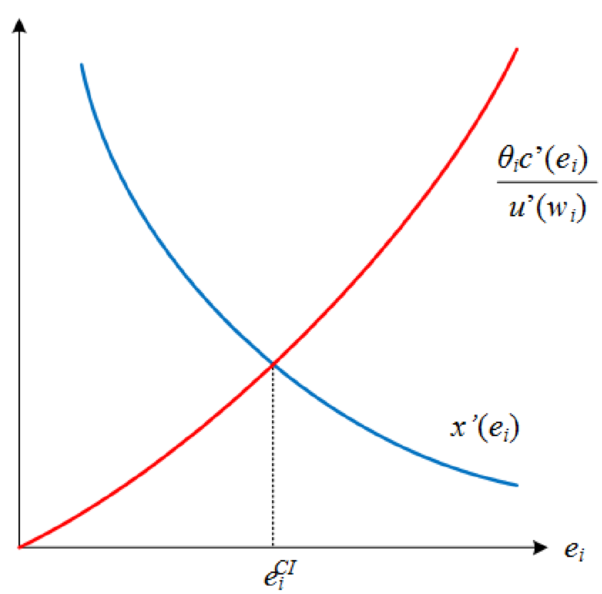
Taking FOC with respect to e_i yields

$$x'(e_i) = \frac{1}{u'(u^{-1}[c(e_i, \theta_i)])} c'(e_i, \theta_i) \tag{2}$$

or more compactly,

$$x'(e_i) = \frac{c'(e_i, \theta_i)}{u'(w)}$$

Hence, effort is increased with the point in which the marginal rate of substitution of effort and wage for the firm (left-hand side, since $MRS_{e,w} = \frac{MU_e}{MU_w} = x'(e)$) coincides with that of the worker (right-hand side). Intuitively, the firm's MRS in this context illustrates by how much the worker's effort needs to increase after a given \$1 increase in wages so that the firm's profits are unaffected. A similar argument applies to the worker, where his MRS represents by how much he can increase effort after receiving a \$1 increase in the salary so that his utility holds constant. The equality between both parties' MRS in the complete information context reveals that increasing/decreasing effort is not Pareto-improving. The following figure separately depicts the left-hand side of the above FOC, $x'(e_i)$, which is decreasing since $x''(e_i) \leq 0$ by definition; and the right-hand side, $\frac{c'(e_i, \theta_i)}{u'(w)}$, which is increasing since $c''(e_i) \geq 0$ by definition. Their crossing point identifies the profit-maximizing effort under complete information, e_i^{CI}.



Example 11.1: Consider a principal and an agent where $p = \frac{1}{2}$, $\theta_L = 1$, $\theta_H = 2$, $x(e_i) = \log(e_i)$, $u(w_i) = w_i$, and cost of effort $c(e_i, \theta_i) = \theta_i e_i^2$. The marginal cost of effort is thus $2\theta_i e_i$, which is positive and increasing in e. Therefore, the principal's profit function is $\pi(w_i, e_i) = \log(e_i) - w_i$ while the agent's utility is $v(w_i, e_i|\theta_i) = w_i - \theta_i e_i^2$.
Under complete information, the FOC we derived earlier becomes

$$x'(e_i) = \frac{c'(e_i, \theta_i)}{u'(w)} \implies \frac{1}{e_i} = \frac{2\theta_i e_i}{1}$$

and solving for e_i yields

$$e_i^2 = \frac{1}{2\theta_i} \implies e_i^{CI} = \left(\frac{1}{2\theta_i}\right)^{\frac{1}{2}}$$

such that wages become

$$w_i = c_i(e_i, \theta_i) = \frac{\theta_i}{2\theta_i} = \frac{1}{2}$$

Plugging in our values for θ_L and θ_H, we obtain optimal contracts of

$$\begin{aligned}(w_H^{CI}, e_H^{CI}) &= \left(\frac{1}{2}, \frac{1}{2}\right) = (0.5, 0.5) \\ (w_L^{CI}, e_L^{CI}) &= \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right) \approx (0.5, 0.707)\end{aligned}$$

Intuitively, the firm will pay both workers the same wage under complete information, but expect a higher effort level from the low cost worker.

1.2. Incomplete information

When the firm cannot observe the worker's type, it seeks to maximize the expected profits by designing a pair of contracts, (w_L, e_L) and (w_H, e_H), which achieve self selection by each type of worker. Hence the firm solves the following profit maximization problem:

$$\begin{aligned}\max_{w_L, e_L, w_H, e_H} \quad & p\,[x(e_L) - w_L] + (1-p)\,[x(e_H) - w_H] \\ \text{subject to} \quad & u(w_H) - c(e_H, \theta_H) \geq 0 && (PC_H) \\ & u(w_L) - c(e_L, \theta_L) \geq 0 && (PC_L) \\ & u(w_H) - c(e_H, \theta_H) \geq u(w_L) - c(e_L, \theta_H) && (IC_H) \\ & u(w_L) - c(e_L, \theta_L) \geq u(w_H) - c(e_H, \theta_L) && (IC_L)\end{aligned}$$

where the first constraint represents the participation constraint for the high-type worker, PC_H, and the second for the low-type worker, PC_L. The third constraint represents the incentive compatibility condition for the high-type worker IC_H so he does not have incentives to choose the contract meant for the low-type worker, and the fourth is the incentive compatibility constraint for the low-type worker IC_L so he prefers the contract meant for him rather than that of the high-type worker. Intuitively, the PC constraints guarantee voluntary participation of all types of workers, while the IC constraints ensure self-selection.

Before finding Kuhn-Tucker conditions, note that PC_L is implied by IC_L and PC_H, since

$$u(w_L) - c(e_L, \theta_L) \underbrace{\geq}_{\text{by } IC_L} u(w_H) - c(e_H, \theta_L) > u(w_H) - c(e_H, \theta_H) \underbrace{\geq}_{\text{by } PC_H} 0$$

where the first inequality originates from IC_L; the second from the assumption on the cost of effort, i.e., $c(e, \theta_L) < c(e, \theta_H)$ for every e; and the third inequality stems from the PC_H condition. As a consequence, we obtain that $u(w_L) - c(e_L, \theta_L) > 0$, as required by PC_L. Therefore, PC_L holds with strict inequality whereas that of the least productive agent, PC_H, binds (which we will prove below).
The Lagrangian of the above maximization problem is

$$\begin{aligned}\mathcal{L} = &\, p\left[x(e_L) - w_L\right] + (1-p)\left[x(e_H) - w_H\right] \\ &+ \lambda_1\left[u(w_H) - c(e_H, \theta_H)\right] \\ &+ \lambda_2\left[u(w_H) - c(e_H, \theta_H) - u(w_L) + c(e_L, \theta_H)\right] \\ &+ \lambda_3\left[u(w_L) - c(e_L, \theta_L) - u(w_H) + c(e_H, \theta_L)\right]\end{aligned}$$

where PC_L is not considered since it holds with strict inequality, as shown above. Taking FOCs, we obtain

$$\begin{aligned}&\frac{\partial \mathcal{L}}{\partial w_L} = -p - \lambda_2 u'(w_L) + \lambda_3 u'(w_L) = 0 \\ &\frac{\partial \mathcal{L}}{\partial w_H} = -(1-p) + \lambda_1 u'(w_H) + \lambda_2 u'(w_H) - \lambda_3 u'(w_H) = 0 \\ &\frac{\partial \mathcal{L}}{\partial e_L} = p x'(e_L) + \lambda_2 c'(e_L, \theta_H) - \lambda_3 c'(e_L, \theta_L) = 0 \\ &\frac{\partial \mathcal{L}}{\partial e_H} = (1-p) x'(e_H) - \lambda_1 c'(e_H, \theta_H) - \lambda_2 c'(e_H, \theta_H) + \lambda_3 c'(e_H, \theta_L) = 0 \\ &\frac{\partial \mathcal{L}}{\partial \lambda_1} = u(w_H) - c(e_H, \theta_H) \leq 0 \\ &\frac{\partial \mathcal{L}}{\partial \lambda_2} = u(w_H) - c(e_H, \theta_H) - u(w_L) + c(e_L, \theta_H) \leq 0 \\ &\frac{\partial \mathcal{L}}{\partial \lambda_3} = u(w_L) - c(e_L, \theta_L) - u(w_H) + c(e_H, \theta_L) \leq 0 \\ &\text{and } \lambda_i \geq 0 \qquad \text{for all } i \in \{1, 2, 3\}\end{aligned}$$

For simplicity, we next consider that the cost of effort takes the form

$$c(e, \theta_i) = \theta_i c(e),\ \forall i = \{H, L\}$$

where $c(e)$ is increasing and convex in effort, $c'(e) \geq 0$ and $c''(e) \geq 0$. (Note that such function satisfies the initial conditions of total and marginal cost of effort being higher for the high than for the low type, i.e., $\theta_H \cdot c(e) > \theta_L \cdot c(e)$ and $\theta_H \cdot c'(e) > \theta_L \cdot c'(e)$ for all $e > 0$.)
Rearranging the first two FOCs,

$$\begin{aligned}-\lambda_2 + \lambda_3 &= \frac{p}{u'(w_L)} \\ \lambda_1 + \lambda_2 - \lambda_3 &= \frac{1-p}{u'(w_H)}\end{aligned}$$

then adding them together yields

$$\lambda_1 = \frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}$$

which gives $\lambda_1 > 0$, implying that the contract associated with the Lagrange multiplier λ_1, PC_H, binds, i.e., $u(w_H) - c(e_H, \theta_H) = 0$.
The third FOC can be rewritten as

$$px'(e_L) = \lambda_3 \theta_L c'(e_L) - \lambda_2 \theta_H c'(e_L)$$

and rearranging, we get

$$\frac{px'(e_L)}{c'(e_L)} = \lambda_3 \theta_L - \lambda_2 \theta_H$$

And the fourth FOC can be represented as

$$(1-p)x'(e_H) = \lambda_1 \theta_H c'(e_H) - \lambda_3 \theta_L c'(e_H) + \lambda_2 \theta_H c'(e_H)$$

which can be rearranged as

$$\frac{(1-p)x'(e_H)}{c'(e_H)} = \lambda_1 \theta_H - (\lambda_3 \theta_L - \lambda_2 \theta_H)$$

Combining the above two FOCs yields

$$\frac{(1-p)x'(e_H)}{c'(e_H)} = \lambda_1 \theta_H - \frac{px'(e_L)}{c'(e_L)}$$

and solving for $\lambda_1 \theta_H$, and using $\lambda_1 = \frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}$ from our above results, we obtain

$$\left[\frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}\right] \theta_H = \frac{px'(e_L)}{c'(e_L)} + \frac{(1-p)x'(e_H)}{c'(e_H)}$$

Moreover, $\lambda_3 > \lambda_2$, since otherwise the first FOC, $(\lambda_3 - \lambda_2)\, u'(w_L) = p$, could not hold. Therefore, $\lambda_3 > 0$, implying that its associated constraint, IC_L, binds. In particular, IC_L can be rewritten as follows:

$$u(w_L) - \theta_L c(e_L) = u(w_H) - \theta_L c(e_H)$$

Rearranging the right side gives

$$u(w_L) - \theta_L c(e_L) = u(w_H) - \theta_H c(e_H) + (\theta_H - \theta_L) c(e_H)$$

and since PC_H binds, $u(w_H) - \theta_H c(e_H) = 0$, the above expression becomes

$$u(w_L) - \theta_L c(e_L) = \underbrace{u(w_H) - \theta_H c(e_H)}_{=0} + (\theta_H - \theta_L) c(e_H) = (\theta_H - \theta_L) c(e_H)$$

entailing that the most efficient agent, θ_L, obtains in equilibrium a positive utility level, $(\theta_H - \theta_L)\, c\,(e_H)$, which increases in his difference with respect to the least efficient worker, $\theta_H - \theta_L$.

In contrast, the incentive compatibility condition of the least efficient worker, IC_H, does not bind, implying that its associated Lagrange multiplier must be zero, $\lambda_2 = 0$. Using this result in the first and third FOCs yields

$$\lambda_3 = \frac{p}{u'(w_L)} \quad \text{and} \quad \frac{px'(e_L)}{c'(e_L)} = \lambda_3 \theta_L$$

or, solving for λ_3 and combining them,

$$\frac{p}{u'(w_L)} = \frac{px'(e_L)}{\theta_L c'(e_L)}$$

which means that, solving for $x'(e_L)$, we obtain

$$x'(e_L) = \frac{\theta_L c'(e_L)}{u'(w_L)}$$

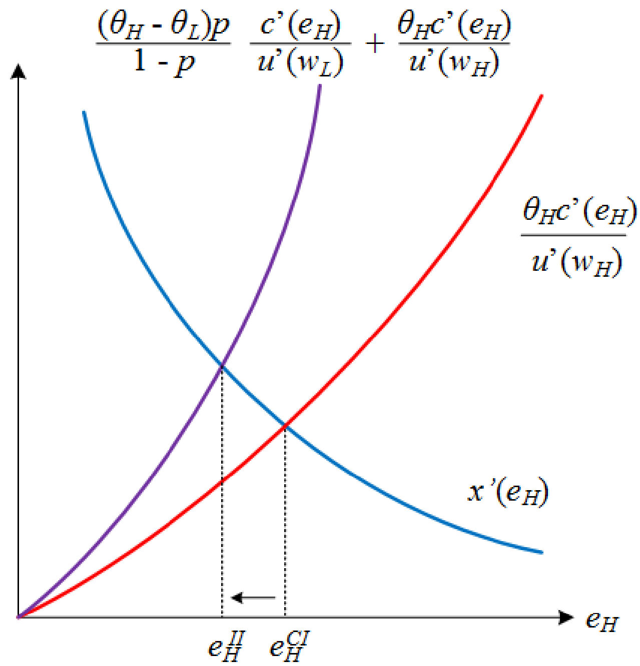
Hence, for the most efficient worker, the equilibrium outcome under incomplete information coincides with the socially optimal result with complete information, namely, effort is increased until the point in which the MRS of the worker coincides with that of the firm. Finally, using $\lambda_1 = \frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}$, $\lambda_2 = 0$, and $\lambda_3 = \frac{p}{u'(w_L)}$ in the fourth FOC yields

$$(1-p)x'(e_H) - \left[\frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}\right] \theta_H c'(e_H) + \frac{p}{u'(w_L)} \theta_L c'(e_H) = 0$$

and rearranging, we obtain,

$$\frac{(\theta_H - \theta_L)p}{1-p} \frac{c'(e_H)}{u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H)$$

This equation entails that effort e_H is not socially optimal, since for that we need $\frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H)$. In other words, the presence of the first term makes e_H suboptimal, and such a term is positive since $\theta_H - \theta_L > 0$, $p \in (0,1)$, $c'(e_H) \geq 0$ and $u'(w_L) > 0$. Hence, $\frac{(\theta_H - \theta_L)pc'(e_H)}{(1-p)u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} > \frac{\theta_H c'(e_H)}{u'(w_H)}$ as depicted in the following figure, implying that the effort level for the high cost worker under incomplete information, e_H^{II}, is lower than that under complete information, e_H^{CI}.



In summary, the pair of contracts (w_L, e_L) and (w_H, e_H) must satisfy the equations we found:

$$u(w_L) - \theta_L c(e_L) = (\theta_H - \theta_L)c(e_H)$$
$$u(w_H) - \theta_H c(e_H) = 0$$
$$x'(e_L) = \frac{\theta_L c'(e_L)}{u'(w_L)}$$
$$\frac{(\theta_H - \theta_L)p}{1-p}\frac{c'(e_H)}{u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H)$$

Monotonicity in effort. Another property is that effort levels satisfy $e_L \geq e_H$, i.e., the worker with the low cost of effort exerts a larger effort level than that with a high cost of effort. In order to show this property, let us combine IC_H and IC_L to obtain

$$u(w_L) - c(e_L, \theta_L) \underbrace{\geq}_{\text{by } IC_L} u(w_H) - c(e_H, \theta_L) \geq u(w_H) - c(e_H, \theta_H) \underbrace{\geq}_{\text{by } IC_H} u(w_L) - c(e_L, \theta_H)$$

where the second inequality is due to the cost of effort satisfying $c(e_H, \theta_L) < c(e_H, \theta_H)$. Hence, the above triple inequality can be rearranged as follows,

$$c(e_H, \theta_L) - c(e_L, \theta_L) \geq u(w_H) - u(w_L) \geq c(e_H, \theta_H) - c(e_L, \theta_H)$$

and multiplying by -1 yields

$$c(e_L, \theta_L) - c(e_H, \theta_L) \leq u(w_L) - u(w_H) \leq c(e_L, \theta_H) - c(e_H, \theta_H)$$

Using $c(e, \theta) = \theta \cdot c(e)$, we obtain

$$\theta_L \left[c(e_L) - c(e_H)\right] \leq w_L - w_H \leq \theta_H \left[c(e_L) - c(e_H)\right]$$

or, more compactly,

$$\theta_L [c(e_L) - c(e_H)] \leq \theta_H [c(e_L) - c(e_H)]$$

Since $\theta_L < \theta_H$ by definition, we must have $c(e_L) \geq c(e_H)$. Therefore, since the cost of effort function increases in e, $c(e_L) \geq c(e_H)$ can only hold if $e_L \geq e_H$. ∎

Example 11.1 (cont'd). Let us now use the same functional forms as in Example 11.1 to calculate the optimal contracts under incomplete information. Taking the FOCs from above, we have

$$\begin{gathered} u(w_L) - \theta_L c(e_L) = (\theta_H - \theta_L)c(e_H) \implies w_L - e_L^2 = e_H^2 \\ u(w_H) - \theta_H c(e_H) = 0 \implies w_H = 2e_H^2 \\ x'(e_L) = \frac{\theta_L c'(e_L)}{u'(w_L)} \implies \frac{1}{e_L} = \frac{2e_L}{1} \\ \frac{(\theta_H - \theta_L)p}{1-p}\frac{c'(e_H)}{u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H) \implies \frac{2e_H}{1} + \frac{4e_H}{1} = \frac{1}{e_H} \end{gathered}$$

which is a system of four equations and four unknowns. We can solve them to obtain our optimal contracts under incomplete information,

$$\begin{aligned} (w_H^{II}, e_H^{II}) &= \left(\frac{1}{3}, \frac{\sqrt{6}}{6}\right) \approx (0.333, 0.408) \\ (w_L^{II}, e_L^{II}) &= \left(\frac{2}{3}, \frac{\sqrt{2}}{2}\right) \approx (0.667, 0.707) \end{aligned}$$

In this context, the utility level that each type of worker obtains under incomplete information is

$$\begin{aligned} v_H^{II} &= w_H - 2e_H^2 = 0 \\ v_L^{II} &= w_L - e_L^2 = \frac{1}{6} \approx 0.167 \end{aligned}$$

which entails that the worker with a low cost of effort captures an information rent of $v_L^{II} - v_L^{CI} = 0.167 - 0 = 0.167 > 0$, while the worker with a high cost of effort does not, i.e., $v_H^{II} = v_H^{CI}$. In addition, we can compare the effort levels and wages between both information contexts as follows:

$$\begin{aligned} e_L^{II} &= 0.707 = 0.707 = e_L^{CI} \\ e_H^{II} &= 0.408 < 0.500 = e_H^{CI} \\ w_L^{II} &= 0.667 > 0.500 = w_L^{CI} \\ w_H^{II} &= 0.333 < 0.500 = w_H^{CI} \end{aligned}$$

Intuitively, the most efficient (i.e., low cost) type has no distortion of effort but earns a positive information rent in order for him to reveal his type (i.e., to choose the contract meant for him). Whereas, the least efficient (i.e., high cost) type provides less effort, and therefore, receives a lower wage in equilibrium, and he is indifferent between participating or not.

2. Regulation and Two Types of Firms

(Based on Macho-Stadler Ch. 4 Ex 15) In this exercise we consider a risk-neutral government that wants to establish a policy of subsidies to firms that carry out efforts to decontaminate. Let e stand for the decontaminating effort. The cost to a firm of the decontamination action is ce^2, where c is a parameter whose value depends on the type of firm at hand. The government's policy consists in a certain decontamination level e and a transfer t that the firm will receive if the decontamination has been carried out. The firm will not accept the subsidy scheme if it does not at least cover the costs. The firm is risk-neutral, and so the utility it earns from accepting the subsidy scheme is

$$t - ce^2.$$

The government bears in mind the social benefits of decontamination, valued at $2e$. On the other hand, the government prefers to pay out the lowest subsidy possible to the firm, and so a payment of t implies a disutility to the government of pt, where $p \in (0,1)$ (it costs the government $(1+p)$ dollars to collect \$1, and hence the social utility of transfering t dollars to a firm is $t - (1-p)\,t$). Given this, the government's objective function is

$$B(e,t) = 2e - pt.$$

2.1. Complete information

Calculate the level of decontamination $e^*(c)$ and transfer $t^*(c)$ that the government would propose to a firm whose decontamination cost is ce^2, when the government knows c.

The government sets up its payoff maximization problem

$$\max_{e(c),t(c)} \quad 2e(c) - pt(c)$$

subject to the participation constraint

$$t(c) - c(e(c))^2 \geq 0$$

(*Note*: There are several ways to approach this part of the problem. A simple argument that the participation constraint binds to maximize social welfare is sufficient, but for the purpose of this exercise, we will solely be using Kuhn-Tucker conditions to prove binding constraints)
Taking first-order conditions yields

$$\begin{aligned} 2 - 2\lambda ce(c) &= 0 \\ -p + \lambda &= 0 \end{aligned}$$

As can be seen in the second first-order condition, $\lambda = p > 0$, which implies that the participation constraint is binding. Thus, we can substitute this value into the first first-order condition to obtain our optimal value for $e(c)$

$$\begin{aligned} 2 - 2pce(c) = 0 \\ \implies e(c) = \frac{1}{pc} \end{aligned}$$

and substitute this value back into the participation constraint to obtain our optimal value of $t(c)$.

$$t(c) = c\left(\frac{1}{pc}\right)^2 = \frac{1}{p^2c}$$

2.2. Incomplete information

Assume now that the government cannot observe c, but knows it can take only one of two values, c_L and c_H with equal probability where $c_L < c_H$. Calculate the menu of decontamination and transfer levels that the government would propose. Interpret the results.

The government sets up its expected payoff maximization problem

$$\max_{e_L, e_H, t_L, t_H} \quad \frac{1}{2}(2e_L - pt_L) + \frac{1}{2}(2e_H - pt_H)$$

subject to the participation and incentive compatibility constraints

$$\begin{aligned} t_L - c_L e_L^2 &\geq 0 && (PC_L) \\ t_H - c_H e_H^2 &\geq 0 && (PC_H) \\ t_L - c_L e_L^2 &\geq t_H - c_L e_H^2 && (IC_L) \\ t_H - c_H e_H^2 &\geq t_L - c_H e_L^2 && (IC_H) \end{aligned}$$

We next use three different approaches to show that constraint PC_L is redundant, and that IC_L and PC_H bind (i.e., hold with equality). We then plug IC_L and PC_H into the objective function to simplify our maximization program before taking first-order conditions. For simplicity, we start with the approach discussed several times in class.

Approach 1: Redundant Constraint. We can show that PC_L is implied by the other three constraints. For example, we have

$$t_L - c_L e_L^2 \underbrace{\geq}_{\text{by } IC_L} t_H - c_L e_H^2 \underbrace{>}_{\text{since } c_L < c_H} t_H - c_H e_H^2 \underbrace{\geq}_{\text{by } PC_H} 0$$

Hence, we can delete that constraint, leaving us with

$$\begin{aligned} t_H - c_H e_H^2 &\geq 0 && (PC_H) \\ t_L - c_L e_L^2 &\geq t_H - c_L e_H^2 && (IC_L) \\ t_H - c_H e_H^2 &\geq t_L - c_H e_L^2 && (IC_H) \end{aligned}$$

Taking first-order conditions yields

$$1 - 2\lambda_2 c_L e_L + 2\lambda_3 c_H e_L = 0 \tag{1}$$
$$1 - 2\lambda_1 c_H e_H + 2\lambda_2 c_L e_H - 2\lambda_3 c_H e_H = 0 \tag{2}$$
$$-\frac{p}{2} + \lambda_2 - \lambda_3 = 0 \tag{3}$$
$$-\frac{p}{2} + \lambda_1 - \lambda_2 + \lambda_3 = 0 \tag{4}$$

Rearranging,

$$\lambda_2 c_L - \lambda_3 c_H = \frac{1}{2e_L} \tag{1}$$
$$\lambda_1 c_H - \lambda_2 c_L + \lambda_3 c_H = \frac{1}{2e_H} \tag{2}$$
$$\lambda_2 - \lambda_3 = \frac{p}{2} \tag{3}$$
$$\lambda_1 - \lambda_2 + \lambda_3 = \frac{p}{2} \tag{4}$$

Adding first-order conditions (3) and (4) together, we obtain

$$\lambda_1 = p > 0$$

which implies that PC_H binds. Also, if IC_L were not to bind, we would have a contradiction in first-order condition 1 ($\lambda_3 < 0$). Hence, PC_H and IC_L must be our binding constraints.

Approach 2: Mechanism Design Approach. We can use the information from the complete information section to determine which type of constaint binds. Recall that under complete information,

$$(e_i, t_i) = \left(\frac{1}{pc_i}, \frac{1}{p^2 c_i}\right)$$

Plugging these values into IC_H gives

$$\begin{aligned} t_H - c_H e_H^2 &\geq t_L - c_H e_L^2 \\ \frac{1}{p^2 c_H} - c_H \left(\frac{1}{pc_H}\right)^2 &\geq \frac{1}{p^2 c_L} - c_H \left(\frac{1}{pc_L}\right)^2 \\ 0 &\geq \frac{1}{p^2 c_L} - \frac{c_H}{c_L}\frac{1}{p^2 c_L} = \frac{1}{p^2 c_L}\left(1 - \frac{c_H}{c_L}\right) \end{aligned}$$

and since $c_L < c_H$, this constraint is satisfied. This implies that the high type consumer would prefer his own contract, and not the contract of the low type individual, and thus, his incentive compatibility constraint will not bind. Plugging these values into IC_L gives

$$\begin{aligned} t_L - c_L e_L^2 &\geq t_H - c_L e_H^2 \\ \frac{1}{p^2 c_L} - c_L \left(\frac{1}{pc_L}\right)^2 &\geq \frac{1}{p^2 c_H} - c_L \left(\frac{1}{pc_H}\right)^2 \\ 0 &\geq \frac{1}{p^2 c_H} - \frac{c_L}{c_H}\frac{1}{p^2 c_H} = \frac{1}{p^2 c_H}\left(1 - \frac{c_L}{c_H}\right) \end{aligned}$$

and since $c_L < c_H$, this constraint is violated. This implies that the low type consumer would prefer to pretend to be a high type individual, and thus, his incentive compatibility constraint must bind. Lastly, we look at participation constraints, and since the low type would always prefer the high type's contract, we know that he will enter the market whenever the high type will enter. Thus PC_L does not bind and PC_H binds.

Approach 3: Kuhn-Tucker Conditions. Taking first-order conditions,

$$1 - 2\lambda_1 c_L e_L - 2\lambda_3 c_L e_L + 2\lambda_4 c_H e_L = 0 \quad (1)$$
$$1 - 2\lambda_2 c_H e_H + 2\lambda_3 c_L e_H - 2\lambda_4 c_H e_H = 0 \quad (2)$$
$$-\frac{p}{2} + \lambda_1 + \lambda_3 - \lambda_4 = 0 \quad (3)$$
$$-\frac{p}{2} + \lambda_2 - \lambda_3 + \lambda_4 = 0 \quad (4)$$

and rearranging,

$$\lambda_1 c_L + \lambda_3 c_L - \lambda_4 c_H = \frac{1}{2e_L} \quad (1)$$
$$\lambda_2 c_H - \lambda_3 c_L + \lambda_4 c_H = \frac{1}{2e_H} \quad (2)$$
$$\lambda_1 + \lambda_3 - \lambda_4 = \frac{p}{2} \quad (3)$$
$$\lambda_2 - \lambda_3 + \lambda_4 = \frac{p}{2} \quad (4)$$

We know that at most, one constraint for each type of firm will bind in equilibrium. We will now proceed to test cases.

Case 1: $\lambda_1, \lambda_3 > 0$, $\lambda_2 = \lambda_4 = 0$. In this case, consider first-order condition (2)

$$-\lambda_3 c_L = \frac{1}{2e_H}$$

this implies that $\lambda_3 < 0$, which is a contradiction.

Case 2: $\lambda_1, \lambda_4 > 0$, $\lambda_2 = \lambda_3 = 0$. In this case, IC_H binds. Substituting into PC_H, we have

$$t_L - c_H e_L^2 \geq 0$$

and PC_L binding, which implies

$$t_L - c_L e_L^2 = 0$$

Howerver, $c_L < c_H$ by definition, which implies

$$t_L - c_H e_L^2 < 0$$

which contradicts PC_H.

Case 3: $\lambda_2, \lambda_4 > 0$, $\lambda_1 = \lambda_3 = 0$. In this case, consider first-order condition (1)

$$-\lambda_4 c_H = \frac{1}{2e_L}$$

this implies that $\lambda_4 < 0$, which is a contradiction.

Case 4: $\lambda_2, \lambda_3 > 0$, $\lambda_1 = \lambda_4 = 0$. In this case, IC_L and PC_H bind.

Continuing with our maximization problem...
Using either of the above three approaches, we obtain that IC_L and PC_H bind. Rearranging these two constraints, we have

$$t_H = c_H e_H^2 \tag{PC_H}$$
$$t_L = (c_H - c_L)e_H^2 + c_L e_L^2 \tag{IC_L}$$

We can substitute these expressions back into our objective function

$$\max_{e_L, e_H} \quad \frac{1}{2}(2e_L - p\left[(c_H - c_L)e_H^2 + c_L e_L^2\right]) + \frac{1}{2}(2e_H - pc_H e_H^2)$$

Taking first-order conditions with respect to e_L and e_H yields

$$1 - pc_L e_L = 0$$
$$-p(c_H - c_L)e_H + 1 - pc_H e_H = 0$$

which is a system of two equations and two unknowns. Solving for e_L and e_H, we obtain

$$e_L = \frac{1}{pc_L}$$
$$e_H = \frac{1}{p(2c_H - c_L)}$$

and using our binding constraints, we can obtain our transfers

$$t_H = \frac{c_H}{p^2(2c_H - c_L)^2}$$
$$t_L = \frac{c_H - c_L}{p^2(2c_H - c_L)^2} + \frac{1}{p^2 c_L}$$

Intuitively, the low cost firm would try to pretend to have high costs in order to induce a higher transfer from the government. Thus, in exchange for revealing its type, it extracts an information rent, and its transfer under incomplete information is higher than under complete information. At the same time, the high cost firm finds itself having to exert less effort and receives less of a transfer under the asymmetric information case.