EconS 503 - Advanced Microeconomics II
Handout on Game Theory (Basics)

1. MWG 7.E.1

Consider the two-player game whose extensive form representation (excluding payoffs) is depicted in the figure 1 below.

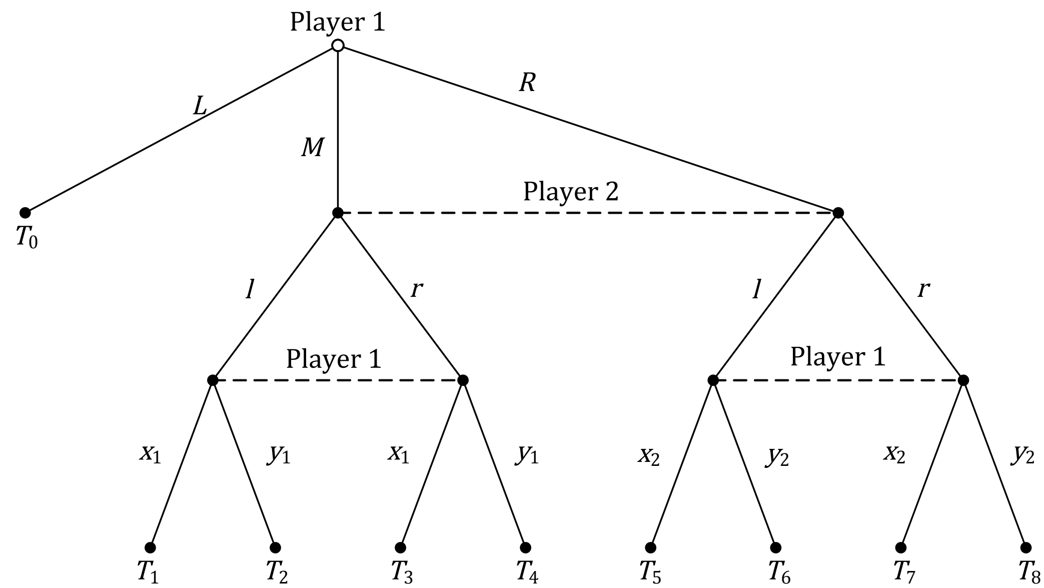

Figure 1

a) What are player 1's possible strategies? Player 2's?

Answer:

In order to specify a strategy for player 1, we need to determine all of his possible actions from the three information sets in which he moves. Thus a typical strategy for player 1 can be written as a triple. The set of strategies for player 1 are:

$$S_1 = \left\{ \begin{array}{c} (L, x_1, x_2), (L, x_1, y_2), (L, y_1, x_2), (L, y_1, y_2), (M, x_1, x_2), (M, x_1, y_2), \\ (M, y_1, x_2), (M, y_1, y_2), (R, x_1, x_2), (R, x_1, y_2), (R, y_1, x_2), (R, y_1, y_2) \end{array} \right\}$$

If player 1 uses strategy (L, x_1, y_2), he plays L at the root of the game, x_1 in his information set following action M (we refer to this information set at "Information Set 2") and y_2 in his information set following action R (we refer to this information set as "Information Set 3").

Similarly, player 2's strategies specify her actions at her single information set (we refer to this information set as "Information Set 1") Thus,

$$S_2 = \{(l), (r)\}.$$

b) Show that for any behavior strategy that player 1 might play, there is realization equivalent mixed strategy; that is, a mixed strategy that generates the same probability distribution over the terminal nodes for *any* mixed strategy choice by player 2.

Answer:

A behavior strategy for player 1 consists of a randomization of his possible moves at each information set in which he has to move. Suppose that at the root, player 1 plays L, M, and R with probabilities of p_1, p_2, and p_3 respectively $(p_1 + p_2 + p_3 = 1)$; at information set 2, player 1 plays x_1, y_1 with probabilities of q_1 and q_2 respectively $(q_1 + q_2 = 1)$; at information set 3, player 1 plays x_2, y_2 with probabilities of s_1 and s_2 respectively.

Assume that player 2 plays l and r with probabilities $\sigma(l)$ and $\sigma(r)$ respectively $(\sigma(l) + \sigma(r) = 1)$. Thus, if player 1 is using the above behavioral strategy and player 2 is using this mixed strategy, the probability that we reach each terminal node will be:

$$\begin{array}{lll} \Pr(T_0) = p_1 & \Pr(T_1) = p_2\sigma(l)\, q_1 & \Pr(T_2) = p_2\sigma(l)\, q_2 \\ \Pr(T_3) = p_2\sigma(r)\, q_1 & \Pr(T_4) = p_2\sigma(r)\, q_2 & \Pr(T_5) = p_3\sigma(l)s_1 \\ \Pr(T_6) = p_3\sigma(l)\, s_2 & \Pr(T_7) = p_3\sigma(r)\, s_1 & \Pr(T_8) = p_3\sigma(r)\, s_2. \end{array}$$

Now the following mixed strategy for player 1 is realization equivalent to the above behavior strategy:

(L, x_1, x_2) with probability p_1, (M, x_1, x_2) with probability $p_2 q_1$. (M, y_1, x_2) with probability $p_2 q_2$,

(R, x_1, x_2) with probability $p_3 s_1$, (R, x_1, y_2) with probability $p_3 s_2$

[Note: $p_1 + p_2 q_1 + p_2 q_2 + p_3 s_1 + p_3 s_2 = p_1 + p_2 (q_1 + q_2) + p_3 (s_1 + s_2) = 1$]

Why these values? Note that for strategy (M, x_1, x_2) there are only two possible outcomes: T_1 if player 2 plays l or T_3 if player 2 plays r. Hence, the total probability associated with strategy (M, x_1, x_2) is

$$\Pr(T_1) + \Pr(T_3) = p_2\sigma(l)q_1 + p_2\sigma(r)q_1 = p_2 q_1(\sigma(l) + \sigma(r)) = p_2 q_1$$

If player 1 is using the above mixed strategy and player 2 is using the mixed strategy σ, the probability that we reach each terminal node will be

$$\begin{aligned}
\Pr(T_0) &= \Pr(L, x_1, x_2) = p_1 \quad \Pr(T_1) = \Pr(M, x_1, x_2) \cdot \Pr(l) = p_2 q_1 \sigma(l) \\
\Pr(T_2) &= \Pr(M, y_1, x_2) \cdot \Pr(l) = p_2 q_2 \sigma(l) \quad \Pr(T_3) = \Pr(M, x_1, x_2) \cdot \Pr(r) = p_2 q_1 \sigma(r) \\
\Pr(T_4) &= \Pr(M, y_1, x_2) \cdot \Pr(r) = p_2 q_2 \sigma(r) \quad \Pr(T_5) = \Pr(R, x_1, x_2) \cdot \Pr(l) = p_3 s_1 \sigma(l) \\
\Pr(T_6) &= \Pr(R, x_1, y_2) \cdot \Pr(l) = p_3 s_2 \sigma(l) \quad \Pr(T_7) = \Pr(R, x_1, x_2) \cdot \Pr(r) = p_3 s_1 \sigma(r) \\
\Pr(T_8) &= \Pr(R, x_1, y_2) \cdot \Pr(r) = p_3 s_2 \sigma(r)
\end{aligned}$$

which is the same as shown before in the behavior strategy (Note that many of the strategies could be replaced with another. For example, $\Pr(M, x_1, x_2) = \Pr(M, x_1, y_2)$ since information set 3 is never reached. Hence, it is irrelevant which strategy we choose for that set and either can be used). Therefore, the above mixed strategy is realization equivalent to the behavior strategy.

c) Show that the converse is also true: For any mixed strategy that player 1 might play, there is a realization equivalent to the behavior strategy.

Answer:

(L, x_1, x_2) with probability p_1; (L, x_1, y_2) with probability p_2; (L, y_1, x_2) with probability p_3, (L, y_1, y_2) with probability p_4; (M, x_1, x_2) with probability p_5; (M, x_1, y_2) with probability p_6, (M, y_1, x_2) with probability p_7; (M, y_1, y_2) with probability p_8; (R, x_1, x_2) with probability p_9, (R, y_1, x_2) with probability p_{10}; (R, x_1, y_2) with probability p_{11}; (R, y_1, y_2) with probability p_{12} [$p_i \geq 0$ for all i and $\sum p_i = 1$]

If player 2 uses the mixed strategy σ, the probability that we reach each terminal node will be:

$$\begin{aligned}
&\Pr(T_0) = p_1 + p_2 + p_3 + p_4; \ \Pr(T_1) = (p_5 + p_6)\,\sigma(l); \ \Pr(T_2) = (p_7 + p_8)\,\sigma(l); \\
&\Pr(T_3) = (p_5 + p_6)\,\sigma(r); \ \Pr(T_4) = (p_7 + p_8)\,\sigma(r); \ \Pr(T_5) = (p_9 + p_{10})\,\sigma(l); \\
&\Pr(T_6) = (p_{11} + p_{12})\,\sigma(l); \ \Pr(T_7) = (p_9 + p_{10})\,\sigma(r); \ \Pr(T_8) = (p_{11} + p_{12})\,\sigma(r).
\end{aligned}$$

The following behavioral strategy for player 1 is realization equivalent: At the root of the game, player 1 plays L, M, R with probabilities of $(p_1 + p_2 + p_3 + p_4), (p_5 + p_6 + p_7 + p_8)$ and $(p_9 + p_{10} + p_{11} + p_{12})$ respectively; at information set 2, player 1 plays x_1, y_1 with probabilities of $\frac{p_5+p_6}{(p_5+p_6+p_7+p_8)}$ and $\frac{p_7+p_8}{(p_5+p_6+p_7+p_8)}$ respectively; at information set 3, player 1 plays x_2, y_2 with probabilities of $\frac{p_9+p_{10}}{(p_9+p_{10}+p_{11}+p_{12})}$ and $\frac{p_{11}+p_{12}}{(p_9+p_{10}+p_{11}+p_{12})}$ respectively. For example

$$\begin{aligned}
\Pr(T_4) &= \Pr(M) \cdot \Pr(r) \cdot \Pr(y_1) = (p_5 + p_6 + p_7 + p_8) \cdot \sigma(r) \cdot \frac{p_7 + p_8}{p_5 + p_6 + p_7 + p_8} \\
&= (p_7 + p_8)\sigma(r) \text{ (same value as above)}
\end{aligned}$$

d) Suppose that we change the game by merging the information sets at player 1's second round of moves (so that all four nodes are now in a single information set). Argue that the game is no longer one of perfect recall. Which of the two results in (*b*) and (*c*) still holds?

Answer:

Note that if player 1 reaches his (only) information set after player 2 moves, he will not remember whether he chose M or R. Thus, the game is not of perfect recall.

The result of part (b) still holds: there exists a mixed strategy for player 1 which is realization equivalent to any behavior strategy. Suppose player 1 uses the following behavior strategy: At information set 1, player 1 plays L, M, R with probabilities of p_1, p_2 and p_3 respectively: at information set 2, player 1 plays x, y with probabilities of q_1 and q_2 respectively. If player 2 is using the mixed strategy σ, then the probability that we reach each terminal node will be:

$$\Pr(T_0) = p_1; \Pr(T_1) = p_2\sigma(l)\,q_1; \Pr(T_2) = p_2\sigma(l)\,q_2; \Pr(T_3) = p_2\sigma(r)\,q_1; \Pr(T_4) = p_2\sigma(r)\,q_2;$$
$$\Pr(T_5) = p_3\sigma(l)\,q_1;\ \ \Pr(T_6) = p_3\sigma(l)\,q_2; \Pr(T_7) = p_3\sigma(r)\,q_1; \Pr(T_8) = p_3\sigma(r)\,q_2.$$

The following mixed strategy for player 1 is realization equivalent: (L, x) with probability p_1, (M, x) with probabilityp_2q_1, (M, y) with probability p_2q_2, (R, x) with probability p_3q_1, (R, y) with probability p_3q_2.
However, there does not always exist a behavior strategy that is realization equivalent to a mixed strategy. Consider the following example. Player 1 uses the mixed strategy playing (M, x) and (R, y) both with parobability $\frac{1}{2}$. Player 2 uses the pure strategy (l). Suppose there exist a behavior strategy for player 1 which is realization equivalent to the mixed strategy: at the root of the game, player 1 plays L, M, R with probabilities of p_1, p_2 and p_3 respectively: at his information set after player 2 moves, player 1 plays x, y with probabilities of q_1 and q_2 respectively. The mixed strategy generates the following distribution over the terminal nodes:

$$\Pr(T_1) = \Pr(T_6) = \tfrac{1}{2}$$
$$\Pr(T_0) = \Pr(T_2) = \Pr(T_3) = \Pr(T_4) = \Pr(T_5) = \Pr(T_7) = \Pr(T_8) = 0$$

The behavior strategy generates:

$$\Pr(T_3) = \Pr(T_4) = \Pr(T_7) = \Pr(T_8) = 0$$
$$\Pr(T_0) = p_1,\ \ \Pr(T_1) = p_2q_1,\ \ \Pr(T_2) = p_2q_2, \Pr(T_5) = p_3q_1,\ \ \Pr(T_6) = p_3q_2$$

in order for these distributions to be equivalent, we need: $\Pr(T_1) = p_2q_1 = \frac{1}{2} \Rightarrow p_2$ and $q_1 \neq 0$, $\Pr(T_2) = p_2q_2 = 0 \Rightarrow q_2 = 0$ since $p_2 \neq 0$, $\Pr(T_6) = p_3q_2 = \frac{1}{2}$ which cannot hold since $q_2 = 0$, a contradiction. There exists no behavior strategy that is realization equivalent to the above mixed strategy.
In a game that is not of perfect recall the following holds:

- For any behavior strategy there exists a mixed strategy that is realization equivalent,
- This proof does not show the general case of mixed strategies implying that there exists a realization equivalent behavior strategy. [Note: for a general proof of these results refer to Fudenberg and Tirole (1991), *Game Theory.* MIT press, p. 87]

2. MWG 8.C.4

Consider a game Γ_N with players 1, 2 and 3 in which $S_1 = \{L, M, R\}, S_2 = \{U, D\}$, and $S_3 = \{l, r\}$. Player 1's payoffs from each of his three strategies conditional on the strategy choices of players 2 and 3 are dipicted as (u_L, u_M, u_R) in each of the four boxes shown below, where $(\pi, \varepsilon, \eta) >> 0$. Assume that $\eta < 4\varepsilon$.

Player 3's Strategy

		l	r
Player 2's Strategy	U	$\pi + 4\varepsilon, \pi - \eta, \pi - 4\varepsilon$	$\pi - 4\varepsilon, \pi + \frac{\eta}{2}, \pi + 4\varepsilon$
	D	$\pi + 4\varepsilon, \pi + \frac{\eta}{2}, \pi - 4\varepsilon$	$\pi - 4\varepsilon, \pi - \eta, \pi + 4\varepsilon$

A strategy for player 2 is to play U with probability α and D with probability $1 - \alpha$, and for player 3 is to play with l with probability β and r with probability $1 - \beta$. Denote by P_A the expected payoff of player 1 when action $A \in \{L, M, R\}$ is taken given α and β. Direct calculation and simple algebra yield:

$$\begin{aligned} P_L &= \pi + 4\varepsilon (2\beta - 1) \\ P_M &= \pi + \left(\tfrac{3\alpha + 3\beta}{2} - 3\alpha\beta - 1\right)\eta \\ P_R &= \pi + 4\varepsilon (1 - 2\beta) \end{aligned}$$

a) Argue that (pure) strategy M is never a best response for player 1 to any independent randomizations by players 2 and 3.

Answer:

To show that M is never a best response to any pair of strategies of players 2 and 3, (α, β), we have three cases:

$\left[\textbf{Case 1: } \beta > \frac{1}{2}\right]$
Note that in this case $\frac{\partial P_M}{\partial \alpha} = \eta\left[\frac{3}{2} - 3\beta\right] < 0$. Thus the highest payoff for player 1 if he plays M is obtained when $\alpha = 0$, because $\alpha \in [0, 1]$. His payoff will be $P_M(\alpha = 0) = \pi + \eta\left[\frac{3}{2}\beta - 1\right] < \pi + 4\varepsilon\left[\frac{3}{2}\beta - 1\right] < \pi + 4\varepsilon[2\beta - 1] = P_L$. Further note that P_L is independent of α, so that these inequailities hold for all α. Therefore, M cannot be a best response in this case.

$\left[\textbf{Case 2: } \beta < \frac{1}{2}\right]$
Now, $\frac{\partial P_M}{\partial \alpha} > 0$, the highest payoff for player 1 if he plays M is obtained when $\alpha = 1$, and his payoff is $P_M(\alpha = 1) = \pi + \eta\left[\frac{3}{2} + \frac{3}{2}\beta - 3\beta - 1\right] = \pi + \eta\left[\frac{1}{2} - \frac{3}{2}\beta\right] < \pi + \eta\left[\frac{1}{2} - \frac{3}{2}\beta + \frac{1}{2} - \beta\right] < \pi + 4\varepsilon[1 - 2\beta] = P_R$. Further note that P_R is independent of α, so that these inequalities hold for all α. Therefore, M cannot be a best response in this case

$\left[\textbf{Case 3: } \beta = \frac{1}{2}\right]$
In this case $P_M = \pi - \frac{\eta}{4} < \pi = P_R = P_L$. This concludes that M can never be a best response.

b) Show that (pure) strategy M is not strictly dominated.

Answer:

Suppose in negation that there exists a mixed strategy, in which player 1 plays R with probability γ and L with probability $1-\gamma$, that strictly dominates M.
$\left[\textbf{Case 1: } \gamma \leq \frac{1}{2}\right]$
If $\beta = 0$ and $\alpha = 1$ then $P_M = \pi + \frac{\eta}{2} > \pi$. The mixed strategy will give a payoff of $\pi - 4\varepsilon(1-2\gamma) \leq \pi < P_M$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case.

$\left[\textbf{Case 2: } \gamma > \frac{1}{2}\right]$
If $\beta = 1$ and $\alpha = 0$ then $P_M = \pi + \frac{\eta}{2} > \pi$. The mixed strategy will give a payoff of $\pi + 4\varepsilon(1-2\gamma) \leq \pi < P_M$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case. This implies a contradiction, so that M cannot be strictly dominated.

Alternative Proof:
Assume that there exists a mixed strategy of L and R that strictly dominates strategy M. For strategy M to be strictly dominated, we must find a single value for γ for which the mixed strategy of L and R strictly dominates strategy M. We can reduce our normal-form game to show the new payoffs

Player 3's Strategy

		l	r
Player 2's Strategy	U	$\pi-\eta, \pi-4\varepsilon+8\varepsilon\gamma$	$\pi+\frac{\eta}{2}, \pi+4\varepsilon-8\varepsilon\gamma$
	D	$\pi+\frac{\eta}{2}, \pi-4\varepsilon+8\varepsilon\gamma$	$\pi-\eta, \pi+4\varepsilon-8\varepsilon\gamma$

where the first term represents Player 1's payoff from playing strategy M and the second term represents the linear combination of strategies L and R which represents Player 1's payoff from selecting the mixed strategy of L and R. Hence, we need the four conditions

$$\begin{aligned}
\pi - 4\varepsilon + 8\varepsilon\gamma &> \pi - \eta \\
\pi - 4\varepsilon + 8\varepsilon\gamma &> \pi + \frac{\eta}{2} \\
\pi + 4\varepsilon - 8\varepsilon\gamma &> \pi + \frac{\eta}{2} \\
\pi + 4\varepsilon - 8\varepsilon\gamma &> \pi - \eta
\end{aligned}$$

to all hold for at least 1 value of γ. Looking at conditions 2 and 3, we have

$$\begin{aligned}
\pi - 4\varepsilon + 8\varepsilon\gamma &> \pi + \frac{\eta}{2} \implies \gamma > \frac{1}{2} + \frac{\eta}{16\varepsilon} \\
\pi + 4\varepsilon - 8\varepsilon\gamma &> \pi + \frac{\eta}{2} \implies \gamma < \frac{1}{2} - \frac{\eta}{16\varepsilon}
\end{aligned}$$

and it is clear that there is no value for γ in which both of these conditions hold at the same time. Hence, we have reached a contradiction and a mixed strategy of L and R does not exist that strictly dominates strategy M.

c) Show that (pure) strategy M can be a best response if player 2's and player 3's randomizations are allowed to be correlated.

Answer:

Suppose players correlate in the following way: Players 2 and 3 play (U, r) with probability $\frac{1}{2}$ and (D, l) with probability $\frac{1}{2}$.

Dl (Pr $= \frac{1}{2}$)			Ur (Pr $= \frac{1}{2}$)		
$\pi + 4\varepsilon$	$\pi + \frac{\eta}{2}$	$\pi - 4\varepsilon$	$\pi - 4\varepsilon$	$\pi + \frac{\eta}{2}$	$\pi + 4\varepsilon$
L	M	R	L	M	R

Any mixed strategy for player 1 involving only L and R will give him a payoff of π.

$$\begin{aligned} & EU_1(\gamma L + (1-\gamma)R) \\ = \; & \gamma(\frac{1}{2}(\pi + 4\varepsilon) + \frac{1}{2}(\pi - 4\varepsilon)) + (1-\gamma)(\frac{1}{2}(\pi - 4\varepsilon) + \frac{1}{2}(\pi + 4\varepsilon)) = \pi \end{aligned}$$

However, playing M will yield him a payoff of $\pi + \frac{\eta}{2}$.

$$EU_1(M) = \frac{1}{2}(\pi + \frac{\eta}{2}) + \frac{1}{2}(\pi + \frac{\eta}{2}) = \pi + \frac{\eta}{2}$$

Thus M is a best-response to the above correlated strategy of player 2 and 3.

3. Applying IDSDS in three-player games

Consider the following anti-coordination game in figure 2. played by three potential entrants seeking to enter into a new industry, such as the development of software applications for smartphones. Every firm (labeled as A, B, and C) has the option of entering or staying out (i.e., remain in the industry they have been traditionally operating, e.g., software for personal computers). The normal form game in figure 2 depicts the market share that each firm obtains, as a function of the entering decision of its rivals. Firms simultaneously and independently choose whether or not to enter. As usual in simultaneous-move games with three players, the triplet of payoffs describes the payoff for the row player (firm A) first, for the column player (firm B) second, and for the matrix player (firm C) third. Find the set of strategy profiles that survive the iterative deletion of strictly dominated strategies (IDSDS). Is the equilibrium you found using this solution concept unique?

Firm C chooses Enter

Firm A \ Firm B	Enter	Stay Out
Enter	14,24,32	8,30,27
Stay Out	30,16,24	13,12,50

Firm C chooses Stay Out

Firm A \ Firm B	Enter	Stay Out
Enter	16,26,30	31,16,24
Stay Out	31,23,14	14,26,32

Figure 2. Normal-form representation of a three-player game

Answer:

We can start by looking at the payoffs for firm C (the matrix player). [Recall that the application of IDSDS is insensitive to the deletion order. Thus, we can start deleting strictly dominated strategies for the row, column or matrix player, and still reach the same equilibrium result.] In particular, let us compare the third payoff of every cell across both matrices. Figure 3 provides a visual illustration of how to do this pairwise comparison across matrices.

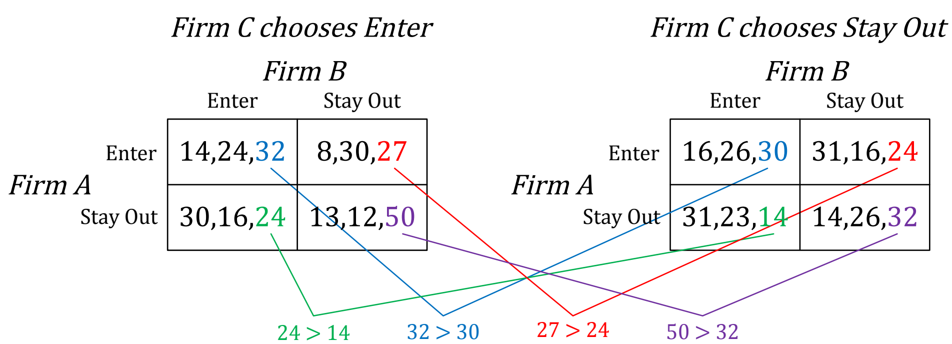

Figure 3. Pairwise payoff comparison for firm C

We find that for firm C (matrix player), entering strictly dominates staying out, i.e., $u_C(s_A, s_B, E) > u_C(s_A, s_B, O)$ for any strategy of firm A, s_A, and firm B, s_B, $32 > 30, 27 > 24, 24 > 14$ and $50 > 32$ in the pairwise payoff comparison depicted in figure 3. This allows us to delete the right-hand side matrix (corresponding to firm C choosing to stay out) since it would not be selected by firm C. We can, hence, focus on the left-hand matrix alone (where firm C chooses to enter), which we reproduce in figure 4.

		Firm B: Enter	Firm B: Stay Out
Firm A	Enter	14,24,32	8,30,27
	Stay Out	30,16,24	13,12,50

Figure 4. Reduced Normal-form game.

We can now check that entering is strictly dominated for the row player (firm A), i.e., $u_A(E, s_B, E) < u_A(O, s_B, E)$ for any strategy of firm B, s_B, once we take into account that firm C selects its strictly dominant strategy of entering. Specifically, firm A prefers to stay out both when firm B enters (in the left-hand column, since $30 > 14$), and when firm B stays out (in the right-hand column, since $13 > 8$). In other words, regardless of firm B's decision, firm A prefers to stay out. This allows us to delete the top row from the above matrix, since the strategy "Enter" would never be used by firm A, which leaves us with a single row and two columns, as illustrated in figure 5.

		Firm B: Enter	Firm B: Stay Out
Firm A	Stay Out	30,16,24	13,12,50

Figure 5. Reduced Normal-form game.

Once we have done that, the game becomes an individual-decision making problem, since only one player (firm B) must select whether to enter or stay out. Since entering yields a payoff of 16 to firm B, while staying out only entails 12, firm B chooses to enter, given that it regards staying out as a strictly dominated strategy, i.e., $u_B(O, E, E) > u_B(O, O, E)$ where we fix the strategies of the other two firms at their strictly dominant strategies: staying out for firm A and entering for firm C. We can thus delete the column corresponding to staying out in the above matrix, as depicted in figure 6.

		Firm B: Enter
Firm A	Stay Out	30,16,24

Figure 6.

As a result, the only surviving cell (strategy profile) that survives the application of the iterative deletion of strictly dominated strategies (IDSDS) is that corresponding to (Stay Out, Enter, Enter), which predicts that firm A stays out, while both firms B and C choose to enter.