EconS 503 - Advanced Microeconomics II
Handout on Perfect Bayesian Equilibrium

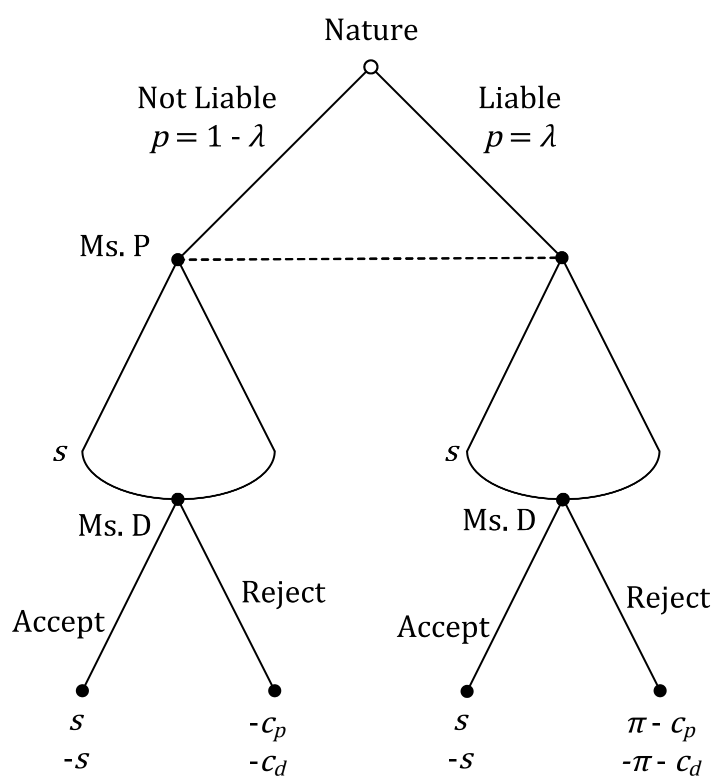
1. Based on MWG 9.C.4

A plaintiff, Ms. P, files a suit against Ms. D (the defendant). If Ms. P wins, she will collect π dollars in damages from Ms. D. Ms. D knows the likelihood that Ms. P will win, $\lambda \in [0,1]$, but Ms. P does not (Ms. D might know if she was actually at fault). They both have strictly positive costs of going to trial of c_p and c_d.
Suppose pretrial settlement negotiations work as follows: Ms. P makes a take-it-or-leave-it settlement offer (s, a dollar amount) to Ms. D. If Ms. D accepts, she pays Ms. P and the game is over. If she does not accept, they go to trial.

a) What are the (pure strategy) weak perfect Bayesian equilibria of this game?

Answer:

We can express the extensive form of this game as follows.



Using backward induction, we know that if Ms. D knows she will be held liable (on the right-hand side) she will accept any settlements $-s \geq -\pi - c_d$, or $s \leq \pi + c_d$. Thus, Ms. P will set $s = \pi + c_d$. By similar logic, when Ms. D won't be held liable (on the left-hand side), she will accept if $s \leq c_d$ and Ms. P will set $s = c_d$. From here, Ms. P must decide which settlement offer to make based on her beliefs on whether Ms. D will be held liable. Similar to the market for lemons, Ms. D will accept any settlement $s = c_d$ regardless of whether she is going to be held liable or not, but will only accept $s = \pi + c_d$ if she is going to be held liable. Thus, Ms. P's expected payoff from each settlement offer can be expressed as

$$\begin{aligned} EU_p(c_d) &= (1-\lambda)(c_d) + \lambda(c_d) = c_d \\ EU_p(\pi + c_d) &= (1-\lambda)(-c_p) + \lambda(\pi + c_d) \end{aligned}$$

comparing these two expected payoffs, and letting $\bar{\lambda}$ solve them with equality, we have

$$\bar{\lambda} = \frac{c_d + c_p}{c_d + c_p + \pi}$$

and Ms. P will choose $s = \pi + c_d$ if $\lambda \geq \bar{\lambda}$. Otherwise, she will choose $s = c_d$.

b) What effects do change in c_p, c_d, and π have?

Answer:

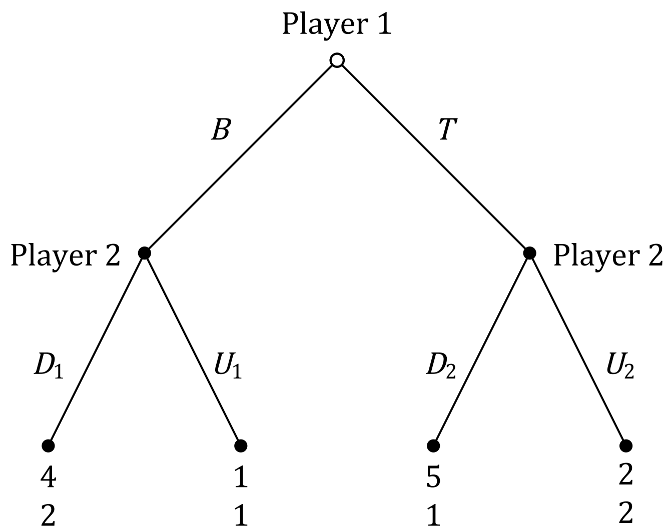
The parameters all affect the cutoff value, $\bar{\lambda}$. Taking derivatives,

$$\begin{aligned} \frac{\partial \bar{\lambda}}{\partial c_d} &= \frac{\partial \bar{\lambda}}{\partial c_p} = \frac{\pi}{(c_d + c_p + \pi)^2} > 0 \\ \frac{\partial \bar{\lambda}}{\partial \pi} &= -\frac{c_d + c_p}{(c_d + c_p + \pi)^2} < 0 \end{aligned}$$

which implies that if either participants court costs rise, Ms. P will more likely want to make a smaller offer and settle out of court. If the amount of damages increases, Ms. P will less likely want to make a settlement offer, and accept the additional risk that she'll be taken to court.

2. MWG 9.C.7

Consider the extensive form game depicted below.



a) Find a subgame perfect Nash equilibrium of this game. Is it unique? Are there any other Nash equilibria (not necessarily subgame perfect, or in pure strategies)?

Answer:

The set of pure strategies for player 1 is $S_1 = \{B, T\}$, and for player 2 is

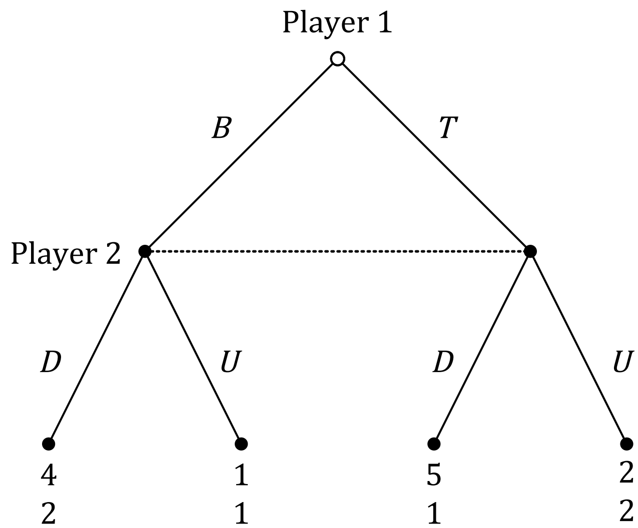
$$S_2 = \{D_1 D_2, D_1 U_2, U_1 D_2, U_1 U_2\}.$$

By backward induction it is easy to see that the unique SPNE is $(B, D_1 U_2)$. There are two more NE: (i) Player 1 plays T, and player 2 plays $U_1 U_2$ with probability p and $D_1 U_2$ with probability $1 - p$, with $p \geq \frac{2}{3}$, and (ii) Player 1 plays B, and player 2 plays $D_1 U_2$ with probability p and $D_1 D_2$ with probability $1 - p$, with $p \geq \frac{1}{3}$.

b) Now suppose that player 2 cannot observe player 1's move. Write down the new extensive form. What is the set of Nash equilibria?

Answer:

We can now represent the game as shown below



which is equivaled to a simultaneous move game with the following normal form

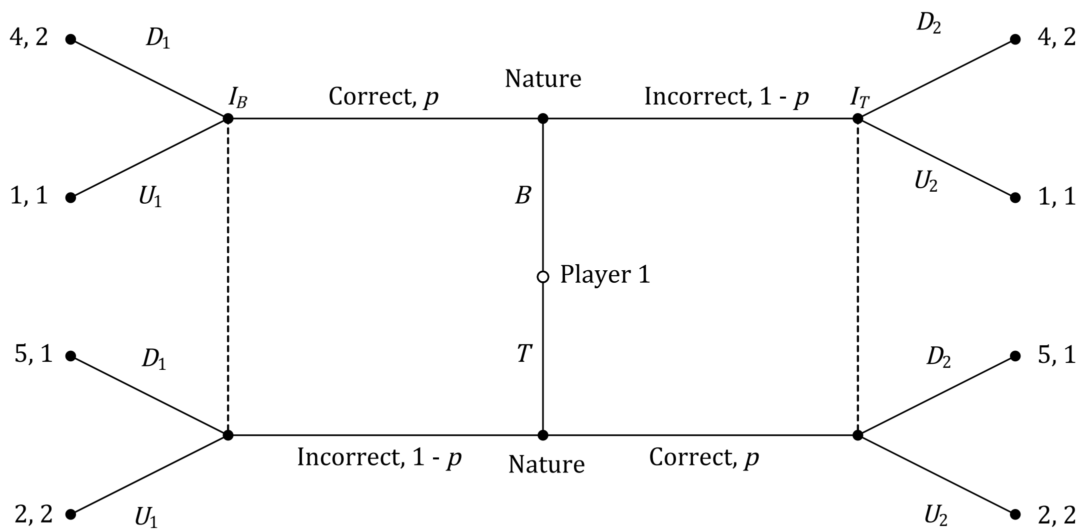
		Player 2	
		D	U
Player 1	B	$4, 2$	$1, 1$
	T	$5, 1$	$2, 2$

It is clear that strategy B is strictly dominated by strategy T for player 1, and then strategy D is strictly dominated by strategy U for player 2 in the reduced form game. Thus, the only Nash equilibrium in this game is (T, U).

c) Now suppose that player 2 observes player 1's move correctly with probability $p \in (0, 1)$ and incorrectly with probability $1 - p$ (e.g., if player 1 plays T, player 2 observes T with probability p and observes B with probability $1 - p$). Suppose that player 2's propensity to observe incorrectly (i.e., given by the value of p) is common knowledge to the two players. What is the extensive form now? Show that there is a unique weak perfect Bayesian equilibrium. What is it?

Answer:

The extensive form of the game now becomes



I_k denotes player 2's information set after she observes $k \in \{B, T\}$, r is the probability she assigns to the event that player 1 played B after she finds herself in formation set I_B, and similarly q is the probability she assigns to the event that player 1 played B after she finds herself in information set I_T. Let $s \in [0, 1]$ denote the probability that player 1 plays B. We can have three possible situations in a WPBE: First, player 1 playing $s = 1$; second, player 1 playing $s = 0$; and third, player 1 playing $s \in (0, 1)$. Player 1 playing $s = 1$ cannot be part of a WPBE. Indeed, if this were the case we must have $q = r = 1$, which implies that player 2 will always play D. But given that 2 always plays D, player 1 will prefer to deviate and play T. Second, player 1 playing $s = 0$ is part of a WPBE. Indeed, if this is the case we must have $q = r = 0$, which implies that player 2 will always play U, and given that 2 always plays U, player 1 will prefer to play T. Thus, player 1 playing T and player 2 playing U in each of her information sets is a WPBE.

To consider the possibility of a WPBE with $s \in (0, 1)$, we first note that this will induce a unique pair of probability beliefs q and r derived by Bayes rule. In particular, in such an equilibrium we must have:

$$\begin{aligned} r &= \frac{s \cdot p}{(1-s)(1-p) + s \cdot p}, \quad \text{and} \\ q &= \frac{s(1-p)}{s(1-p) + p(1-s)}. \end{aligned}$$

Simple algebra shows that $s \underset{<}{\overset{>}{=}} p$ if and only if $q \underset{<}{\overset{>}{=}} \frac{1}{2}$, and that $s \underset{<}{\overset{>}{=}} (1-p)$ if and only if $r \underset{<}{\overset{>}{=}} \frac{1}{2}$. This observation allows us to concentrate on 4 cases as follows:

i) $s > p$ and $s > (1-p)$: In this case we must have $q > \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will always play D, which in turn implies that player 1's best response is $s = 0$. Therefore there cannot be a WPBE in this case.

ii) $s < p$ and $s < (1-p)$: In this case we must have $q < \frac{1}{2}$ and $r < \frac{1}{2}$. This implies that player 2 will always play U, which in turn implies that player 1's best response is $s = 0$. This coincides with the pure strategy WPBE described earlier.

iii) $(1-p) < s < p$: (which implies $p > \frac{1}{2}$) In this case we must have $q < \frac{1}{2}$ and $r > \frac{1}{2}$. This implies that player 2 will play U in information set I_T and will play D in information set I_B. Player 1's best response will now depend on p. Playing B will give player 1 an expected payoff of $4p + 1(1-p)$, and playing T will give him $2p + 5(1-p)$. If $p \neq \frac{2}{3}$ then player 1 will have a unique best response which rules out such WPE. However, if $p = \frac{2}{3}$ then we have a mixed strategy WPBE as follows: player 1 plays B with probability $s \in \left(\frac{1}{3}, \frac{2}{3}\right)$, and player 2 will play U in information set I_T and will play D in information set I_B.

iv) $p < s < (1-p)$: (which implies $p < \frac{1}{2}$) This case is symmetric to case (iii) above. If $p \neq \frac{1}{3}$ then player 1 will have a unique best response which rules out such WPBE. However, if $p = \frac{1}{3}$ the we have a mixed strategy WPBE as follows: player 1 plays B with probability $s \in \left(\frac{1}{3}, \frac{2}{3}\right)$, and player 2 will play D in information set I_T and will play U in information set I_B.

To conclude, there exists a unique pure strategy WPBE as described earlier, and if p is randomly drawn from the interval $(0, 1)$ then the pure strategy WPBE is the unique WPBE with probability 1. However, if $p = \frac{1}{3}$ or $p = \frac{2}{3}$ then in addition there exists a mixed strategy WPBE as described in cases (iii) and (iv) above.

3. Signalling with a Spaniard - Based on *The Princess Bride*

In *The Princess Bride*, the Dread Pirate Roberts is climbing up a rocky cliff in pursuit of Princess Buttercup. At the top of the cliff awaits Inigo Montoya, who has been ordered to duel Roberts to the death once he ascends the cliff. Being impatient, Montoya would like Roberts to hurry up the cliff, and has offered to throw down a rope to assist with the climb. The Dread Pirate Roberts does not know if he can trust Inigo Montoya, however, and must decide whether to accept or reject the help. Accepting help from an untrustworthy Montoya would likely injure Roberts since he will likely fall (We'll assume not to his death though, since he is the Dread Pirate Roberts, after all!).

The game proceeds as follows: First, Nature determines whether Inigo Montoya is trustworthy (with probability $p = 0.5$) or not. Montoya can then send a signal to The Dread Pirate Roberts of "I could give you my word as a Spaniard" (denoted as "Spaniard") or "I swear on the soul of my father, Domingo Montoya, you will reach the top alive" (denoted as "Father"). Finally, Roberts chooses whether to accept or reject the help from Montoya.

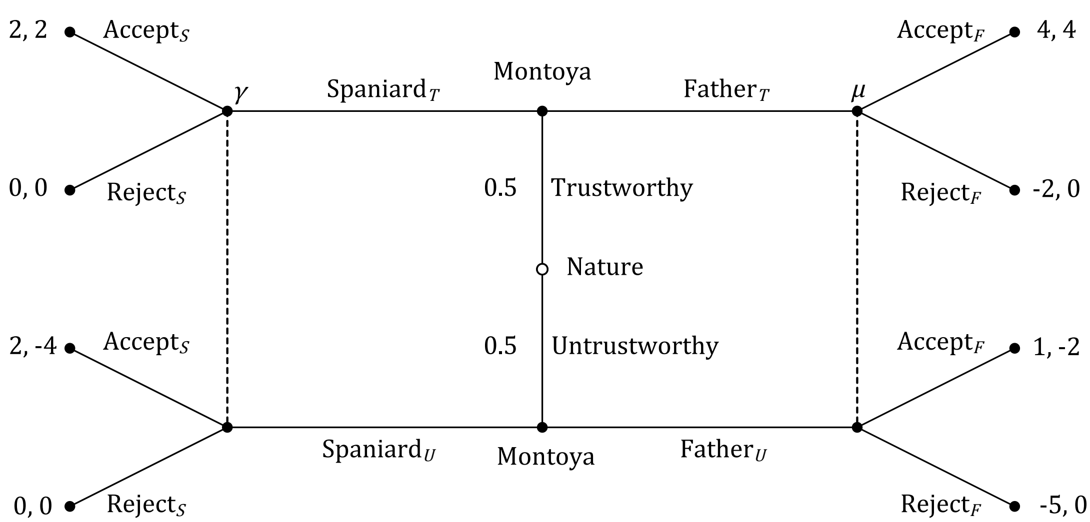
Inigo Montoya's payoffs are as follows. He receives 2 if his help is accepted and 0 otherwise. Montoya takes great pride in his father and receives a benefit of 2 for invoking his father's honor. If, however, he is untrustworthy, he must pay a cost of 3, as he has brought his father shame. Likewise, if he invokes his father's soul and his help is rejected, he must pay a cost of 2 due to the offense he receives.

The Dread Pirate Roberts receives 2 for accepting help from a trustworthy Montoya, -4 from accepting help from an untrustworthy Montoya, and 0 from rejecting help. If Roberts hears the story of Montoya's father, he receives an additional benefit of 2 if he accepts help since the story moves him.

a) Draw the extensive form of the game. Let γ represent Roberts' belief that Montoya is trustworthy given that he has invoked being a Spaniard and μ represent Roberts' belief that Montoya is trustworthy given that he has invoked his father's soul.

Answer:

The extensive form of the game is as follows:



b) For what values of γ can the pooling strategy (Father$_T$Father$_U$) be supported as a perfect Bayesian equilibrium?

Answer:

When Montoya uses the pooling strategy (Father$_T$Father$_U$), Roberts' belief that Montoya is trustworthy given that he has invoked his father's soul is equal to the prior probabilities, i.e., $\mu = 0.5$. This can be shown with Bayes Rule, as follows.

$$\mu = \frac{0.5 * \Pr(Father|T)}{0.5 * \Pr(Father|T) + 0.5 * \Pr(Father|U)} = \frac{0.5 * 1}{0.5 * 1 + 0.5 * 1} = 0.5$$

Roberts then chooses his response based on whichever action gives the highest expected value

$$\begin{aligned} EU_R(Accept|Father) &= 4\mu - 2(1-\mu) = 1 \\ EU_R(Reject|Father) &= 0\mu + 0(1-\mu) = 0 \end{aligned}$$

Thus, Roberts will accept the help from Montoya. Off the equilibrium, Roberts is not able to determine his beliefs if he were to observe a Spaniard signal, i.e.,

$$\gamma = \frac{0.5 * \Pr(Father|T)}{0.5 * \Pr(Father|T) + 0.5 * \Pr(Father|U)} = \frac{0.5 * 0}{0.5 * 0 + 0.5 * 0} = \frac{0}{0}$$

which implies that $\gamma \in (0, 1)$. From the Father signal, however, it can be shown that Montoya will only wish to deviate if a Spaniard signal is met with acceptance (The Untrustworthy Montoya would deviate, obtaining a payoff of 2 as opposed to 1 from signalling Father). Thus, if Roberts' best response is to never accept assistance upon receiving a Spaniard signal, this equilibrium is a PBE. Never accepting will be a best response for Roberts if

$$\begin{aligned} EU_R(Accept|Spaniard) &< EU_R(Reject|Spaniard) \\ 2\gamma - 4(1-\gamma) &< 0\gamma + 0(1-\gamma) \\ \gamma &< \frac{2}{3} \end{aligned}$$

Therefore, for values of $\gamma < \frac{2}{3}$, the strategy profile ($\text{Father}_T\text{Father}_U$) can be supported as a PBE.

As an extension, practice calculating the values for μ for which the pooling strategy profile ($\text{Spaniard}_T\text{Spaniard}_U$) can be supported as a PBE. The solution is $\mu < \frac{1}{3}$.