EconS 424 - Games with Incomplete Information II and Auction Theory

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April 28, 2014

Watson, Ch. 26 Exercise 2

- Two players have to simultaneously and independently decide how much to contribute to a public good. If player 1 contributes x_1 and player 2 contributes x_2 then the value of the public good is

$$v = 2(x_1 + x_2 + x_1 x_2),$$

  which they each receive. Assume that x_1 and x_2 are positive numbers.
- Player 1 must pay a cost x_1^2 of contributing; thus, player 1's payoff in the game is:

$$u_1 = 2(x_1 + x_2 + x_1 x_2) - x_1^2$$

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- Player 2 pays the cost tx_2^2 so that player 2's payoff is:

$$u_2 = 2(x_1 + x_2 + x_1 x_2) - tx_2^2$$

The number t is private information to player 2; player 1 does not observe the precise value of t, but knows that:

$$t = 2 \text{ with probability } \frac{1}{2}, \text{ and } t = 3 \text{ with probability } \frac{1}{2}.$$

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- Compute the Bayesian Nash equilibrium of this game.

Watson, Ch. 26 Exercise 2

- This exercise is very similar to the exercise on Cournot competition where one firm is privately informed about its marginal cost of production we did in class.
  - Recall that in the exercise on Cournot competition, we first focused on the privately informed agent (we found its BRF for each possible type) which we can then combine with our results from the expected utility maximization problem of the uninformed player. Here we will follow a similar methodology.
- Let us hence focus first on the informed player (player 2). For the informed player 2 we have to analyze two cases, when he is low type and high type.

- **Low Type:**

$$u_{2L} = 2(x_1 + x_{2L} + x_1 x_{2L}) - 2x_{2L}^2$$

For this type of player 2 the F.O.C. with respect to x_{2L} that maximizes his utility is:

$$2 + 2x_1 - 4x_{2L} = 0$$

Solving for x_{2L}, we obtain

$$x_{2L}^* = \frac{1}{2} + \frac{1}{2}x_1 \tag{1}$$

This is player 2's BRF when his costs of contributing to the public good are low.

- **High type:**

$$u_{2H} = 2(x_1 + x_{2H} + x_1 x_{2H}) - 3x_{2H}^2$$

For this type of player 2 the F.O.C. with respect to x_{2H} that maximizes his utility is:

$$2 + 2x_1 - 6x_{2H} = 0$$

Solving for x_{2H}:

$$x_{2H}^* = \frac{1}{3} + \frac{1}{3}x_1 \tag{2}$$

This is player 2's BRF when his costs of contributing to the public good are high.

Watson, Ch. 26 Exercise 2

- Let us now examine the uninformed player (player 1).
- Player 1's utility depends on the particular contribution that player 2 makes, and such contribution is potentially different for player 2 when he is low type or high type.
- Since player 1 does not observe player 2's type, he must maximize his expected utility taking into account the probability that player 2 is low and high type.

- In particular, player 1' sexpected payoff is given by the expected value $E(v)$ minus the cost $c(x)$:

$$E(u_1) = E(v) - c(x)$$

where $E(v)$ is the sum of the payoff for player 1 when player 2 is low type, times the probability that this event happens $(1/2)$, plus the payoff for player 1 when player 2 is high type, times the probability that this event happens $(\frac{1}{2})$, as follows:

$$E(v) = \frac{1}{2}\left[2(x_1 + x_{2L} + x_1 x_{2L})\right] + \frac{1}{2}\left[2(x_1 + x_{2H} + x_1 x_{2H})\right]$$

- Then:

$$E(u_1) = \frac{1}{2}[2(x_1 + x_{2L} + x_1 x_{2L})] + \frac{1}{2}[2(x_1 + x_{2H} + x_1 x_{2H})] - x_1^2$$

Simplifying

$$E(u_1) = (x_1 + x_{2L} + x_1 x_{2L}) + (x_1 + x_{2H} + x_1 x_{2H}) - x_1^2$$

and

$$E(u_1) = 2x_1 + x_{2L} + x_1 x_{2L} + x_{2H} + x_1 x_{2H} - x_1^2$$

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- Thus, it is easy to find the value of x_1 for which player 1 maximizes his expected utility:

$$\max_{x_1} \left[2x_1 + x_{2L} + x_1 x_{2L} + x_{2H} + x_1 x_{2H} - x_1^2\right]$$

Taking F.O.C. with respect to x_1 we obtain:

$$2 + x_{2L} + x_{2H} - 2x_1 = 0$$

And solving for x_1:

$$x_1^* = \frac{1}{2}(2 + x_{2L} + x_{2H}) \qquad (3)$$

This is player 1's BRF. Note that there is only one, since he cannot condition on player 2's type being high or low (since player 1 cannot observe such information).

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- We have now a system of three equations (1,2 and 3) and three unknowns $(x_1^*, x_{2L}^*, x_{2H}^*)$ that we can solve by substituting (x_{2L}^*, x_{2H}^*) into x_1^*, as follows:

$$x_1^* = \frac{1}{2}(2 + x_{2L}^* + x_{2H}^*)$$

By inserting x_{2L}^* from expression (1) and x_{2H}^* from expression (2), we obtain

$$x_1^* = \frac{1}{2}\left(2 + \left(\frac{1}{2} + \frac{1}{2}x_1^*\right) + \left(\frac{1}{3} + \frac{1}{3}x_1^*\right)\right)$$

- We can now simplify this expression, as follows:

$$x_1^* = 1 + \frac{1}{4} + \frac{1}{4}x_1^* + \frac{1}{6} + \frac{1}{6}x_1^*$$

$$x_1^* = \left(1 + \frac{1}{4} + \frac{1}{6}\right) + \left(\frac{1}{4} + \frac{1}{6}\right)x_1^*$$

$$x_1^* = \frac{34}{24} + \frac{10}{24}x_1^*$$

$$x_1^* \frac{14}{24} = \frac{34}{24}$$

$$x_1^* = \frac{17}{7}$$

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- Plugging this result into x_{2L}^* from expression (1) and x_{2H}^* from expression (2), we obtain:

$$x_{2L}^* = \frac{1}{2} + \frac{1}{2}x_1^* = \frac{1}{2} + \frac{1}{2}\left(\frac{17}{7}\right) = \frac{12}{7}$$

$$x_{2H}^* = \frac{1}{3} + \frac{1}{3}x_1^* = \frac{1}{3} + \frac{1}{3}\left(\frac{17}{7}\right) = \frac{8}{7}$$

Thus,

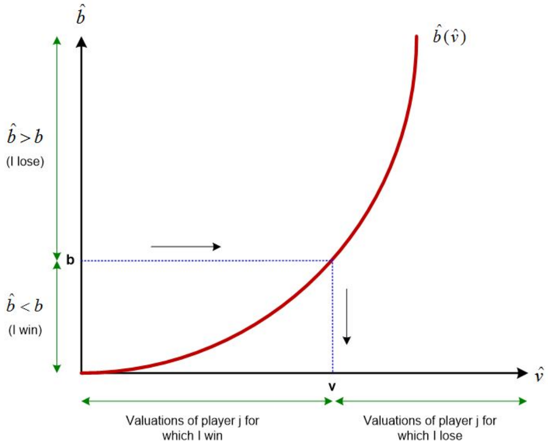
$$\{x_1^*, x_{2L}^*, x_{2H}^*\} = \{\frac{17}{7}, \frac{12}{7}, \frac{8}{7}\}$$

is the Bayesian Nash equilibrium (BNE) of the game.

Watson, Ch. 27 Exercise 2

- Suppose you and one other bidder are competing in a private-value auction. The auction format is sealed bid, first price.
- Let v and b denote your valuation and bid respectively, let $\hat{v}$ and $\hat{b}$ denote the valuation and bid of your opponent.
- Your payoff is $(v - b)$ if $b \geq \hat{b}$ and 0 otherwise.
- Although you do not observe $\hat{v}$, you know that $\hat{v}$ is uniformly distributed over the interval between 0 and 1. That is, v' is the probability that $\hat{v} < v'$.
- You also know that your opponent bids according to the function $\hat{b}(\hat{v}) = \hat{v}^2$.
- Suppose your value is $\frac{3}{5}$. What is your optimal bid?

Watson, Ch. 27 Exercise 2



where $b = x = v^2 \longrightarrow \sqrt{x} = v$

Watson, Ch. 27 Exercise 2

- Note that the probability of winning can be found by using the above figure, as follows:

$$Prob(b > \hat{b}) = Prob(x > \hat{b}) = Prob(\sqrt{x} > \hat{v})$$

And since valuations are uniformly distributed between 0 and 1, then the probability of winning is:

$$Prob(\sqrt{x} > \hat{v}) = \sqrt{x}$$

Watson, Ch. 27 Exercise 2

- Summarizing,

	Probability	Payoff
Winning	$\sqrt{x}$	$(v - x)$
Losing	$1 - \sqrt{x}$	0

- Therefore, bidder i's expected utility from participating in this auction is:

$$EU_i(x|v) = (v - x) \times \sqrt{x} + 0 \times (1 - \sqrt{x})$$

Watson, Ch. 27 Exercise 2

- Hence,

$$EU_i(x|v) = vx^{\frac{1}{2}} - x^{\frac{3}{2}}$$

Taking F.O.C.s with respect to his bid x, we have:

$$\begin{aligned} \frac{1}{2}vx^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} &= 0 \\ \frac{1}{2}\frac{v}{\sqrt{x}} &= \frac{3}{2}\sqrt{x} \implies v = 3x \end{aligned}$$

$$x(v) = \frac{1}{3}v$$

Therefore, the optimal bidding function is $x(v) = \frac{1}{3}v$.

Watson, Ch. 27 Exercise 2

- Finally, if bidder i's valuation is exactly $v = \frac{3}{5}$, we just plug it into the above optimal bidding function:

$$x = \frac{1}{3}v = \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$$

Exercise 3 - FPA with Risk Averse Bidders

- What is the equilibrium bidding function $b_i(v_i)$ for every bidder i ?

Exercise 3 - FPA with Risk Averse Bidders

- We know that in a symmetric BNE where $b_i(v_i) = av_i$ for every bidder i, the probability of bidder i of winning the auction is:

$$Prob(b_i > b_j) = Prob(x > b_j) = Prob\left(\frac{x}{a} > v_j\right) = \frac{x}{a}$$

(see lecture notes for an expression of these three steps)

	Probability	Payoff
Winning	$\frac{x}{a}$	$(v-x)^{\alpha}$
Losing	$1-\frac{x}{a}$	0

Exercise 3 - FPA with Risk Averse Bidders

- Therefore, bidder i's expected utility from participating in this auction is:
$$EU_i(x|v_i) = \frac{x}{a} \times (v - x)^{\alpha} + 0 \times \left(1 - \frac{x}{a}\right)$$
Taking F.O.C. with respect to the bid $b_i = x$, we have:
$$\frac{1}{a}(v - x)^{\alpha} - \frac{x}{a}\alpha(v - x)^{\alpha - 1} = 0$$
$$\implies v - x = \alpha x$$
$$\implies (1 + \alpha)x = v$$

Exercise 3 - FPA with Risk Averse Bidders

- Solving for x, we can find the optimal bidding function,

$$x(v) = \frac{v}{1+\alpha}$$

Exercise 3 - FPA with Risk Averse Bidders

- Hence,
  - When $\alpha = 1$ (risk neutral bidder): $x(v) = \frac{v}{2}$
  - When $\alpha \to 0$ (extremely risk averse bidder): $x(v) = v$

Exercise 3 - FPA with Risk Averse Bidders

- Does function $b_i(v_i)$ increase or decrease in his degree of risk aversion, α?
  - Provide an intuitive explanation for your result.

Exercise 3 - FPA with Risk Averse Bidders

- Consider a particular bidder i with valuation v_i.
- Fix the strategies of all other bidders and suppose that he bids b_i.
- Now suppose that bidder is considering reducing his bid to $(b_i - \varepsilon)$.
  - (1) If he wins the auction, he obtains an additional profit of ε, since he has to pay a lower price for the object he acquires, but...
  - (2) Lowering his bid, he increases the probability of losing the auction.

Exercise 3 - FPA with Risk Averse Bidders

- For a risk averse bidder, the effect of slightly lowering his bid on his wealth level (getting the object at a cheaper price, as described in point (1) has smaller utility consequence than does the possible loss if this lower bid were, in fact, to result in him losing the auction (as described in point 2).
- Therefore, since the possible loss from losing the auction dominates the benefit from acquiring the object at a cheaper price...
  - the risk adverse bidder decides to not reduce his bid, but rather to increase it, relative to the more risk neutral bidders.