EconS 424 - Signalling Games III

Félix Muñoz-García

Washington State University

fmunoz@wsu.edu

February 5, 2018

Watson, Ch. 29 Exercise 9

- A manager and a worker interact as follows:
  - The manager would like the worker to exert some effort on a project. Let e denote the worker's effort.
  - Each unit of effort produces a unit of revenue for the firm; that is, revenue is e.
  - The worker bears a cost of effort given by αe^2, where α is a positive constant.
  - The manager can pay the worker some money, which enters their payoffs in an additive way.
  - Thus, if the worker picks effort level e and the manager pays the worker x, then the manager's payoff is $e - x$ and the worker's payoff is $x - \alpha e^2$.
  - Assume that effort is verifiable and externally enforceable, meaning that the parties can commit to a payment and effort level.

Watson, Ch. 29 Exercise 9

- Imagine that the parties interact as follows:
  - First, the manager makes a contract offer to the worker.
  - The contract is a specification of effort $\hat{e}$ and a wage $\hat{x}$.
  - Then the worker accepts or rejects the offer.
  - If she rejects, then the game ends and both parties obtain payoffs of 0.
  - If she accepts, then the contract is enforced (effort $\hat{e}$ is taken and $\hat{x}$ is paid).
- Because the contract is externally enforced, you do not have to concern yourself with the worker's incentive to exert effort.

Watson, Ch. 29 Exercise 9

- Find the manager's optimal contract offer.

Watson, Ch. 29 Exercise 9

- Setting up the maximization problem,

$$\max_{\hat{e},\hat{x}} \quad \hat{e} - \hat{x} \tag{1}$$

$$\text{s.t. } \hat{x} - \alpha\hat{e}^2 \geq 0 \implies \hat{x} = \alpha\hat{e}^2 \tag{2}$$

- The constraint (2) on the maximization problem for the manager is the minimum payment that the worker requires in order to participate (Participation Constraint).
- The Manager wants to minimize the payment that he makes to the Worker (to maximize profits), so we can assume that the constraint is binding and he selects $\hat{x} = \alpha\hat{e}^2$.

- We can now substitute that value back into the maximization problem in order to make it unconstrained, as follows:

$$\max_{\hat{e}} \quad \hat{e} - \alpha \hat{e}^2$$

Taking F.O.C.s with respect to $\hat{e}$, yields

$$1 - 2\alpha\hat{e} = 0$$

- Solving for $\hat{e}$ we obtain an optimal effort $\hat{e} = \frac{1}{2\alpha}$.
- Plugging this effort back into equation (2), $\hat{x} = \alpha\hat{e}^2$, yields $\hat{x} = \frac{1}{4\alpha}$

Watson, Ch. 29 Exercise 9

- How do $\hat{e}$ and $\hat{x}$ depend on parameters?

Watson, Ch. 29 Exercise 9

- As α is in the denominator, an increase in α decreases both $\hat{e}$ and $\hat{x}$.
- This is also the efficient outcome from a Social Planner's perspective:
  - The sum of both players' utility functions results in the Manager's unconstrained maximization problem (which we just solved).
  - Indeed, the sum of the Manager's and Worker's utilities yields:

    $$(\hat{e} - \hat{x}) + (\hat{x} - \alpha \hat{e}^2) = \hat{e} - \alpha \hat{e}^2$$

    which coincides with the objective function in the above maximization problem.)
  - Hence, the efficient outcome and the Manager's ideal outcome coincide.

Watson, Ch. 29 Exercise 9

- Let $\underline{e}$ and $\underline{x}$ denote the equilibrium contract in the case in which $\alpha = \frac{1}{8}$, and
- let $\bar{e}$ and $\bar{x}$ denote the equilibrium contract in the case in which $\alpha = \frac{3}{8}$.
    - Calculate these four values.

Watson, Ch. 29 Exercise 9

- High type, $\alpha_H = \frac{3}{8}$, equilibrium values:

$$\bar{e} = \frac{1}{2 * \alpha_H} = \frac{4}{3} \quad \text{and} \quad \bar{x} = \frac{1}{4 * \alpha_H} = \frac{2}{3}$$

- Low type, $\alpha_L = \frac{1}{8}$, equilibrium values:

$$\bar{e} = \frac{1}{2 * \alpha_L} = 4 \quad \text{and} \quad \bar{x} = \frac{1}{4 * \alpha_L} = 2$$

Watson, Ch. 29 Exercise 9

- Suppose that α is private information to the worker.
- The manager knows only that:
  - $\alpha = \frac{1}{8}$ with probability $\frac{1}{2}$, and
  - $\alpha = \frac{3}{8}$ with probability $\frac{1}{2}$.
- Suppose that the manager offers the worker a choice between contracts $(\underline{e}, \underline{x})$ and $(\bar{e}, \bar{x})$–that is, the manager offers a menu of contracts–in the hope that the high type will choose $(\bar{e}, \bar{x})$ and the low type will choose $(\underline{e}, \underline{x})$.
  - Will each type pick the contract intended for him? If not, what will happen and why?

Watson, Ch. 29 Exercise 9

- **High type:** $\rightarrow \alpha_H = \frac{3}{8}$, When he chooses the contract meant for the low-type worker, $(\underline{e}, \underline{x})$, his payoff is:

$$\underline{x} - \alpha_H \underline{e}^2 = 2 - \frac{3}{8} * 4^2 = -4 < 0$$

If, in contrast, he chooses the contract meant for him, $(\bar{e}, \bar{x})$, the high-type worker obtains:

$$\bar{x} - \alpha_H \bar{e}^2 = \frac{2}{3} - \frac{3}{8}\left(\frac{4}{3}\right)^2 = 0$$

So the high type chooses the contract meant for him $(\bar{e}, \bar{x})$, since it yields a higher utility (0) than the contract meant for the low-type $(\underline{e}, \underline{x})$, -4.

Watson, Ch. 29 Exercise 9

- **Low Type:** $\rightarrow \alpha_L = \frac{1}{8}$ When he chooses the contract meant for the him, $(\underline{e}, \underline{x})$, his payoff is:

$$\underline{x} - \alpha_L \underline{e}^2 = 2 - \frac{1}{8} * 4^2 = 0$$

When he chooses the contract meant for the high-type worker, $(\bar{e}, \bar{x})$, his payoff is:

$$\bar{x} - \alpha_L \bar{e}^2 = \frac{2}{3} - \frac{1}{8}\left(\frac{4}{3}\right)^2 = \frac{4}{9} > 0$$

So the low type chooses the contract meant for the high type, $(\bar{e}, \bar{x})$, since it yields a higher utility $(\frac{4}{9})$ than the contract meant for himself $(\underline{e}, \underline{x})$, 0.

- Suppose that the manager offers a menu of two contracts (e_L, x_L) and (e_H, x_H), where he hopes that the first contract will be accepted by the low type and the second will be accepted by the high type.
  - Under what conditions will each type accept the contract intended for him?

Watson, Ch. 29 Exercise 9

- The *Incentive Compatibility Conditions* for the low and high types, respectively, are:

$$\textbf{Low Type:} \;\rightarrow\; \overbrace{x_L - \frac{1}{8}e_L^2}^{\substack{\text{Payoff from being}\\ \text{a low type}}} \geq \overbrace{x_H - \frac{1}{8}e_H^2}^{\substack{\text{Payoff from pretending}\\ \text{to be a high type}}} \qquad (3)$$

$$\textbf{High Type:} \;\rightarrow\; \underbrace{x_H - \frac{3}{8}e_H^2}_{\substack{\text{Payoff from being}\\ \text{a high type}}} \geq \underbrace{x_L - \frac{3}{8}e_L^2}_{\substack{\text{Payoff from pretending}\\ \text{to be a low type}}} \qquad (4)$$

Watson, Ch. 29 Exercise 9

- The *Participation Constraints* are:

$$\textbf{Low Type:} \rightarrow \overbrace{x_L - \frac{1}{8}e_L^2}^{\substack{\text{Low type worker's} \\ \text{payoff from participating}}} \geq \overbrace{0}^{\substack{\text{Low type worker's payoff} \\ \text{from not participating}}} \quad (5)$$

$$\textbf{High Type:} \rightarrow \underbrace{x_H - \frac{3}{8}e_H^2}_{\substack{\text{High type worker} \\ \text{payoff from participating}}} \geq \underbrace{0}_{\substack{\text{High type worker's payoff} \\ \text{from not participating}}} \quad (6)$$

- When inequalities 3-6 hold, each type will accept the contract that is intended for him.

Watson, Ch. 29 Exercise 9

- Compute the manager's optimal menu (e_L, x_L) and (e_H, x_H).

Watson, Ch. 29 Exercise 9

- The Manager wants to maximize his expected payoff:

$$\frac{1}{2}[e_H - x_H] + \frac{1}{2}[e_L - x_L]$$

where $\left(\frac{1}{2}\right)$ represents the probabilities of the Worker being either the High or Low types...

  - something that the Manager cannot observe when offering the two possible contracts to the Worker.

- From part (d), equation (3) is binding (Recall that the low type would rather pretend to be the high type) and yields:

$$x_L = x_H + \frac{1}{8}e_L^2 - \frac{1}{8}e_H^2$$

and equation (6) is binding (The high type has the higher reservation utility) and yields:

$$x_H = \frac{3}{8}e_H^2$$

Then substituting the latter equation into the former we obtain:

$$x_L = \frac{3}{8}e_H^2 + \frac{1}{8}e_L^2 - \frac{1}{8}e_H^2$$

Combining terms:

$$x_L = \frac{1}{4}e_H^2 + \frac{1}{8}e_L^2$$

Watson, Ch. 29 Exercise 9

- Substituting for x_L and x_H into the Manager's expected payoff yields the following unconstrained maximization problem:

$$\max_{e_L, e_H} \ \frac{1}{2}\left[e_H - \frac{3}{8}e_H^2\right] + \frac{1}{2}\left[e_L - \frac{1}{4}e_H^2 - \frac{1}{8}e_L^2\right]$$

Watson, Ch. 29 Exercise 9

- F.O.C.s with respect to e_L:

$$\frac{\partial \mathcal{L}}{\partial e_L} = \frac{1}{2} - \frac{1}{8}e_L = 0 \;\longrightarrow\; e_L^* = 4$$

F.O.C.s with respect to e_H:

$$\frac{\partial \mathcal{L}}{\partial e_H} = \frac{1}{2} - \frac{5}{8}e_H = 0 \;\longrightarrow\; e_H^* = \frac{4}{5}$$

which implies

$$x_H^* = \frac{3}{8}e_H^2 = \frac{3}{8} * \left(\frac{4}{5}\right)^2 = \frac{6}{25}$$

$$x_L^* = \frac{1}{4}e_H^2 + \frac{1}{8}e_L^2 = \frac{1}{4} * \left(\frac{4}{5}\right)^2 + \frac{1}{8} * 4^2 = \frac{54}{25}$$

Watson, Ch. 29 Exercise 9

- Comment on the relation between the solution to the manager's problem when there is complete versus incomplete information.
  - How does the optimal menu under asymmetric information distort away from efficiency?

Watson, Ch. 29 Exercise 9

	Complete	Information	Incomplete	Information
	$(\underline{e}, \underline{x})$ $(4, 2)$	$(\bar{e}, \bar{x})$ $\left(\frac{4}{3}, \frac{2}{3}\right)$	(e_L^*, x_L^*) $\left(4, \frac{54}{25}\right)$	(e_H^*, x_H^*) $\left(\frac{4}{5}, \frac{6}{25}\right)$
Low Type $\alpha_L = \frac{1}{8}$	0		$\frac{1}{4}$	$\frac{4}{25}$
High Type $\alpha_H = \frac{3}{8}$		0	$-\frac{15}{4}$	0

Watson, Ch. 29 Exercise 9

- We can now use the above summarizing table in order to compare the effort for each type of worker:
- **Low-type worker:**
  - The low-type worker exerts the same effort under both information contexts $(e_L = 4)$,
  - However, he is paid more under incomplete info, $\frac{54}{25}$, than under complete info, 2, in order for him to have incentives to reveal his type by voluntarily selecting the contract meant for him.
- **High-type worker:**
  - The high-type worker exerts less effort under incomplete information, $(\frac{4}{5} < \frac{4}{3})$,
  - and is paid less, $(\frac{6}{25} < \frac{2}{3})$.