EconS 424 - Strategy and Game Theory
Midterm Exam #1 - Anwer key

1. **[IDSDS, psNE and msNE]** [32 points].Consider the following simultaneous-move game played by player 1 (in rows) and player 2 (in columns).

		Player 2		
		x	y	z
	a	$2,3$	$1,4$	$3,2$
Player 1	b	$5,1$	$2,3$	$1,2$
	c	$3,7$	$4,6$	$5,4$
	d	$4,2$	$1,3$	$6,1$

(a) [5 points] Which strategy pairs survive the application of iterative deletion of strictly dominated strategies (IDSDS)?

- For player 2 (column player), strategy z is strictly dominated by y. We can then delete column z, leaving us with the following reduced-form matrix.

		Player 2	
		x	y
	a	$2,3$	$1,4$
Player 1	b	$5,1$	$2,3$
	c	$3,7$	$4,6$
	d	$4,2$	$1,3$

For player 1 (row player), strategy a is strictly dominated by b. After deleting the row corresponding to a, we obtain

		Player 2	
		x	y
	b	$5,1$	$2,3$
Player 1	c	$3,7$	$4,6$
	d	$4,2$	$1,3$

At this point, we cannot delete any more strategies for players 1 or 2 if we restrict them to use pure strategies. However, if we allow player 1 to randomize between the strategies that provide the highest payoff, b and c. In particular, assigning a probability p to strategy b and the remaining probability $1-p$ to strategy c, player 1's expected payoff when player 2 chooses strategy x (in the left-hand column of the above matrix) is

$$5p + 3(1-p) = 2p + 3$$

which is larger than player 1's payoff from strategy d, 4, as long as $2p+3 > 4$, or solving for p, if $p > \frac{1}{2}$. Similarly, when player 2 chooses straegy y (in the right-hand column of the above matrix), player 1's expected payoff from randomizing between b and c becomes

$$2p + 4(1-p) = 4 - 2p$$

which is larger than player 1's payoff from strategy d, 1, as long as $4-2p>1$, or solving for p, if $p<\frac{3}{2}$. This condition holds by assumption since probability p must be a number between 0 and 1. Therefore, any randomization between strategies b and c that assigns more than 50% probability on strategy b (that is, $p>1/2$) yields a expected utility larger than the utility player 1 receives from strategy d. We can therefore claim that strategy d is strictly dominated, and delete the bottom row of the above matrix, leaving us with the followed reduced-form matrix.

		Player 2	
		x	y
Player 1	b	$5,1$	$2,3$
	c	$3,7$	$4,6$

At this point, we cannot delete any further strategies for players 1 or 2. Then, the strategy profiles surviving IDSDS are those in the four cells of the above matrix:

$$IDSDS=\{(b,x),(b,y),(c,x),(c,y)\}.$$

(b) [5 points] Using your results from part (a), show that there is no pure strategy Nash equilibrium (psNE) in this game.

- Using the strategy profiles that survived IDSDS, we can next underline best response payoffs, as depicted in the matrix below.

		Player 2	
		x	y
Player 1	b	$\underline{5},1$	$2,\underline{3}$
	c	$3,\underline{7}$	$\underline{4},6$

Since there is no cell where both players' payoffs are underlined, we can claim that there is no pure strategy Nash equilibrium in this game. There is, however, a mixed strategy Nash equilibrium, as we show in the next part of the exercise!

(c) [9 points] Using your results from part (a), find a mixed strategy Nash equilibrium (msNE) in this game.

- *Player 1.* If player 1 is randomizing, he must be indifferent between pure strategies b and c. His expected utility from choosing b (in the top row of the above matrix) is

$$EU_1(b)=5q+2(1-q)=3q+2$$

while his expected utility from selecting c (in the bottom row of the matrix) is

$$EU_1(c)=5q+4(1-q)=4-q.$$

Then, player 1 is indifferent between b and c if and only if $EU_1(b)=EU_1(c)$, which implies that

$$3q+2=4-q$$

and, after rearranging, $4q=2$, or $q=\frac{1}{2}$.

- *Player 2.* If player 2 is randomizing, he must be indifferent between his pure strategies x and y. His expected utility from choosing x (in the left-hand column of the above matrix) is

$$EU_2(x) = 1p + 7(1-p) = 7 - 6p$$

while his expected utility from selecting y (in the right-hand column of the matrix) is

$$EU_2(y) = 3p + 6(1-p) = 6 - 3p.$$

Then, player 2 is indifferent between x and y if and only if $EU_2(x) = EU_2(y)$, which implies that

$$7 - 6p = 6 - 3p$$

or, after rearranging, $1 = 3p$, or $p = \frac{1}{3}$.

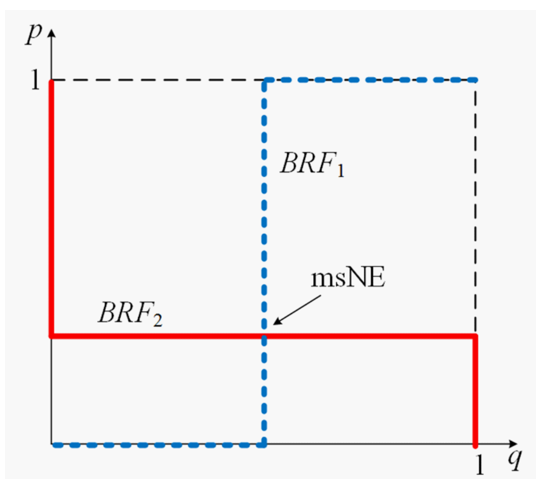
- Therefore, the mixed strategy Nash equilibrium of the game is

$$\left\{ \left(\frac{1}{3}b, \frac{2}{3}c \right), \left(\frac{1}{2}x, \frac{1}{2}y \right) \right\}$$

where the first pair indicates player 1's randomization between b and c with probabilities 1/3 and 2/3 respectively, while the second pair represents player 2's randomization between x and y, each with 50% probability.

(d) [11 points] In a figure with p in the vertical axis and q in the horizontal axis, draw player 1's best response function, $p(q)$, and player 2's best response function, $q(p)$. (Use different colors, if possible).

- The next figure depicts the best response functions for each player. They only have a crossing point, which illustrates the mixed strategy Nash equilibrium of the game.



- For player 1, note that when $q = 1$ (player 2 chooses x), his best response is to choose b, implying that he assigns full probability to b, that is, $p = 1$ at the top right-hand corner of the figure. When $q = 0$ (player 2 selects y), player 1's best response is to choose c, implying that he assigns no probability to b, that is, $p = 0$, at the bottom left-hand corner of the figure.

- For player 2, note that when $p = 1$ (player 1 chooses b), his best response is to choose y, implying that he assigns no probability to x, that is, $q = 0$ at the top left-hand corner of the figure. When $p = 0$ (player 1 selects c), player 2's best response is to choose x, implying that he assigns full probability to x, that is, $q = 1$, at the bottom right-hand corner of the figure.

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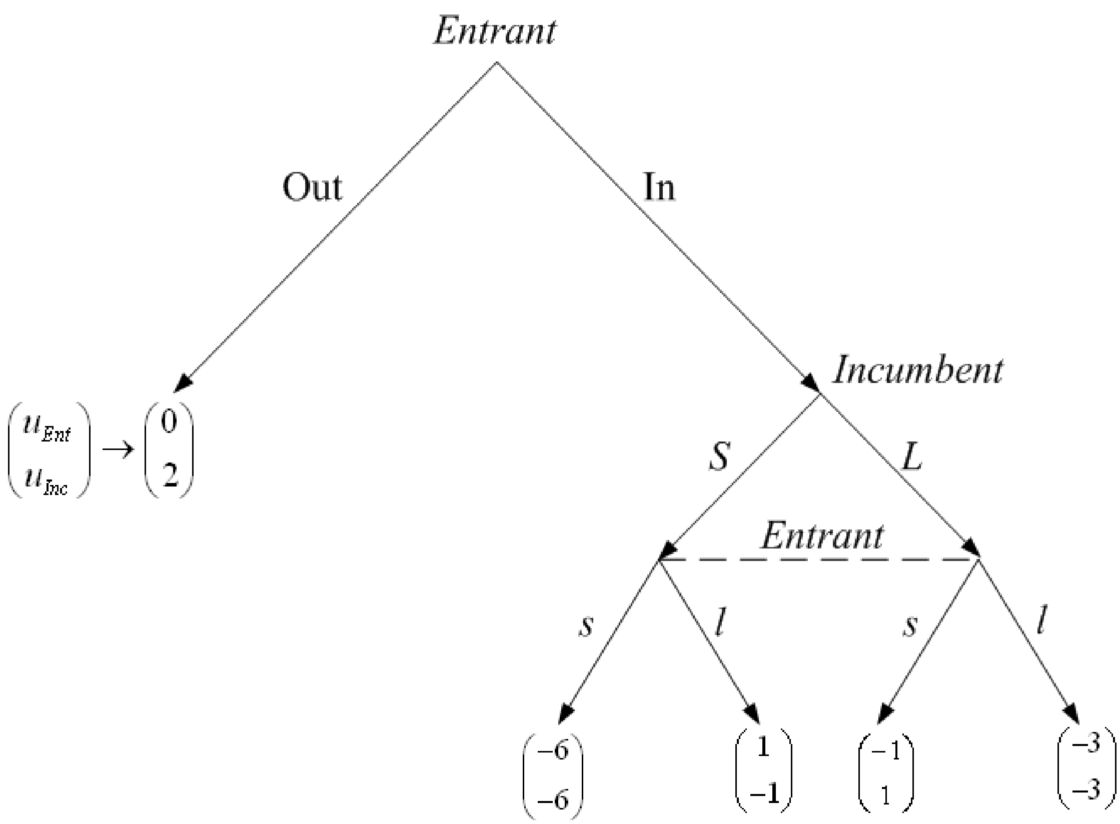
Midterm exam #1 – Answer key

Exercise #2 – Entry deterrence game with simultaneous decisions after entry

Consider the following extensive form game. It represents the entry-deterrence game that we discussed in class, but with a slight modification. The entrant firm decides whether to enter the market where an incumbent is currently operating, or to remain out of the market. If the entrant decides to enter, then a simultaneous move game is played, where both the entrant and the incumbent must decide whether they will take either a small or large niche of the market.



a) Operating by backwards induction, firstly find all the Nash equilibria for the subgame initiated after the entrant firm decides to enter (simultaneous move game). Consider both the pure strategy Nash equilibria and the mixed strategy Nash equilibrium.

Notice that the subgame induced after the entrant decides to enter can be represented in its normal form representation (since it is a simultaneous move game) as follows:

		Entrant	
Incumbent		*Small, s*	*Large, l*
	Small, S	-6,-6	-1,1

	Large, L	1,-1	-3,-3

This game has two equilibria in pure strategies: (Large, Small) and (Small, Large).

In addition, there exists a mixed strategy Nash equilibrium, where the entrant randomizes with a probability *q* that makes the incumbent indifferent between choosing a Small or Large niche:

EU_I(Small)= EU_I(Large), that is

-6q+(-1)(1-q)=1q+(-3)(1-q), and rearranging we obtain

2=9q

which implies that the entrant randomizes using a probability *q*=2/9

And similarly, the incumbent chooses a probability p such that the entrant is indifferent between selecting Small or Large. That is,

EU_E(Small)= EU_E(Large), that is

6p+(-1)(1-p)=1p+(-3)(1-p), and rearranging we obtain

2=9p

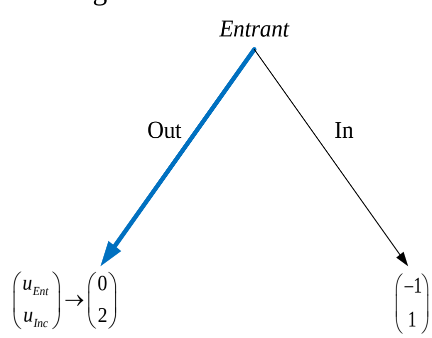
which implies that the incumbent randomizes with probability *p*=2/9

Then, the mixed strategy Nash equilibrium for this subgame is

{(2/9Small, 7/9Large),(2/9Small 7/9 Large)}

b) Once you have identified all the equilibria for the proper subgame where firms compete in what niche they will capture, find the subgame perfect Nash equilibria of the entire game. Notice that you just need to take into account the utility resulting from playing each of the possible Nash equilibria of the subgame, and then check what is optimal for the entrant to do (either In or Out). *Hint: there are three different SPNE.*
Let's analyze each of the different SPNE separately:

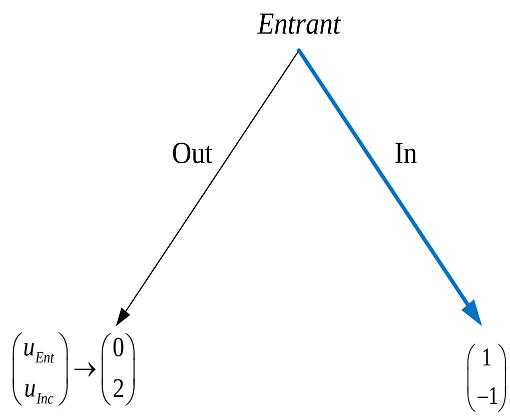
1. If the subgame is played using the pure strategy Nash equilibrium (Large, Small), then the corresponding payoff vector is (1,-1). That is, the Entrant is obtaining a payoff level from the psNE of this subgame of -1. Therefore, the Entrant prefers to remain out of the market (obtaining a payoff of 0) rather than entering and obtaining -1.



The first SPNE is then,

{Out, (Large, Small)}

2. If the subgame is played using the pure strategy Nash equilibrium (Small, Large), then the corresponding payoff vector is (-1,1). That is, the Entrant is obtaining a payoff level from the psNE of this subgame of 1. Therefore, the Entrant prefers to enter into the market (obtaining 1) rather than remaining outside (and obtaining a payoff of 0).



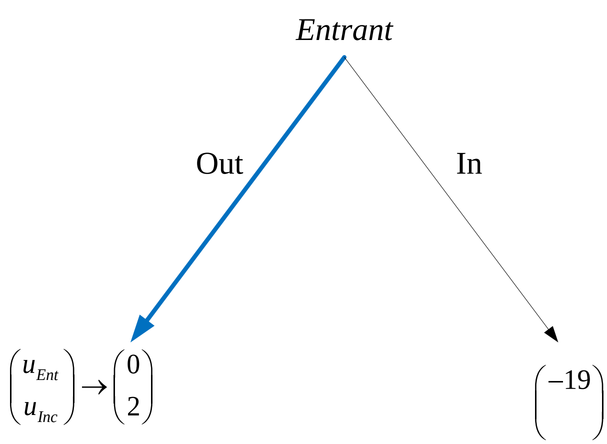
The second SPNE is then,

{In, (Small, Large)}

3. If the subgame induced after the Entrant decided to enter is played using the mixed strategy {(2/9Small, 7/9Large),(2/9Small 7/9 Large)}, then the corresponding expected utility level for the Entrant from playing such a mixed strategy Nash equilibrium in this subgame is

$$\frac{2}{9}\left((-6)\frac{2}{9}+1\frac{7}{9}\right)+\frac{7}{9}\left((-1)\frac{2}{9}+(-3)\frac{7}{9}\right)=-19$$

Therefore, the expected utility level from playing such mixed strategy Nash equilibrium for the entrant is so low, that it is better off by not entering into the market.



Then, the third SPNE is given by

{Out, {(2/9Small, 7/9Large),(2/9Small 7/9 Large)}}

Exercise #3 – R&D tournaments

Several strategic settings can be modeled as a tournament, whereby the probability of winning a certain prize not only depends on how much effort you exert, but also on how much effort other participants in the tournament exert. For instance, wars between countries, or R&D competitions between different firms in order to develop a new product, not only depend on a participant's own effort, but on the effort put by its competitors. Let's analyze equilibrium behavior in these settings. Consider that the benefit that firm 1 obtains from being the first company to launch a new drug is \$36 million. However, the probability of winning this R&D competition against its rival (i.e., being the first to launch the drug) is $\frac{x_1}{x_1+x_2}$, which it increases with this firm's own expenditure on R&D, x_1, relative to total expenditure, $x_1 + x_2$. Intuitively, this suggests that, while spending more than its rival, i.e., $x_1 > x_2$, increases firm 1's chances of being the winner, the fact that $x_1 > x_2$ does not guarantee that firm 1 will be the winner. That is, there is still some randomness as to which firm will be the first to develop the new drug, e.g., a firm can spend more resources than its rival but be "unlucky" because its laboratory exploits a few weeks before being able to develop the drug. For simplicity, assume that firms' expenditure cannot exceed 25, i.e., $x_i \in [0, 25]$.The cost is simply x_i, so firm 1's profit function is

$$\pi_1(x_1, x_2) = 36\left(\frac{x_1}{x_1 + x_2}\right) - x_1$$

and there is an analogous profit function for country 2:

$$\pi_2(x_1, x_2) = 36\left(\frac{x_2}{x_1 + x_2}\right) - x_2$$

You can easily check that these profit functions are concave in a firm's own expenditure, i.e., $\frac{\partial^2 \pi_i(x_{i,} x_j)}{\partial x_i^2} \leq 0$ for every firm i={1, 2} where $j \neq i$. Intuitively, this indicates that, while profits increase in the firm's R&D, the first million dollar is more profitable than the 10th million dollar, e.g., the innovation process is more exhausted.

a. Find each firm's best-response function.

b. Find a symmetric Nash equilibrium, i.e., $x_1^* = x_2^* = x^*$.

Answer:

Firm 1's optimal expenditure is the value of x_1 for which the first derivative of its profit function equals zero. That is,

$$\frac{\partial \pi_1(x_1, x_2)}{\partial x_1} = 36\left[\frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2}\right] - 1 = 0$$

Rearranging, we find

$$36\left[\frac{x_2}{(x_1 + x_2)^2}\right] - 1 = 0$$

$$36x_2 = (x_1 + x_2)^2$$

and further rearranging

$$6\sqrt{x_2} = x_1 + x_2$$

Solving for x_1, we obtain firm 1's best response function

$$x_1(x_2) = 6\sqrt{x_2} - x_2$$

Figure 1 depicts firm 1's best response function, $x_1(x_2) = 6\sqrt{x_2} - x_2$ as a function of its rival's expenditure, x_2 in the horizontal axis for the admissible set $x_2 \in [0, 25]$.

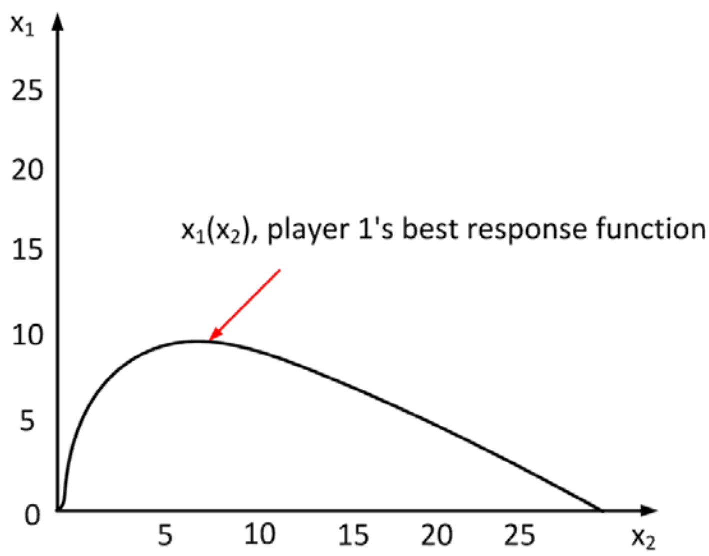


Figure 1. Firm 1's Best Response Function

It is straightforward to show that, for all values of $x_2 \in [0,25]$, firm 1's best response also lies in the admissible set $x_1 \in [0,25]$. In particular, the maximum of BRF_1 occurs at x_2=9 since

$$\frac{\partial BR_1(x_2)}{\partial x_2} = \frac{\partial[6\sqrt{x_2} - x_2]}{\partial x_2} = 3(x_2)^{-\frac{1}{2}} - 1$$

Hence, the point at which this best response function reaches its maximum is that in which its derivative is zero, i.e., $3(x_2)^{-\frac{1}{2}} - 1 = 0$, which yields a value of $x_2 = 9$. At this point, firm 1's best response function informs us that firm 1 optimally spends $6\sqrt{9} - 9 = 9$. Finally, note that the best response function is concave in its rival expenditure, x_2, since

$$\frac{\partial^2 BR_1(x_2)}{\partial x_2} = -\frac{3}{2}(x_2)^{-\frac{3}{2}} < 0.$$

By symmetry, firm 2's best response function is $x_2(x_1) = 6\sqrt{x_1} - x_1$.

b. Find a symmetric Nash equilibrium, i.e., $x_1^* = x_2^* = x^*$

A symmetric Nash equilibrium is an expenditure level, denoted x^*, such that a firm finds it optimal to spend x^* when the other firm spends x^*. It is then a solution to

$$x^* = 6\sqrt{x^*} - x^*$$

Rearranging, we obtain $2x^* = 6\sqrt{x^*}$, and solving for x^*, we find $x^* = 9$. Hence, the unique symmetric Nash equilibrium has each firm spending 9. As figure 1 depicts, the points at which the best response function of player 1 and 2 cross each other occur at the 45-degree line (so the equilibrium is symmetric). In particular, those points are the origin, i.e., (0,0), but this case is uninteresting since it implies that no firm spends money on R&D, and (9,9).

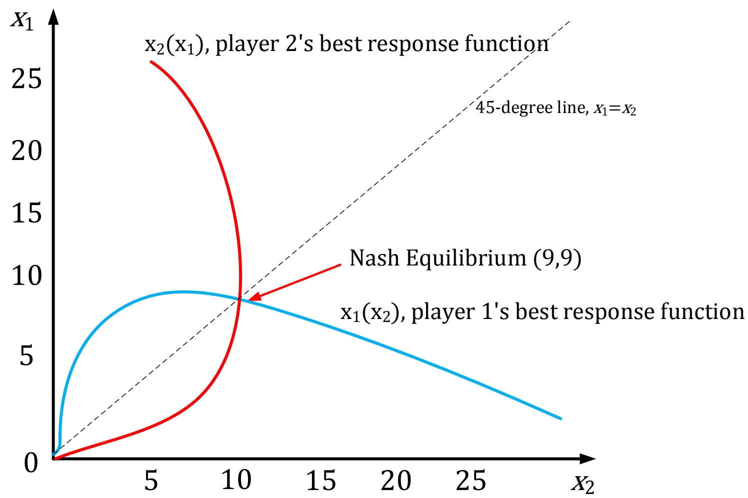


Figure 2. Best Response Functions. Nash-Equilibrium