Advanced Microeconomic Theory

Chapter 3: Demand Theory Applications

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

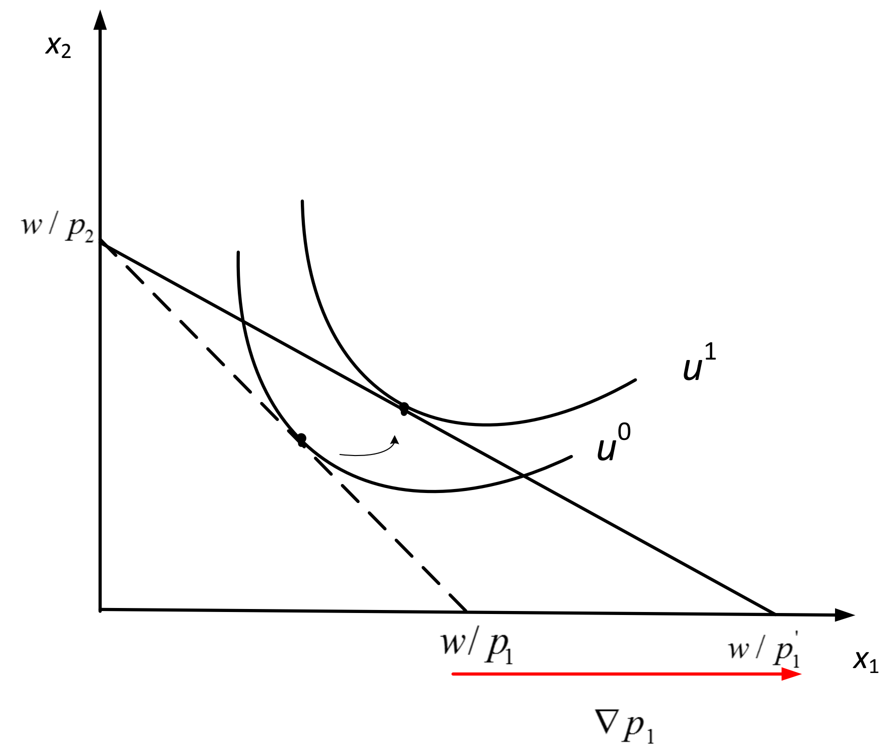
Measuring the Welfare Effects of a Price Change

Measuring the Welfare Effects of a Price Change

- How can we measure the welfare effects of:
  - a price decrease/increase
  - the introduction of a tax/subsidy
- Why not use the difference in the individual's utility level, i.e., from u^0 to u^1?
  - Two problems:
    1) *Within a subject criticism*: Only ranking matters (ordinality), not the difference;
    2) *Between a subject criticism*: Utility measures would not be comparable among different individuals.
- Instead, we will pursue monetary evaluations of such price/tax changes.

Measuring the Welfare Effects of a Price Change

- Consider a price decrease from p_1^0 to p_1^1.
- We cannot compare u^0 to u^1.
- Instead, we will find a money-metric measure of the consumer’s welfare change due to the price change.



Measuring the Welfare Effects of a Price Change

- ***Compensating Variation (CV)***:
  - How much money a consumer would be willing to give up *after* a reduction in prices to be just as well off as *before* the price decrease.
- ***Equivalent Variation (EV)***:
  - How much money a consumer would need *before* a reduction in prices to be just as well off as *after* the price decrease.

Measuring the Welfare Effects of a Price Change

- Two approaches:
  1) Using expenditure function
  2) Using the Hicksian demand

CV using Expenditure Function

- $CV(p^0, p^1, w)$ using $e(p, u)$:

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0)$$

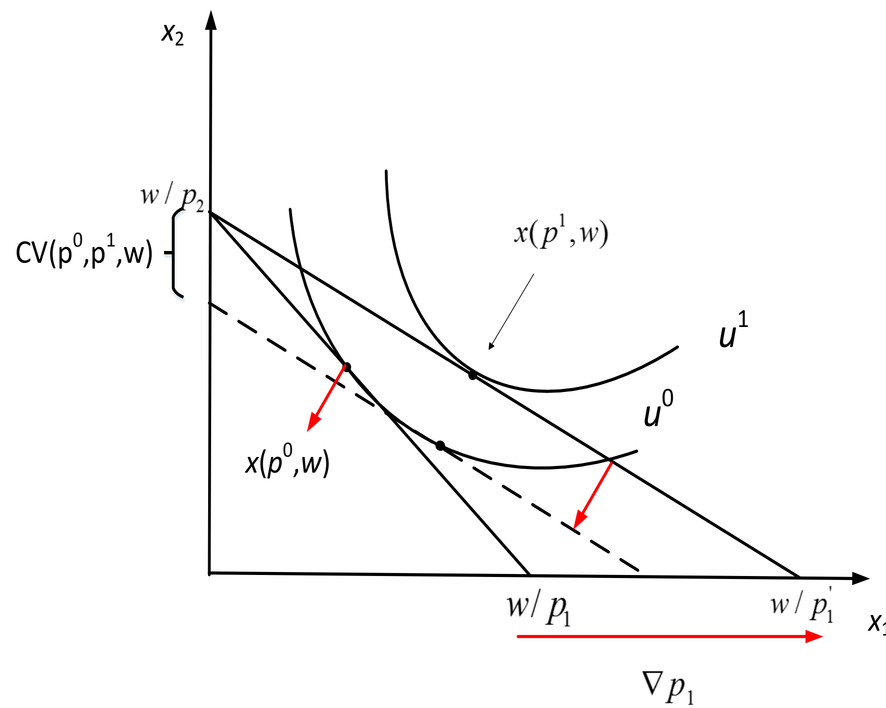
- The amount of money the consumer is willing to give up *after* the price decrease (after price level is p^1 and her utility level has improved to u^1) to be just as well off as *before* the price decrease (reaching utility level u^0).

CV using Expenditure Function

1) When $B_{p^0,w}$, $x(p^0, w)$
2) ∇p_1 and $x(p^1, w)$ under $B_{p^1,w}$
3) Adjust final wealth (*after* the price change) to make the consumer as well off as *before* the price change
4) Difference in expenditure:

$$CV(p^0, p^1, w) = \underbrace{e(p^1, u^1)}_{\text{at } B_{p^1,w}} - \underbrace{e(p^1, u^0)}_{\text{dashed line}}$$

This is Hicksian wealth compensation!



EV using Expenditure Function

- $EV(p^0, p^1, w)$ using $e(p, u)$:

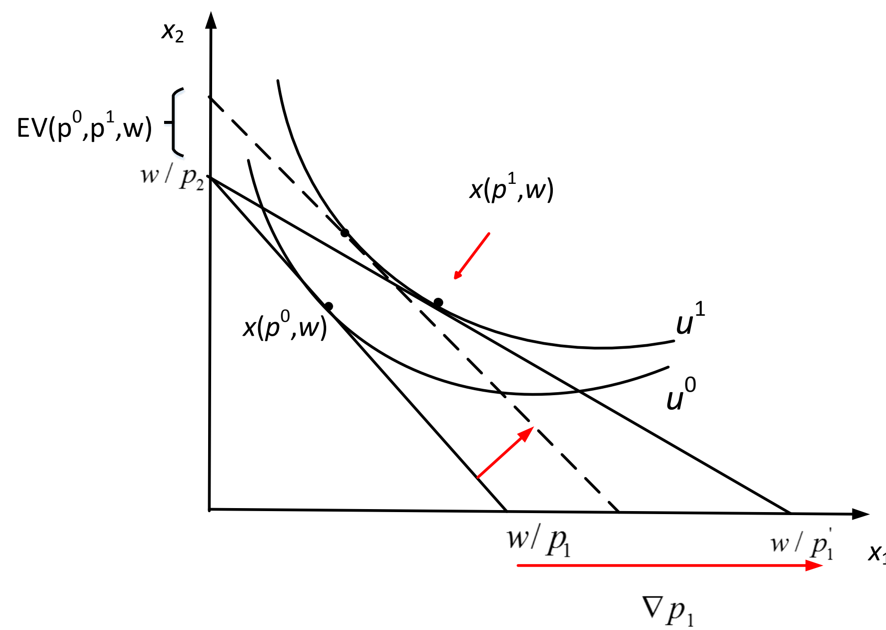
$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0)$$

- The amount of money the consumer needs to receive *before* the price decrease (at the initial price level p^0 when her utility level is still u^0) to be just as well off as *after* the price decrease (reaching utility level u^1).

EV using Expenditure Function

1) When $B_{p^0,w}$, $x(p^0, w)$
2) ∇p_1 and $x(p^1, w)$ under $B_{p^1,w}$
3) Adjust initial wealth (*before* the price change) to make the consumer as well off as *after* the price change
4) Difference in expenditure:

$$EV(p^0, p^1, w) = \underbrace{e(p^0, u^1)}_{\text{dashed line}} - \underbrace{e(p^0, u^0)}_{\text{at } B_{p^0,w}}$$



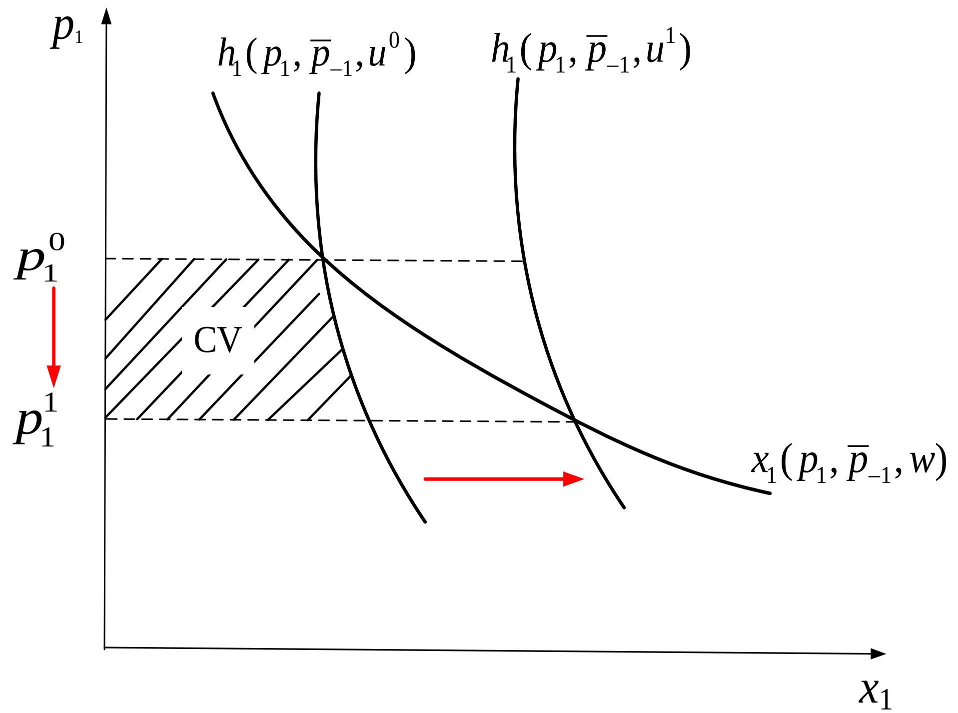
CV using Hicksian Demand

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $p_k^1 = p_k^0$ for all $k \neq 1$, then

$$
\begin{aligned}
CV(p^0, p^1, w) &= e(p^1, u^1) - e(p^1, u^0) \\
&= w - e(p^1, u^0) \\
&= e(p^0, u^0) - e(p^1, u^0) \\
&= \int_{p_1^1}^{p_1^0} \frac{\partial e(p_1, \bar{p}_{-1}, u^0)}{\partial p_1} \, dp_1 \\
&= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0) \, dp_1
\end{aligned}
$$

CV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, CV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^0, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.



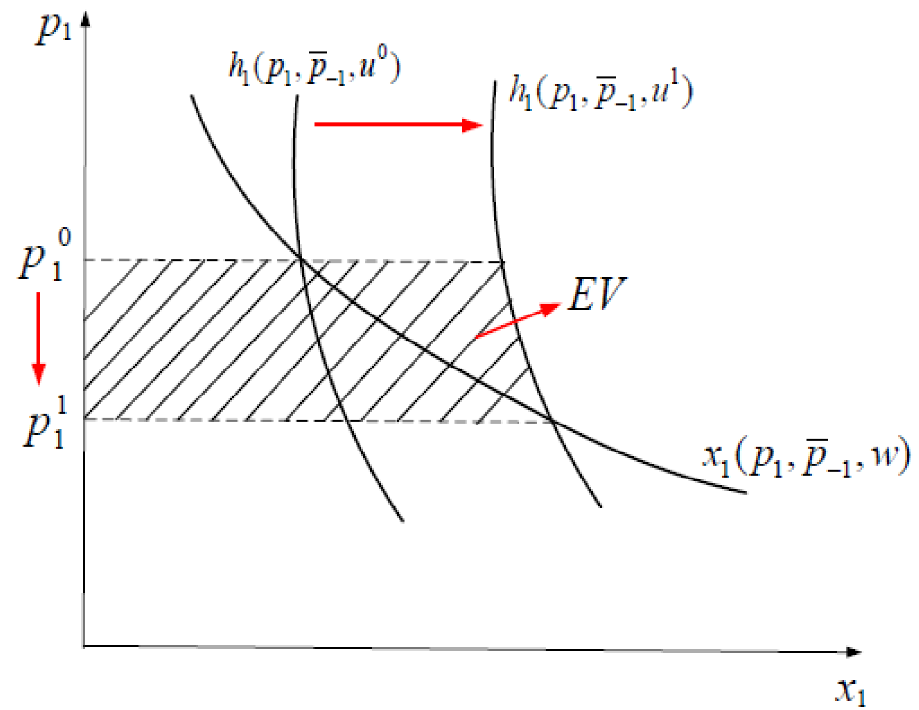
EV using Hicksian Demand

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $p_k^1 = p_k^0$ for all $k \neq 1$, then

$$\begin{aligned} EV(p^0, p^1, w) &= e(p^0, u^1) - e(p^0, u^0) \\ &= e(p^0, u^1) - w \\ &= e(p^0, u^1) - e(p^1, u^1) \\ &= \int_{p_1^1}^{p_1^0} \frac{\partial e(p_1, \bar{p}_{-1}, u^1)}{\partial p_1} dp_1 \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1 \end{aligned}$$

EV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, EV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^1, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.



What about a price increase?

- The Hicksian demand associated with initial utility level u^0 (before the price increase, or before the introduction of a tax) experiences an inward shift when the price increases, or when the tax is introduced, since the consumer's utility level is now u^1, where $u^0 > u^1$. Hence,

$$h_1(p_1, \bar{p}_{-1}, u^0) > h_1(p_1, \bar{p}_{-1}, u^1)$$

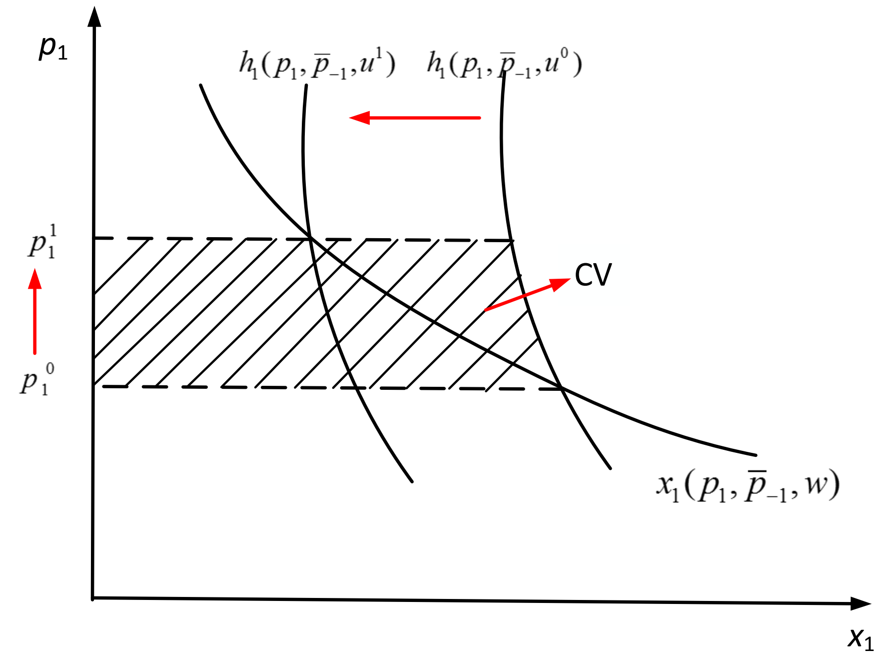
What about a price increase?

- The definitions of CV and EV would now be:
  - CV: the amount of money that a consumer would need *after* a price increase to be as well off as *before* the price increase.
  - EV: the amount of money that a consumer would be willing to give up *before* a price increase to be as well off as *after* the price increase.
- Graphically, it looks like the CV and EV areas have been reversed:
  - CV is associated to the area below $h_1(p_1, \bar{p}_{-1}, u^0)$ as usual
  - EV is associated with the area below $h_1(p_1, \bar{p}_{-1}, u^1)$.

What about a price increase?

- CV is always associated with $h_1(p_1, \bar{p}_{-1}, u^0)$

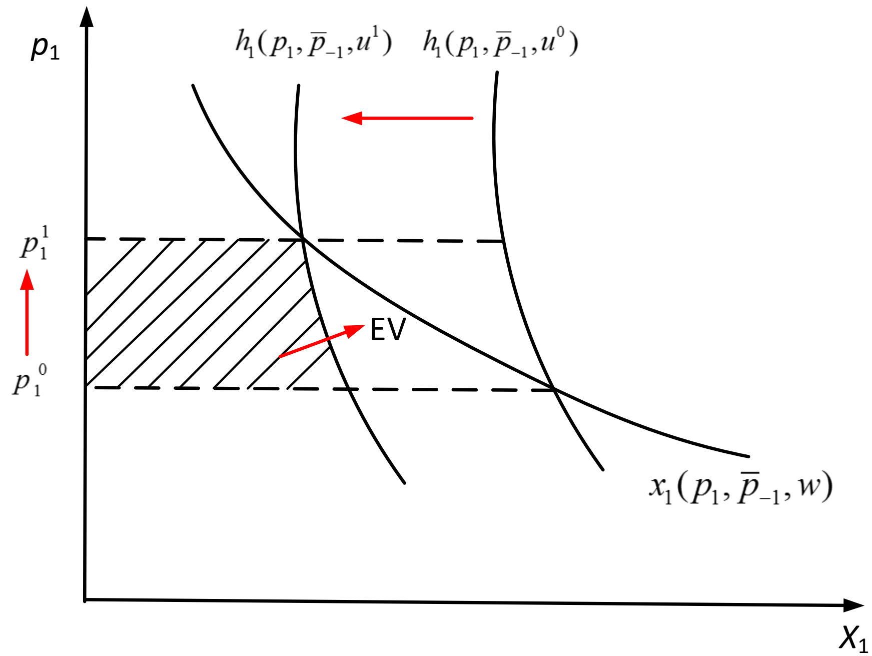
- $CV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$



What about a price increase?

- EV is always associated with $h_1(p_1, \bar{p}_{-1}, u^1)$

- $EV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$



Introduction of a Tax

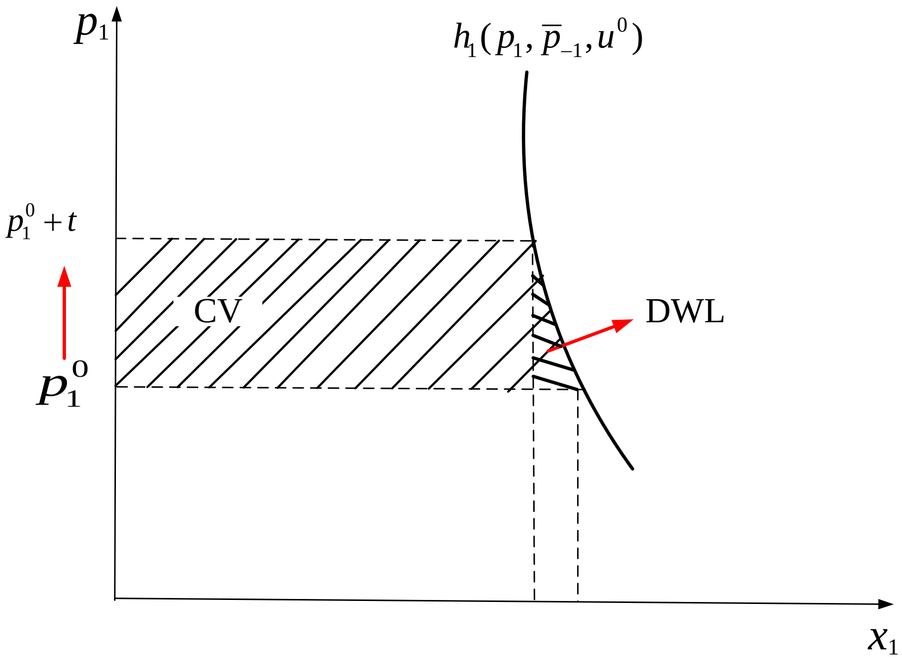
- The introduction of a tax can be analyzed as a price increase.
- *The main difference*: we are interested in the area of CV and EV that is *not* related to tax revenue.
- Tax revenue is:

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^0) \text{ (using CV)}$$

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^1) \text{ (using EV)}$$

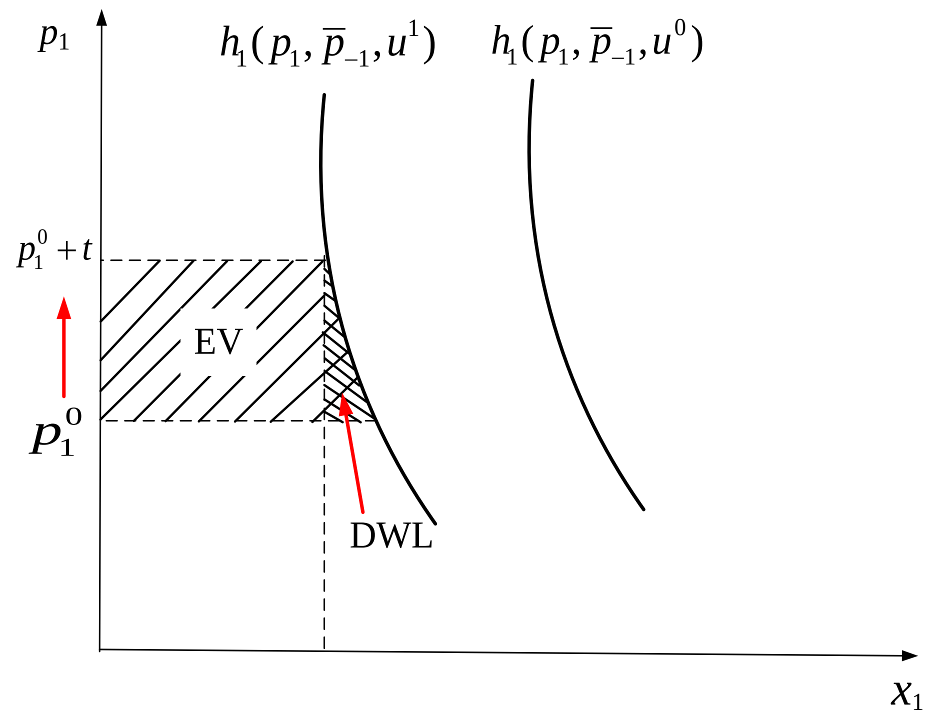
Introduction of a Tax

- CV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^0)$:
  $CV(p^0, p^1, w)$
  $$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$$
- Welfare loss (DWL) is the area of the CV not transferred to the government via tax revenue:
  $$DWL = CV - T$$



Introduction of a Tax

- EV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^1)$:
  $EV(p^0, p^1, w)$
  $$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$$
- Welfare loss (DWL) is the area of the EV not transferred to the government via tax revenue:
  $$DWL = EV - T$$

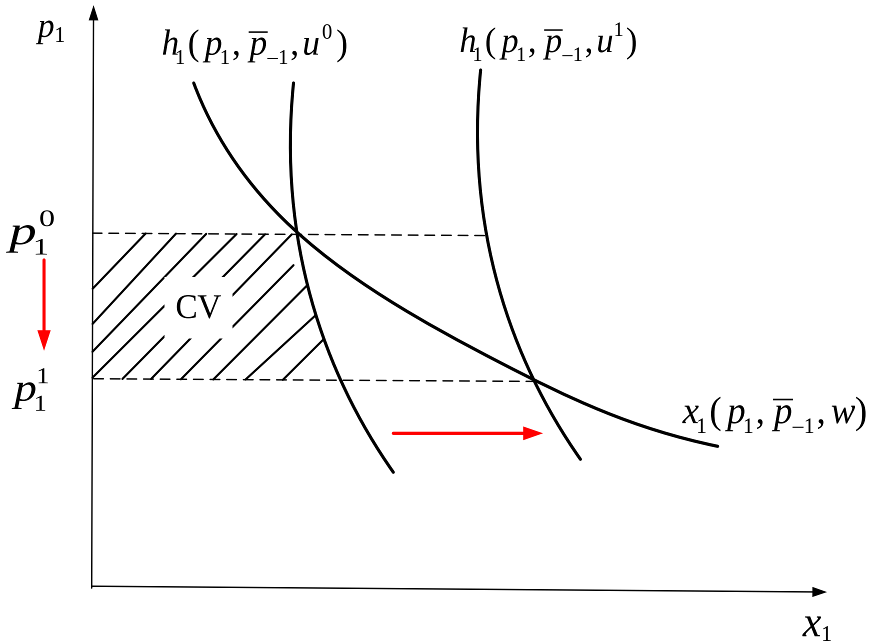


Why not use the Walrasian demand?

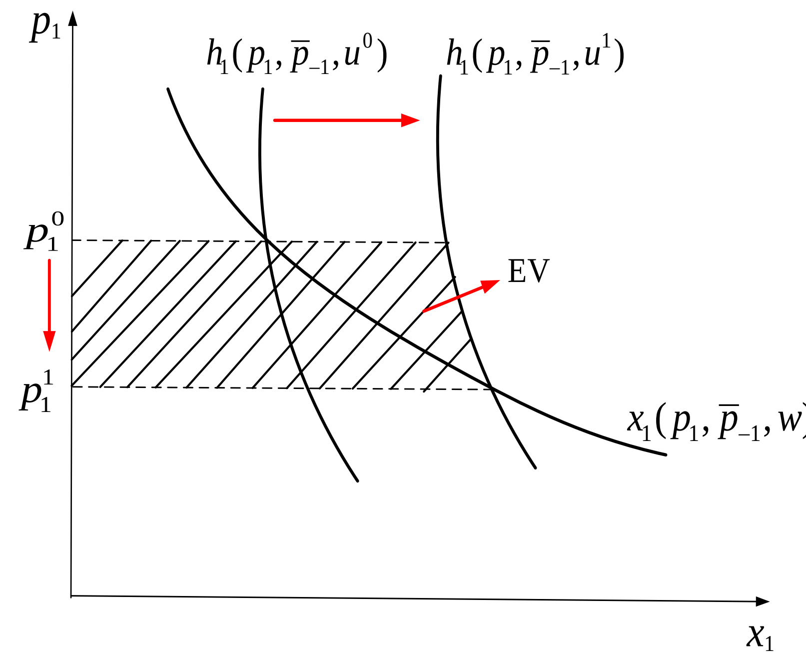
- Walrasian demand is easier to observe, so we could use the variation in consumer's surplus as an approximation of welfare changes.
- This is only valid when income effects are *zero*:
  - Recall that the Walrasian demand measures both income and substitution effects resulting from a price change, while
  - The Hicksian demand measures only substitution effects from such a price change.
- Hence, there will be a difference between CV and CS, and between EV and CS.

Why not use the Walrasian demand?

- Normal goods:



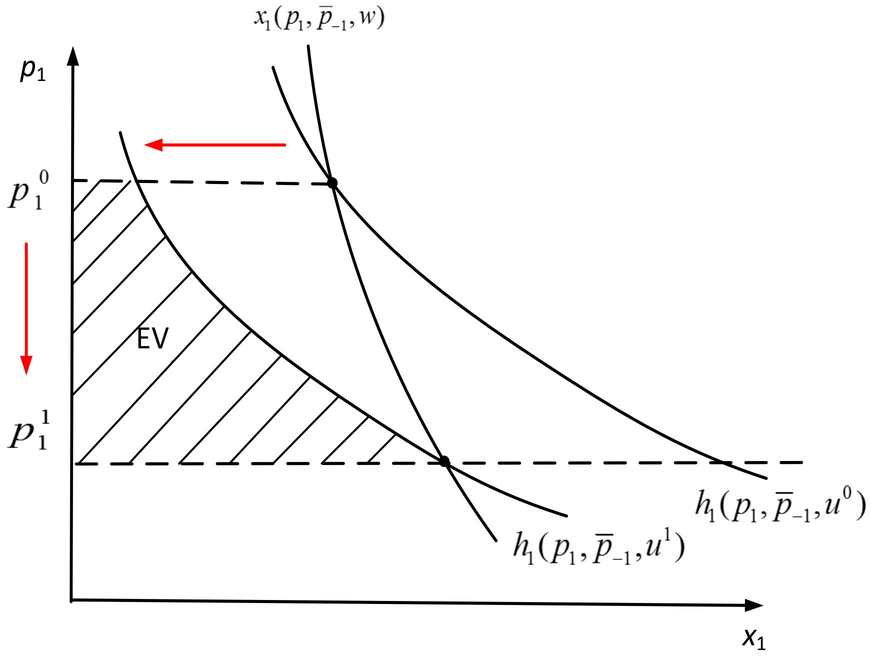
$CV < CS$



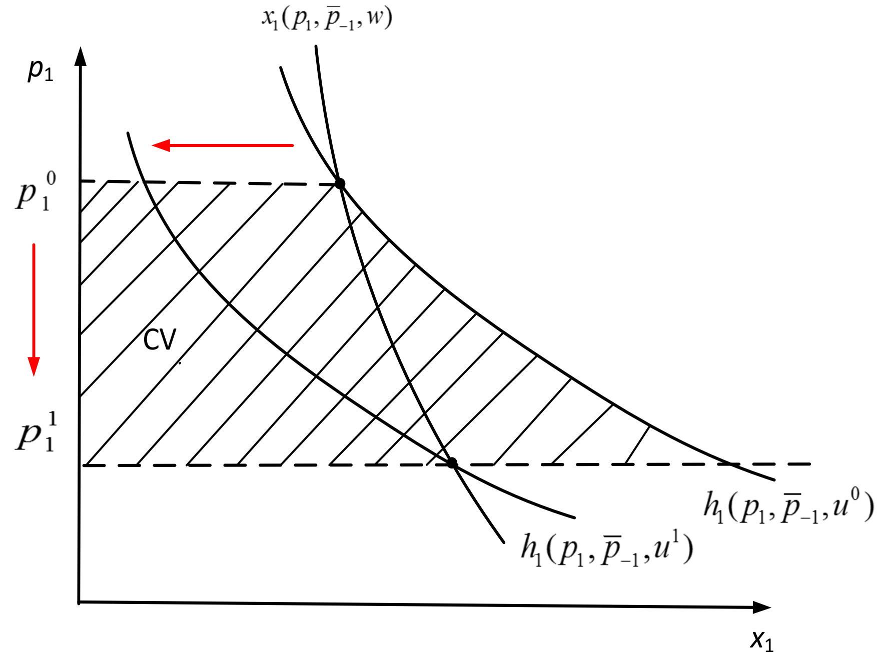
$CS < EV$

Why not use the Walrasian demand?

- Inferior goods:



$EV < CS$



$CS < CV$

Why not use the Walrasian demand?

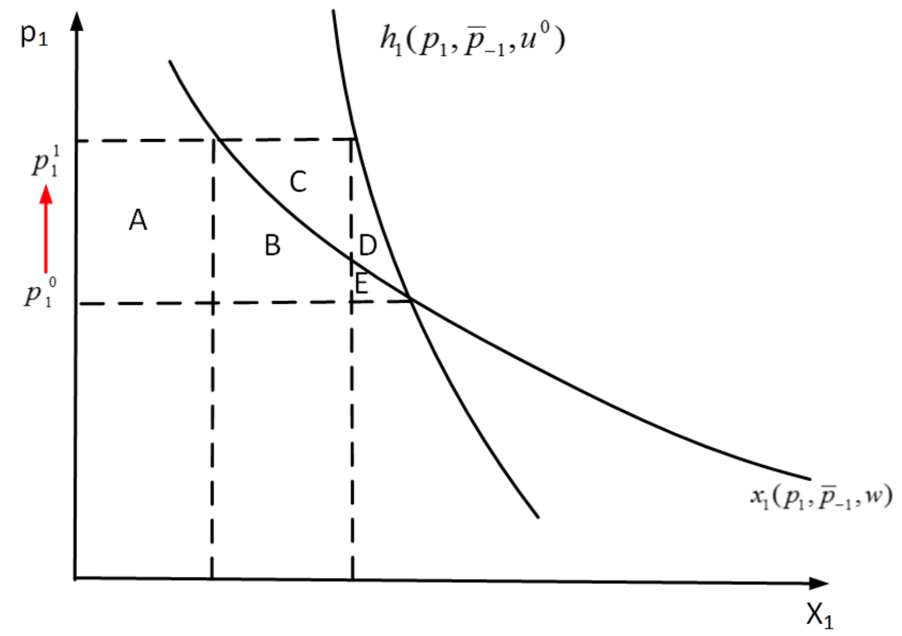
- For normal goods:
  - Price decrease: $CV < CS < EV$
  - Price increase: $CV > CS > EV$

- For inferior goods we find the opposite ranking:
  - Price decrease: $CV > CS > EV$
  - Price increase: $CV < CS < EV$

- NOTE: consumer surplus is also referred to as the ***area variation*** (AV).

When can we use the Walrasian demand?

- When the price change is small (using AV):
  - $CV = A + B + C + D + E$
  - $CS = A + B + E$
  - Measurement error from using CS (or AV) is $C + D$



When can we use the Walrasian demand?

- The measurement difference between CV (and EV) and CS, $C + D$, is relatively small:
  1) When income effects are small:
     - Graphically, $x(p, w)$ and $h(p, u)$ almost coincide.
     - The welfare change using the CV and EV coincide too.
  2) When the price change is very small:
     - The error involved in using *AV*, i.e., areas $C + D$, as a fraction of the true welfare change, becomes small. That is,

$$\lim_{(p_1^1 - p_1^0) \to 0} \frac{C + D}{CV} = 0$$

When can we use the Walrasian demand?

- However, if we measure the approximation error by $\frac{C+D}{DW}$, where $DW = D + E$, then

$$\lim_{(p_1^1 - p_1^0)\to 0} \frac{C + D}{DW}$$

does not necessarily converge to zero.

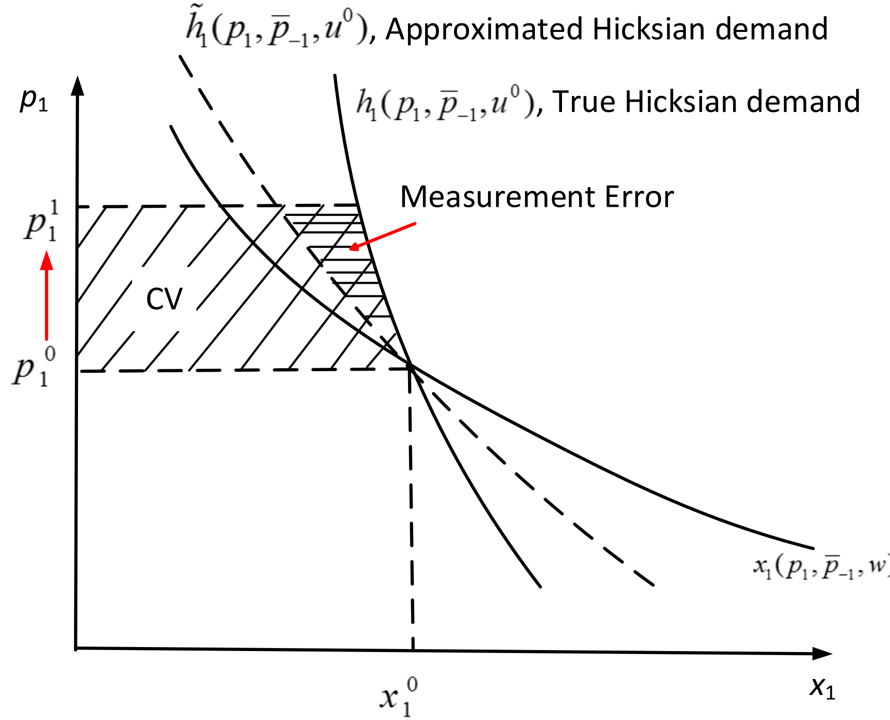
When can we use the Walrasian demand?

- Another possibility when the price change is relatively small:
  - Take a first-order Taylor approximation of $h(p,u^0)$ at p^0,
$$\tilde{h}(p,u^0) = h(p^0,u^0) + D_p h(p^0,u^0)(p - p^0)$$
and then calculate
$$CV = \int_{p_1^0}^{p_1^1} \tilde{h}_1(p_1,\bar{p}_{-1},u^0)dp_1$$
where, since $h(p^0,u^0) = x(p^0,w)$ and $D_p h(p^0,u^0) = S(p^0,w)$, we can rewrite the "approximated" Hicksian as
$$\tilde{h}(p,u^0) = x(p^0,w) + S(p^0,w)(p - p^0)$$

When can we use the Walrasian demand?

- The "approximated" Hicksian demand $\tilde{h}_1(p_1, \bar{p}_{-1}, u^0)$ lies between the true Hicksian demand, $h_1(p_1, \bar{p}_{-1}, u^0)$, and the Walrasian demand, $x(p^0, w)$.

- Since $\tilde{h}(p, u^0)$ has the same slope as $h(p, u^0)$ at price p^0, and a small change in prices will not imply a big error.



Application of IE and SE

- From the Slutsky equation, we know

$$\frac{\partial h_1(p,u)}{\partial p_1} = \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w)$$

- Multiplying both terms by $\frac{p_1}{x_1}$,

$$\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{x_1} = \frac{\partial x_1(p,w)}{\partial p_1}\frac{p_1}{x_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w)\frac{p_1}{x_1}$$

And multiplying all terms by $\frac{w}{w} = 1$,

$$\underbrace{\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{x_1}}_{\substack{\text{Substitution Price} \\ \text{elasticity of demand} \\ \tilde{\varepsilon}_{p,Q}}} = \underbrace{\frac{\partial x_1(p,w)}{\partial p_1}\frac{p_1}{x_1}}_{\substack{\text{Price elasticity} \\ \text{of demand} \\ \varepsilon_{p,Q}}} + \underbrace{\frac{\partial x_1(p,w)}{\partial w} x_1(p,w)\frac{p_1}{x_1}\frac{w}{w}}_{?}$$

Application of IE and SE

- Rearranging the last term, we have

$$\frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \frac{p_1}{x_1} \frac{w}{w}$$

$$= \underbrace{\frac{\partial x_1(p,w)}{\partial w} \frac{w}{x_1}}_{\substack{\text{Income elasticity} \\ \text{of demand} \\ \varepsilon_{w,Q}}} \cdot \underbrace{\frac{p_1 x_1(p,w)}{w}}_{\substack{\text{Share of budget} \\ \text{spent on good 1, } \theta}}$$

- We can then rewrite the Slutsky equation in terms of elasticities as follows

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta$$

Application of IE and SE

- ***Example***: consider a good like housing, with $\theta = 0.4$, $\varepsilon_{w,Q} = 1.38$, and $\varepsilon_{p,Q} = -0.6$.
- Therefore,

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta = -0.6 + 1.38 \cdot 0.4 = -0.05$$

- If price of housing rises by 10%, and consumers do not receive a wealth compensation to maintain their welfare unchanged, consumers reduce their consumption of housing by 6%.
- However, if consumers receive a wealth compensation, the housing consumption will only fall by 0.5%.
  - Intuition: Housing is such an important share of my monthly expenses, that higher prices lead me to significantly reduce my consumption (if not compensated), but to just slightly do so (if compensated).

Application of IE and SE

- Other useful lessons from the previous expression

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta$$

- Price-elasticities very close $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$ if
  - Share of budget spent on this particular good, θ, is very small (Example: garlic).
  - The income-elasticity is really small (Example: pizza).
- Advantages if $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$:
  - The Walrasian and Hicksian demand are very close to each other. Hence, $CV \simeq EV \simeq CS$.

Application of IE and SE

- You can read sometimes “in this study we use the change in CS to measure welfare changes due to a price increase given that income effects are negligible”
  - What the authors are referring to is:
    - Share of budget spent on the good is relatively small and/or
    - The income-elasticity of the good is small
- Remember that our results are not only applicable to price changes, but also to changes in the sales taxes.
- For which preference relations a price change induces no income effect? Quasilinear.

Application of IE and SE

- In 1981 the US negotiated voluntary automobile export restrictions with the Japanese government.
- Clifford Winston (1987) studied the effects of these export restrictions:
  – Car prices: p_{Jap} was 20% higher with restrictions that without. p_{US} was 8% higher with restrictions than without.
  – What is the effect of these higher prices on consumer's welfare?
  – Would you use CS? Probably not, since both θ and $\varepsilon_{w,Q}$ are relatively high.

Application of IE and SE

- Winston did not use CS. Instead, he focused on the CV. He found that CV = -$14 billion.
  - *Intuition*: The wealth compensation that domestic car owners would need after the price change (after setting the export restrictions) in order to be as well off as they were before the price change is $14 billion.
- This implies that, considering that in 1987 there were 179 million car owners in the US, the wealth compensation per car owner should have been $14,000/$179 = $78.
- Of course, this is an underestimation, since we should divide over the number of new care owners during the period of export restriction was active (not the number of all car owners).

Application of IE and SE

- Jerry Hausmann (MIT) measures the welfare gain consumers obtain from the price decrease they experience after a Walmart store locates in their locality/country.
- He used CV. Why? Low-income families spend a non-negligible part of their budget in Wal-Mart.
- Result: welfare improvement of 3.75%.

Consumer as a Labor Supplier

Consumer as a Labor Supplier

- Consider the following UMP, where the consumer chooses the amount of goods, x, and leisure, L, that solve

$$\max_{x,L} \ u(x, L)$$

$$\text{s.t.} \ \sum_{i=1}^{K} p_i x_i \leq M = wz + \bar{M}$$

$$\text{and} \ T = z + L$$

where M is total wealth, coming from the z hours dedicated to work (at a wage w per hour), and the non-work income, $\bar{M}$. Total time T must be either dedicated to work (z) or leisure (L).

Consumer as a Labor Supplier

- Let us now use the Composite Commodity Theorem:
  - If the prices for all goods maintain a constant proportion with respect to the price of labor (wage), i.e., $p_1 = \alpha_1 w, \ldots, p_n = \alpha_n w$, we can represent these goods all by a single (composite) commodity y, with price p.
  - Then, we have only two goods: the composite commodity y and the number of hours dedicated to work, z.

Consumer as a Labor Supplier

- Hence, the UMP can be rewritten as

$$\max_{y,z} \quad v(y, z)$$
$$\text{s.t.} \quad py \leq wz + \bar{M}$$

where py represents the money spent on consumption goods, and $wz + \bar{M}$ reflects the total income originating from labor and non-labor sources.

Consumer as a Labor Supplier

- The Lagrangian of this UMP is

$$L = v(y, z) + \lambda(\bar{M} + wz - py)$$

and the FOCs (for interior optimum) are

$$\frac{\partial L}{\partial y} = v_y - \lambda p = 0 \implies \lambda = \frac{v_y}{p}$$

$$\frac{\partial L}{\partial z} = v_z + \lambda w = 0 \implies \lambda = -\frac{v_z}{w}$$

- Hence, $\frac{v_y}{p} = -\frac{v_z}{w}$.
  - That is, at the optimum the marginal utility per dollar earned working must be equal to the marginal utility per dollar spent on consumption goods.

Consumer as a Labor Supplier

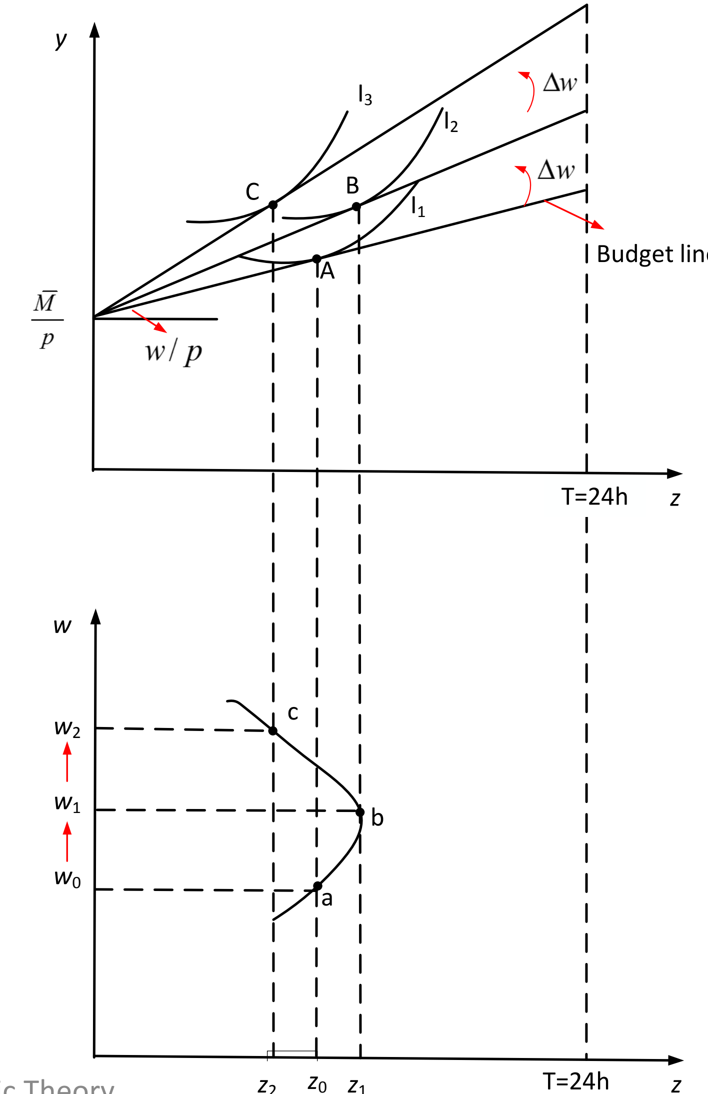
- Rearranging $\frac{v_y}{p} = -\frac{v_z}{w}$, we obtain

$$MRS_{z,y} \rightarrow -\frac{v_z}{v_y} = \frac{w}{p}$$

- Finally, using $\frac{v_y}{p} = -\frac{v_z}{w}$ and the constraint, we obtain the Walrasian demand for the composite commodity, $x_y(w, p, \bar{M})$, and the labor supply function, $x_z(w, p, \bar{M})$.

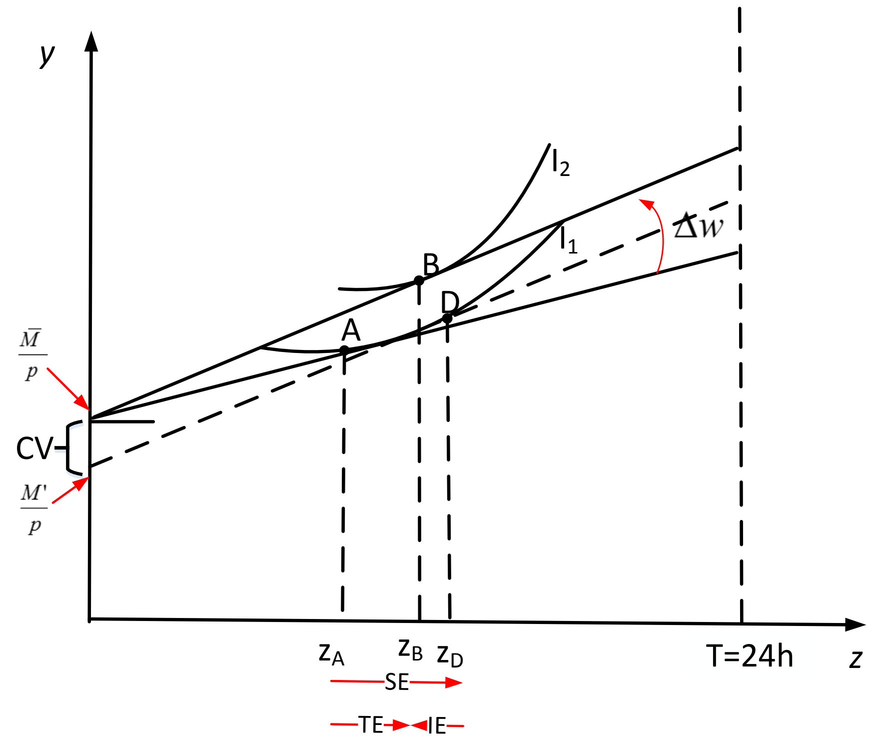
Consumer as a Labor Supplier

- Budget line: upward sloping straight line.
  - an increase in the amount of hours worked entails a larger amount of wealth.
- Indifference curve: increasing utility levels as we move northwest.
  - individual is better off when his consumption of the composite commodity increases and the number of working hours decreases.
- Labor supply: backward bending.
  - labor supply initially increases as a result of higher wages but then decreases.
  - Substitution and Income effects.



Consumer as a Labor Supplier

- SE (+):
  - Δw implies Δz, i.e., more working hours supplied by worker.
- IE (-):
  - Δw implies ∇z, i.e., less working hours supplied by worker.
- If $|IE| > |SE|$, then working hours would become a Giffen good.



Consumer as a Labor Supplier

- How to relate this income and substitution effects with the Slutsky equation?
  - First, let us state the previous problems as a EMP

$$\min_{y,z} \; \bar{M} = py - wz$$
$$\text{s.t.} \quad v(y,z) = v$$

  - From this EMP we can find the optimal hicksian demands, $h_y(w,p,v)$ and $h_z(w,p,v)$.
  - Inserting them into the objective function, we obtain the value function of this EMP (i.e., the expenditure function):

$$e(w,p,v) = ph_y(w,p,v) + wh_z(w,p,v)$$

Consumer as a Labor Supplier

- How to relate this income and substitution effects with the Slutsky equation?
  - We know that

$$x_z\Big(\underbrace{w,p}_{\text{prices}},\underbrace{e(w,p,v)}_{\text{income}}\Big) = h_z(w,p,v)$$

Differentiating both sides with respect to w and using the chain rule

$$\frac{\partial x_z}{\partial w} + \frac{\partial x_z}{\partial e}\frac{\partial e}{\partial w} = \frac{\partial h_z}{\partial w} \Leftrightarrow \frac{\partial x_z}{\partial w} = \frac{\partial h_z}{\partial w} - \frac{\partial x_z}{\partial e}\frac{\partial e}{\partial w}$$

and since we know that $\frac{\partial e(w,p,v)}{\partial w} = -h_z(w,p,v)$, then

$$\frac{\partial x_z}{\partial w} = \frac{\partial h_z}{\partial w} + \frac{\partial x_z}{\partial e} h_z(w,p,v)$$

Consumer as a Labor Supplier

- Using the Slutsky equation (SE and IE) in the analysis of labor markets:

$$\frac{\partial x_z}{\partial w} = \frac{\partial h_z}{\partial w} + \frac{\partial x_z}{\partial e} h_z(w, p, v)$$

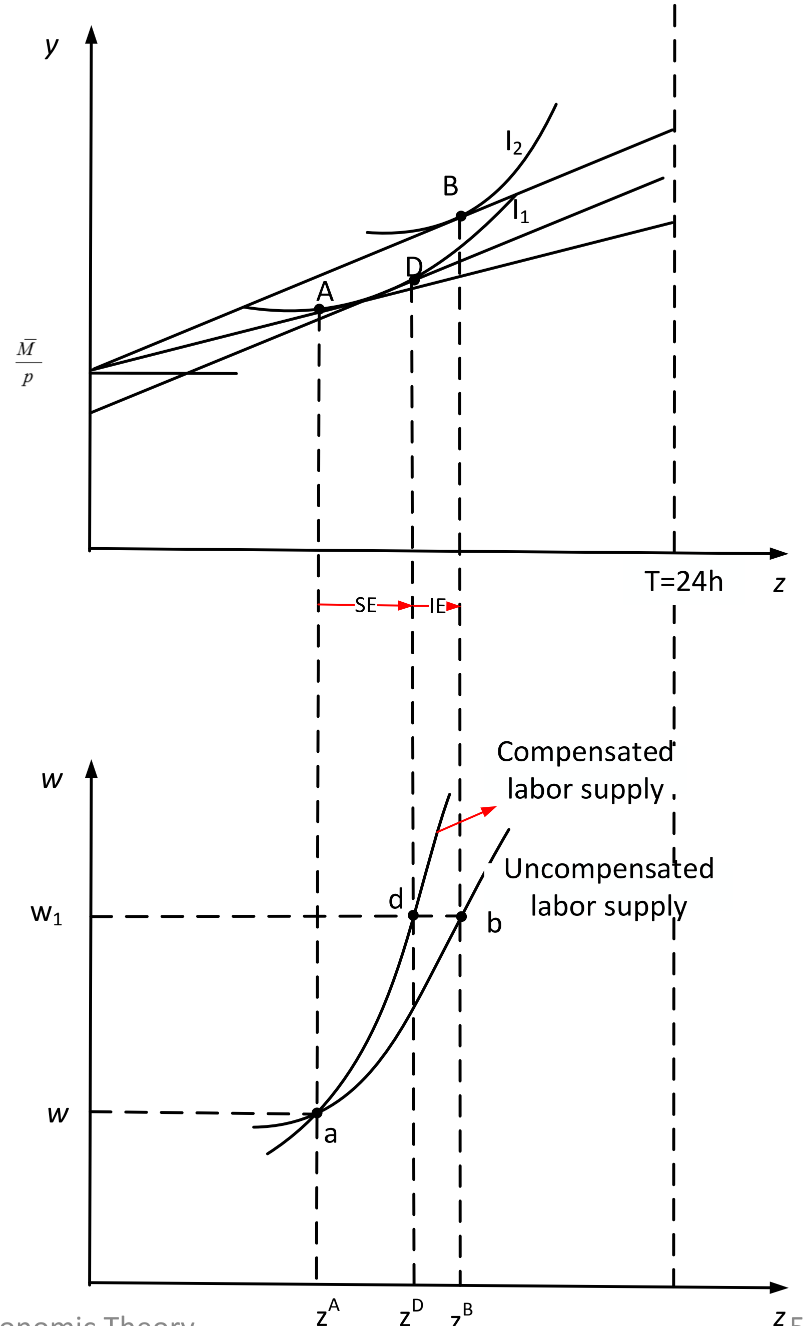
where

1) $\frac{\partial h_z}{\partial w} > 0$ is the SE effect: an increase in wages increases the worker's supply of labor, if we give him a wealth compensation.

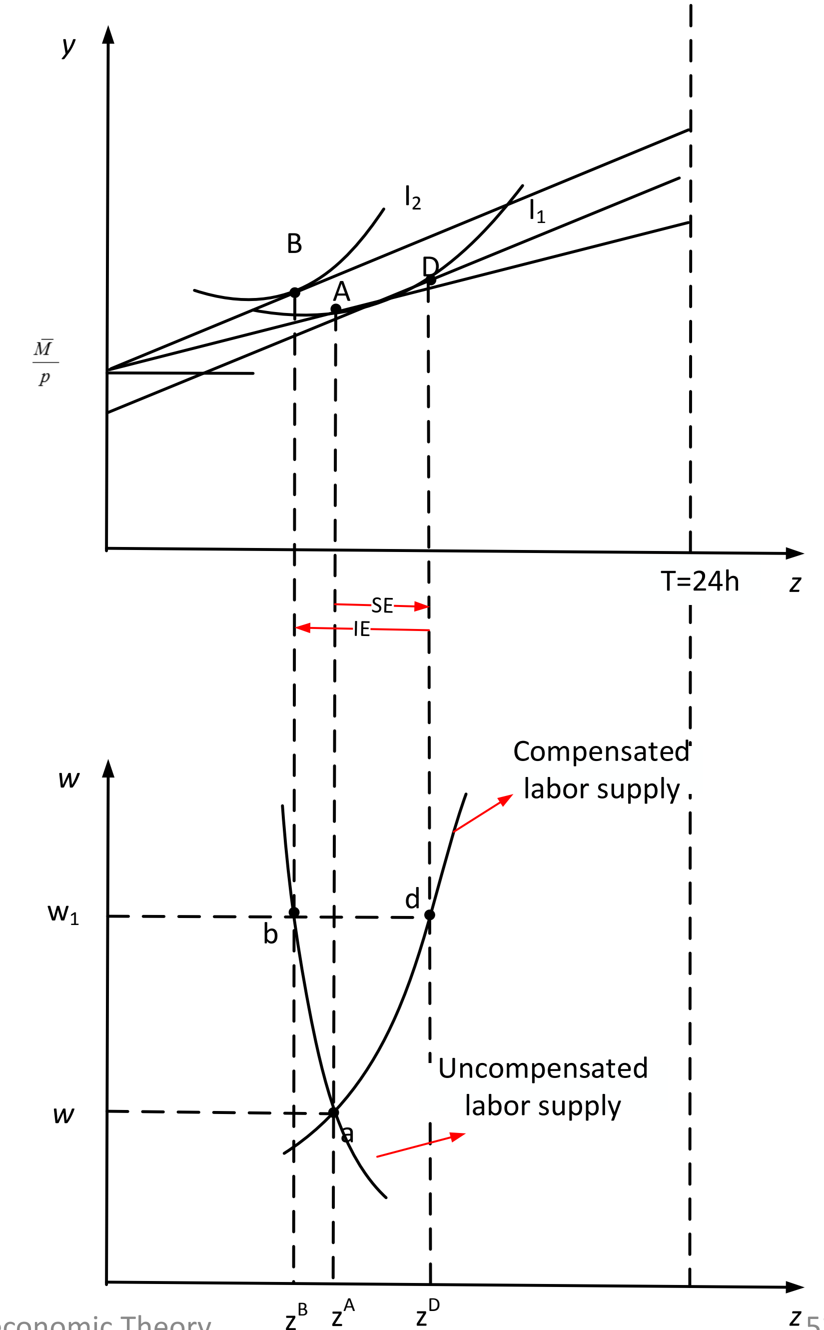
2) $\frac{\partial x_z}{\partial e} h_z(w, p, v)$ is the IE:

- If $\frac{\partial x_z}{\partial e} > 0$, an increase in wages makes that worker richer, and he decides to work ***more*** (this would be an upward bending supply curve);
- If $\frac{\partial x_z}{\partial e} < 0$, an increase in wages makes that worker richer, but he decides to work ***less*** (e.g., nurses in Mass.).

- Income effect from a wage increase is positive, $IE > 0$.
  - a positively sloped labor supply curve for all wages
- The compensated supply curve is positive sloped:
  - It captures the SE due to the wage increase, but not the IE.
- The uncompensated labor supply curve, in contrast, represents both the SE and IE.

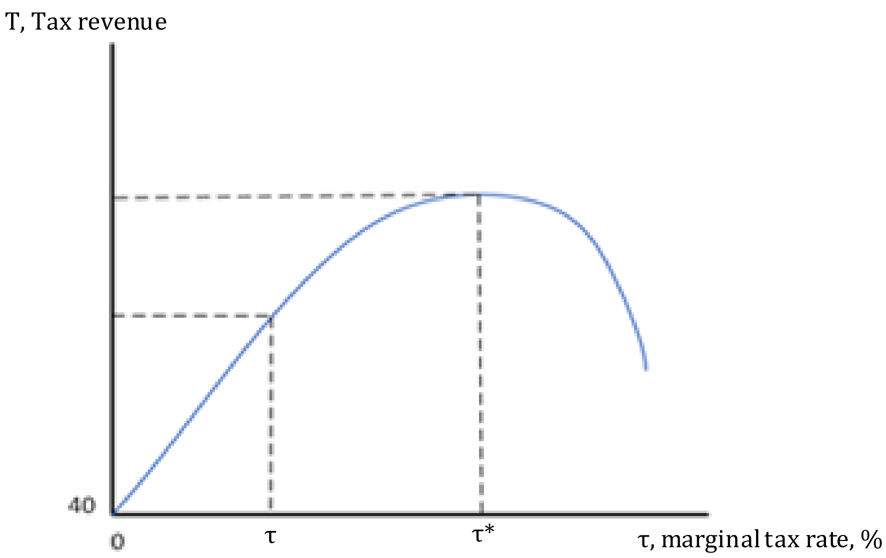


- When $IE < 0$ and $|IE| > |SE|$, implying that $TE < 0$.
- In this case, the uncompensated supply curve becomes negatively sloped.



The Laffer Curve

- An increase in the tax rate might initially *increase* tax revenue but, after a certain rate, further increments might *reduce* the tax revenues.
- Or, alternatively, a decrease in the marginal tax rate can actually increase tax revenues.
- This suggests that there is an optimal tax rate τ^* which will bring in the most tax revenue.



Consumer as a Labor Supplier

- Consider salary w per hour, and a net salary of
$$\omega = (1 - \tau)w$$
after taxes.
- Hence, $H(\omega)$ represents the number of working hours, where workers consider their net wage when deciding how many hours to work.
- Therefore, tax revenue is
$$T = \tau \cdot w \cdot H(\omega)$$

Consumer as a Labor Supplier

- Since total tax revenue is $T = \tau \cdot w \cdot H(\omega)$, the effect of marginally increasing the tax rate is

$$\frac{\partial T}{\partial \tau} = w \cdot H(\omega) + \tau \cdot w \cdot (-w) \frac{\partial H}{\partial \omega}$$

$$= \underbrace{w \cdot H(\omega)}_{\text{Positive effect}} - \underbrace{\tau \cdot w^2 \cdot \frac{\partial H}{\partial \omega}}_{\text{Negative effect}}$$

  - The *positive effect* represents that, for a given supply of working hours, an increase in the tax rate increases tax revenue.
  - The *negative effect* represents that an increase in the tax rate reduces the amount of working hours supplied and, hence, tax revenue.

Consumer as a Labor Supplier

- Therefore, under which conditions we can guarantee that $\frac{\partial T}{\partial \tau} < 0$ (so that an increase in tax rates actually decreases total tax collection, as proposed by the Laffer curve)?
  - We need

$$w \cdot H(\omega) - \tau \cdot w^2 \cdot \frac{\partial H}{\partial \omega} < 0$$

That is,

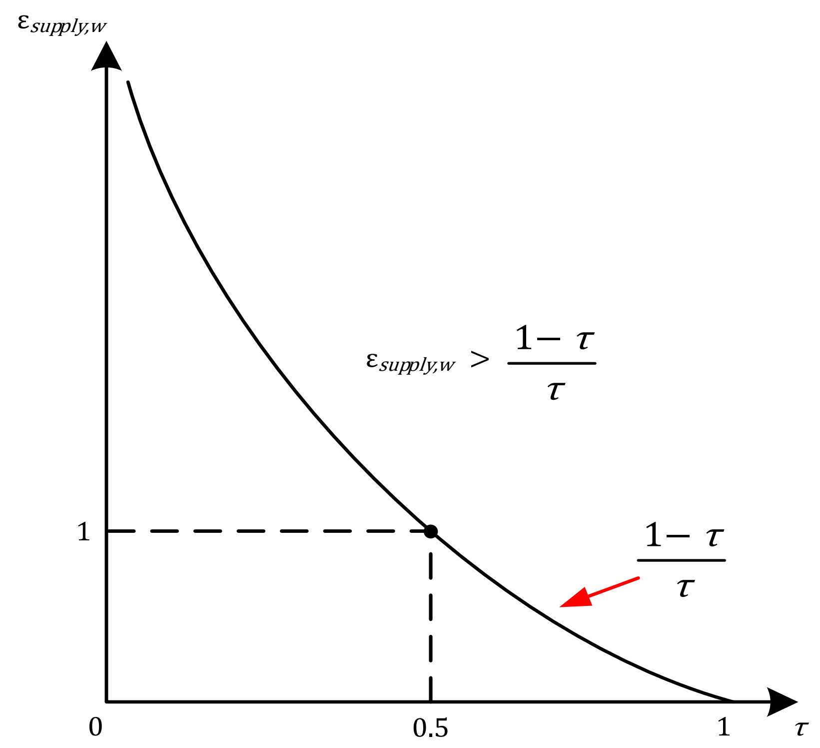
$$w \cdot H(\omega) < \tau \cdot w^2 \cdot \frac{\partial H}{\partial \omega} \quad \text{or} \quad \frac{1}{\tau} < \frac{\partial H}{\partial \omega} \frac{w}{H(\omega)}$$

Multiplying both sides by $(1 - \tau)$ yields

$$\frac{1 - \tau}{\tau} < \frac{\partial H}{\partial \omega} \frac{\overbrace{w(1 - \tau)}^{\omega}}{H(\omega)} \quad \Longrightarrow \quad \frac{1 - \tau}{\tau} < \varepsilon_{\text{supply}, \omega}$$

Consumer as a Labor Supplier

- The area above (below) cutoff $\frac{1-\tau}{\tau}$ represents combinations of the elasticity of labor supply ($\varepsilon_{\text{supply},\,\omega}$) and tax rates ($\tau$) for which a marginal increase in the tax rate yields a smaller (larger, respectively) total tax revenue.



Consumer as a Labor Supplier

- Hence, for total tax revenue to fall after an increase in the tax rate, τ, we need

$$\frac{1-\tau}{\tau} < \varepsilon_{\text{supply},\,\omega}$$

- ***Example 1***: If the marginal tax rate for the most affluent citizens is $\tau = 0.8$, then the above condition implies

$$\frac{1-0.8}{0.8} = 0.25 < \varepsilon_{\text{supply},\,\omega}$$

which is likely to be satisfied.

Consumer as a Labor Supplier

- ***Example 2***: An economy in which the maximum marginal tax rate is $\tau = 0.35$, would need
$$\frac{1-0.35}{0.35} = 1.85 < \varepsilon_{\text{supply},\,\omega}$$
for total tax revenue to increase, which is very unlikely to hold for the average worker in most developed countries.

Gross/Net Complements and Gross/Net Substitutes

Demand Relationships among Goods

- So far, we were focusing on the SE and IE of varying the price of good k on the demand for good k.

- Now, we analyze the SE and IE of varying the price of good k on the demand for other good j.

Demand Relationships among Goods

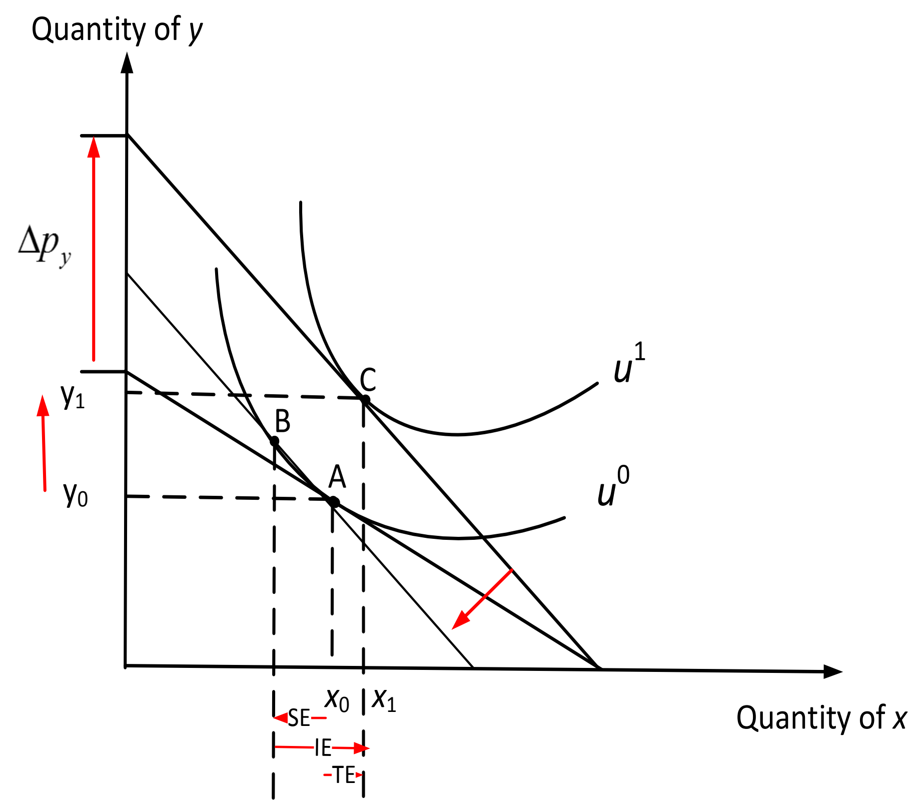
- For simplicity, let us start our analysis with the two-good case.
  - This will help us graphically illustrate the main intuitions.

- Later on we generalize our analysis to $N > 2$ goods.

Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *small* that the consumer purchases more x and more y.
  - In this case, we call x and y ***gross complements***.

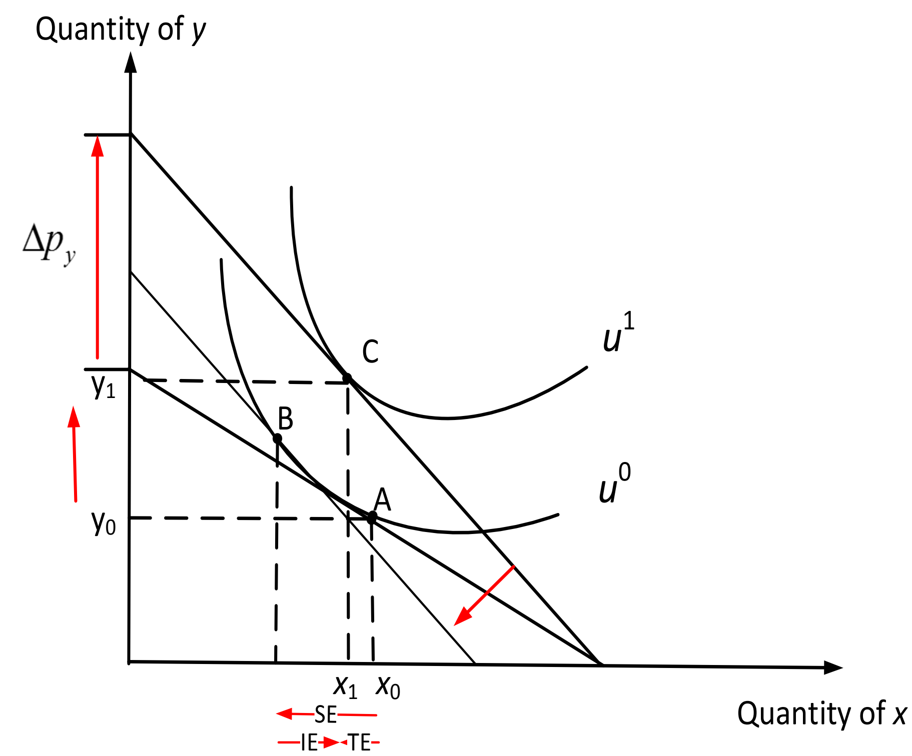
$$\frac{\partial x}{\partial p_y} < 0$$



Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *large* that the consumer purchases less x and more y.
  - In this case, we call x and y ***gross substitutes***.

$$\frac{\partial x}{\partial p_y} > 0$$



Demand Relationships among Goods: The Two-Good Case

- A mathematical treatment
  - The change in x caused by changes in p_y can be shown by a Slutsky-type equation:

$$\frac{\partial x}{\partial p_y} = \underbrace{\underbrace{\frac{\partial h_x}{\partial p_y}}_{SE\ (+)} - \underbrace{y\frac{\partial x}{\partial w}}_{\substack{IE: \\ (-)\text{ if } x \text{ is normal} \\ (+)\text{ if } x \text{ is inferior}}}}_{\text{Combined effect (ambiguous)}}$$

$SE > 0$ is not a typo: Δp_y induces the consumer to buy more of good x, if his utility level is kept constant. Graphically, we are moving along the same indifference curve.

Demand Relationships among Goods: The Two-Good Case

- Or, in elasticity terms

$$\varepsilon_{x,p_y} = \underbrace{\tilde{\varepsilon}_{x,p_y}}_{SE\ (+)} - \underbrace{\theta_y \varepsilon_{x,w}}_{\substack{IE: \\ (-)\text{ if } x \text{ is normal} \\ (+)\text{ if } x \text{ is inferior}}}$$

where θ_y denotes the share of income spent on good y. The combined effect of Δp_y on the observable Walrasian demand, $x(p,w)$, is ambiguous.

Demand Relationships among Goods: The Two-Good Case

- ***Example***: Let's show the SE and IE across different goods for a Cobb-Douglas utility function $u(x, y) = x^{0.5}y^{0.5}$.
  - The Walrasian demand for good x is
$$x(p, w) = \frac{1}{2}\frac{w}{p_x}$$
  - The Hicksian demand for good x is
$$h_x(p, u) = \frac{\sqrt{p_y}}{\sqrt{p_x}} \cdot u$$

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - First, not that differentiating $x(p,w)$ with respect to p_y, we obtain
$$\frac{\partial x(p,w)}{\partial p_y} = 0$$
    i.e., variations in the price of good y do not affect consumer's Walrasian demand.
  - But,
$$\frac{\partial h_x(p,u)}{\partial p_y} = \frac{1}{2}\frac{u}{\sqrt{p_x p_y}} \neq 0$$
  - How can these two (seemingly contradictory) results arise?

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - Answer: the SE and IE completely offset each other.
  - **Substitution Effect**: Given

$$\frac{\partial h_x(p,u)}{\partial p_y} = \frac{1}{2}\frac{u}{\sqrt{p_x p_y}},$$

    plug the indirect utility function $u = \frac{1}{2}\frac{w}{\sqrt{p_x p_y}}$ to obtain a SE of $\frac{1}{4}\frac{w}{p_x p_y}$.
  - **Income Effect**:

$$-y\frac{\partial x}{\partial w} = -\left(\frac{1}{2}\frac{w}{p_y}\right)\left(\frac{1}{2}\frac{1}{p_x}\right) = -\frac{1}{4}\frac{w}{p_x p_y}$$

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - Therefore, the total effect is

$$\overbrace{\frac{\partial x(p,w)}{\partial p_y}}^{TE} = \overbrace{\frac{\partial h_x}{\partial p_y}}^{SE} - \overbrace{y\frac{\partial x}{\partial w}}^{IE}$$
$$= \frac{1}{4}\frac{w}{p_x p_y} - \frac{1}{4}\frac{w}{p_x p_y} = 0$$

  - Intuitively, this implies that the substitution and income effect completely offset each other.

Demand Relationships among Goods: The Two-Good Case

- Common mistake:
  - "$\frac{\partial x(p,w)}{\partial p_y} = 0$ means that good x and y cannot be substituted in consumption. That is, they must be consumed in fixed proportions. Hence, this consumer's utility function is a Leontieff type."

- No! We just showed that

$$\frac{\partial x(p,w)}{\partial p_y} = 0 \implies \frac{\partial h_x}{\partial p_y} = y\frac{\partial x}{\partial w}$$

  i.e., the SE and IE completely offset each other.

- For the above statement to be true, we would need that the IE is zero, i.e., $y\frac{\partial x}{\partial w} = 0$.

Demand Relationships among Goods: The N-Good Case

- We can, hence, generalize the Slutsky equation to the case of $N > 2$ goods as follows:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial w}$$

  for any i and j.

- The change in the price of good j induces IE and SE on good i.

Asymmetry of the Gross Substitute and Complement

- Two goods are ***substitutes*** if one good may replace the other in use.
  - Example: tea and coffee, butter and margarine
- Two goods are ***complements*** if they are used together.
  - Example: coffee and cream, fish and chips.
- The concepts of gross substitutes and complements include both SE and IE.
  - Two goods are gross substitutes if $\frac{\partial x_i}{\partial p_j} > 0$.
  - Two goods are gross complements if $\frac{\partial x_i}{\partial p_j} < 0$.

Asymmetry of the Gross Substitute and Complement

- The definitions of gross substitutes and complements are *not* necessarily symmetric.
  - It is possible for x_1 to be a substitute for x_2 and at the same time for x_2 to be a complement of x_1.
- Let us see this potential asymmetry with an example.

Asymmetry of the Gross Substitute and Complement

- Suppose that the utility function for two goods is given by

$$U(x, y) = \ln x + y$$

- The Lagrangian of the UMP is

$$L = \ln x + y + \lambda(w - p_x x - p_y y)$$

- The first order conditions are

$$\frac{\partial L}{\partial x} = \frac{1}{x} - \lambda p_x = 0$$

$$\frac{\partial L}{\partial y} = y - \lambda p_y = 0$$

$$\frac{\partial L}{\partial \lambda} = w - p_x x - p_y y = 0$$

Asymmetry of the Gross Substitute and Complement

- Manipulating the first two equations, we get

$$\frac{1}{p_x x} = \frac{1}{p_y} \implies p_x x = p_y$$

- Inserting this into the budget constraint, we can find the Marshallian demand for y

$$\underbrace{p_x x}_{p_y} + p_y y = w \implies p_y y = w - p_y \implies$$

$$y = \frac{w - p_y}{p_y}$$

Asymmetry of the Gross Substitute and Complement

- An increase in p_y causes a decline in spending on y
  - Since p_x and w are unchanged, spending on x must rise $\left(\frac{\partial x}{\partial p_y} > 0\right)$.
    - Hence, x and y are gross substitutes.
  - But spending on y is independent of p_x $\left(\frac{\partial y}{\partial p_x} = 0\right)$.
    - Thus, x and y are neither gross substitutes nor gross complements.
  - This shows the asymmetry of gross substitute and complement definitions.
    - While good y is a gross substitute of x, good x is neither a gross substitute or complement of y.

Asymmetry of the Gross Substitute and Complement

- Depending on how we check for gross substitutability or complementarities between two goods, there is potential to obtain different results.

- Can we use an alternative approach to check if two goods are complements or substitutes in consumption?
  – Yes. We next present such approach.

Net Substitutes and Net Complements

- The concepts of net substitutes and complements focus solely on SE.
  - Two goods are ***net (or Hicksian) substitutes*** if

$$\frac{\partial h_i}{\partial p_j} > 0$$

  - Two goods are ***net (or Hicksian) complements*** if

$$\frac{\partial h_i}{\partial p_j} < 0$$

where $h_i(p_i, p_j, u)$ is the Hicksian demand of good i.

Net Substitutes and Net Complements

- This definition looks only at the shape of the indifference curve.
- This definition is unambiguous because the definitions are perfectly symmetric

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

  – This implies that every element above the main diagonal in the Slutsky matrix is symmetric with respect to the corresponding element below the main diagonal.

Net Substitutes and Net Complements

$$S(p,w) = \begin{pmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \frac{\partial h_1(p,u)}{\partial p_2} & \frac{\partial h_1(p,u)}{\partial p_3} \\ \frac{\partial h_2(p,u)}{\partial p_1} & \frac{\partial h_2(p,u)}{\partial p_2} & \frac{\partial h_2(p,u)}{\partial p_3} \\ \frac{\partial h_3(p,u)}{\partial p_1} & \frac{\partial h_3(p,u)}{\partial p_2} & \frac{\partial h_3(p,u)}{\partial p_3} \end{pmatrix}$$

$$\frac{\partial h_1(p,u)}{\partial p_2} = \frac{\partial h_2(p,u)}{\partial p_1}, \quad \frac{\partial h_1(p,u)}{\partial p_3} = \frac{\partial h_3(p,u)}{\partial p_1}, \quad \frac{\partial h_2(p,u)}{\partial p_3} = \frac{\partial h_3(p,u)}{\partial p_2}$$

Net Substitutes and Net Complements

- Proof:
  - Recall that, from Shephard's lemma, $h_k(p,u) = \frac{\partial e(p,u)}{\partial p_k}$. Hence,

$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_k \partial p_j}$$

  - Using Young's theorem, we obtain

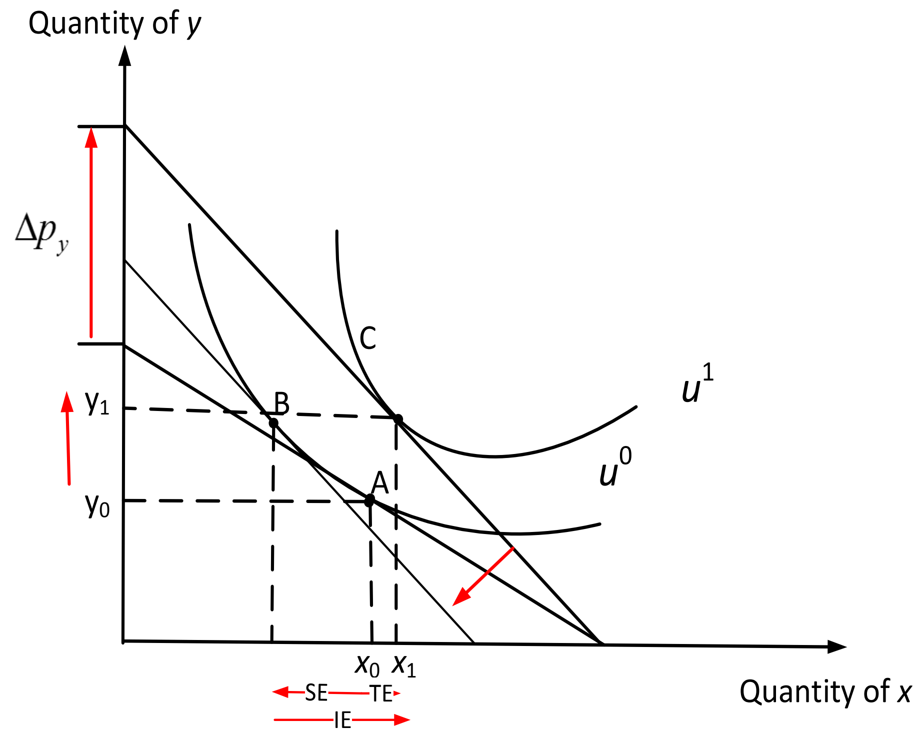
$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_j \partial p_k}$$

which implies

$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_k}$$

Net Substitutes and Net Complements

- Even though x and y are gross complements, they are net substitutes.
- Since MRS is diminishing, the own-price SE must be negative $(SE < 0)$ so the cross-price SE must be positive $(TE > 0)$.



A Note on the Euler's Theorem

- We say that a function $f(x_1, x_2)$ is homogeneous of degree k if
$$f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$$
- Differentiating this expression with respect to x_1, we obtain
$$\frac{\partial f(tx_1, tx_2)}{\partial x_1} \cdot t = t^k \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$
or, rearranging,
$$\frac{\partial f(tx_1, tx_2)}{\partial x_1} = t^{k-1} \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$

A Note on the Euler's Theorem

- Last, denoting $f_1 \equiv \frac{\partial f}{\partial x_1}$, we obtain

$$f_1(tx_1, tx_2) = t^{k-1} \cdot f_1(x_1, x_2)$$

- Hence, if a function is homogeneous of degree k, its first-order derivative must be homogeneous of degree $k - 1$.

A Note on the Euler's Theorem

- Differentiating the left-hand side of the definition of homogeneity, $f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$, with respect to t yields

$$\frac{\partial(tx_1, tx_2)}{\partial t} = f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2$$

- Differentiating the right-hand side produces

$$\frac{\partial(t^k \cdot f(x_1, x_2)}{\partial t} = k \cdot t^{k-1} f(x_1, x_2)$$

A Note on the Euler's Theorem

- Combining the differentiation of LHS and RHS,
$$f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2$$
$$= k \cdot t^{k-1} f(x_1, x_2)$$
- Setting $t = 1$, we obtain

$$f_1(x_1, x_2)x_1 + f_2(x_1, x_2)x_2 = k \cdot f(x_1, x_2)$$

  where k is the homogeneity order of the original function $f(x_1, x_2)$.
  - If $k = 0$, the above expression becomes 0.
  - If $k = 1$, the above expression is $f(x_1, x_2)$.

A Note on the Euler's Theorem

- ***Application***:
  - The Hicksian demand is homogeneous of degree zero in prices, that is,
  $$h_k(tp_1, tp_2, \dots, tp_n, u) = h_k(p_1, p_2, \dots, p_n, u)$$
  - Hence, multiplying all prices by t does not affect the value of the Hicksian demand.
  - By Euler's theorem,
  $$\frac{\partial h_i}{\partial p_1} p_1 + \frac{\partial h_i}{\partial p_2} p_2 + \cdots + \frac{\partial h_i}{\partial p_n} p_n$$
  $$= 0 \cdot t^{0-1} h_i(p_1, p_2, \dots, p_n, u) = 0$$

Substitutability with Many Goods

- ***Question:*** Is net substitutability or complementarity more prevalent in real life?
- To answer this question, we can start with the compensated demand function

$$h_k(p_1, p_2, \dots, p_n, u)$$

- Applying Euler's theorem yields

$$\frac{\partial h_k}{\partial p_1} p_1 + \frac{\partial h_k}{\partial p_2} p_2 + \cdots + \frac{\partial h_k}{\partial p_n} p_n = 0$$

- Dividing both sides by h_k, we can alternatively express the above result using compensated elasticities

$$\tilde{\varepsilon}_{i1} + \tilde{\varepsilon}_{i2} + \cdots + \tilde{\varepsilon}_{in} \equiv 0$$

Substitutability with Many Goods

- Since the negative sign of the SE implies that $\tilde{\varepsilon}_{ii} \leq 0$, then the sum of Hicksian cross-price elasticities for all other $j \neq i$ goods should satisfy

$$\sum_{j \neq i} \tilde{\varepsilon}_{ij} \geq 0$$

- Hence, “most” goods must be substitutes.
- This is referred to as ***Hick’s second law of demand***.

Aggregate Demand

Aggregate Demand

- We now move from individual demand, $x_i(p, w_i)$, to aggregate demand,

$$\sum_{i=1}^{I} x_i(p, w_i)$$

which denotes the total demand of a group of I consumers.

- Individual i's demand $x_i(p, w_i)$ still represents a vector of L components, describing his demand for L different goods.

Aggregate Demand

- We know individual demand depends on prices and individual's wealth.
  - When can we express aggregate demand as a function of prices and aggregate wealth?
  - In other words, when can we guarantee that aggregate demand defined as

    $$x(p, w_1, w_2, \dots, w_I) = \sum_{i=1}^{I} x_i(p, w_i)$$

    satisfies

    $$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

Aggregate Demand

- This is satisfied if, for any two distributions of wealth, $(w_1, w_2, \dots, w_I)$ and $(w_1', w_2', \dots, w_I')$ such that $\sum_{i=1}^{I} w_i = \sum_{i=1}^{I} w_i'$, we have

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} x_i(p, w_i')$$

- For such condition to be satisfied, let's start with an initial distribution $(w_1, w_2, \dots, w_I)$ and apply a differential change in wealth $(dw_1, dw_2, \dots, dw_I)$ such that the aggregate wealth is unchanged, $\sum_{i=1}^{I} dw_i = 0$.

Aggregate Demand

- If aggregate demand is just a function of aggregate wealth, then we must have that

  $$\sum_{i=1}^{I} \frac{\partial x_i(p,w_i)}{\partial w_i} dw_i = 0 \text{ for every good } k$$

  In words, the wealth effects of different individuals are compensated in the aggregate. That is, in the case of two individuals i and j,

  $$\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$$

  for every good k.

Aggregate Demand

- This result *does not* imply that $IE_i > 0$ while $IE_j < 0$.
- In addition, it indicates that its absolute values coincide, i.e., $|IE_i| = \left|IE_j\right|$, which means that any redistribution of wealth from consumer i to j yields

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} dw_i + \frac{\partial x_{kj}(p, w_j)}{\partial w_j} dw_j = 0$$

which can be rearranged as

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} \underbrace{dw_i}_{-} = -\frac{\partial x_{kj}(p, w_j)}{\partial w_j} \underbrace{dw_j}_{+}$$

- Hence, $\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$, since $|dw_i| = \left|dw_j\right|$.

Aggregate Demand

- In summary, for any
  - fixed price vector p,
  - good k, and
  - wealth level any two individuals i and j

  the wealth effect is the same across individuals.
- In other words, the wealth effects arising from the distribution of wealth across consumers cancel out.
- This means that we can express aggregate demand as a function of aggregate wealth

$$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

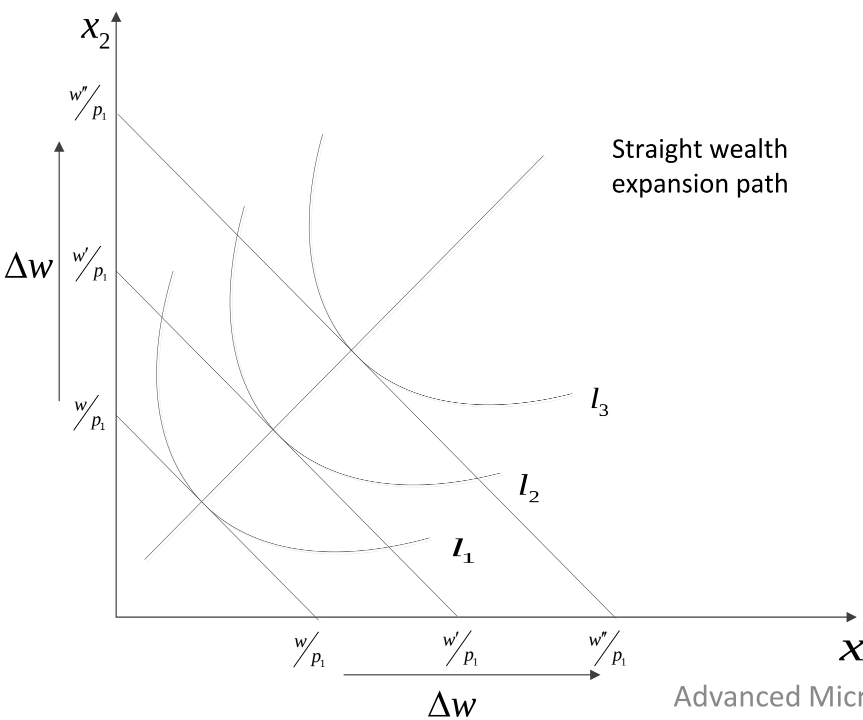
Aggregate Demand

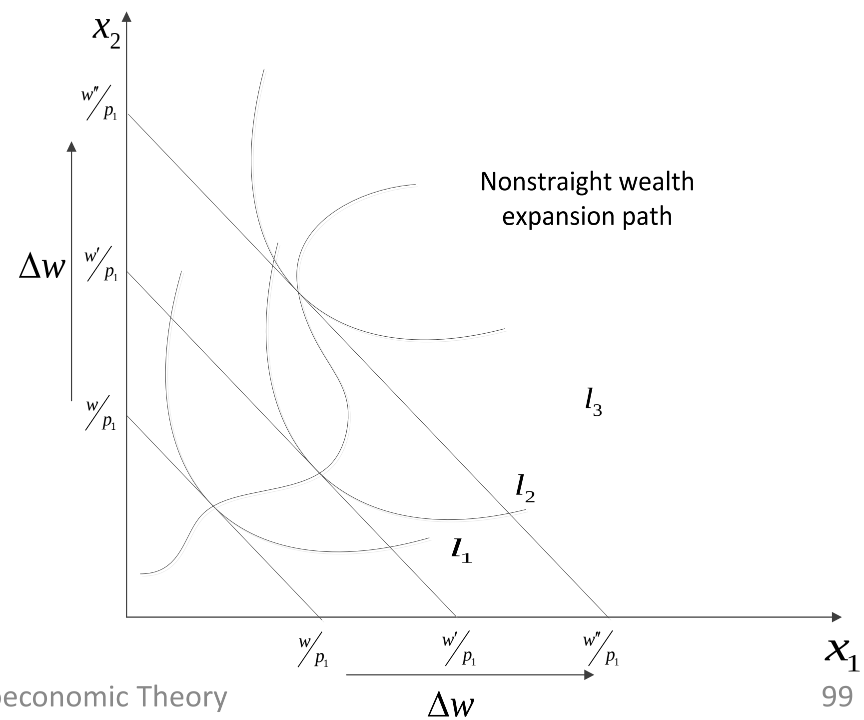
- Graphically, this condition entails that all consumers exhibit *parallel*, *straight* wealth expansion paths.
  - ***Straight***: wealth effects do not depend on the individuals' wealth level.
  - ***Parallel***: individuals' wealth effects must coincide across individuals.
    - Recall that wealth expansion paths just represent how an individual demand changes as he becomes richer.

Aggregate Demand

A given increase in wealth leads the same change in the consumption of good x_i, *regardless* of the initial wealth of the individual

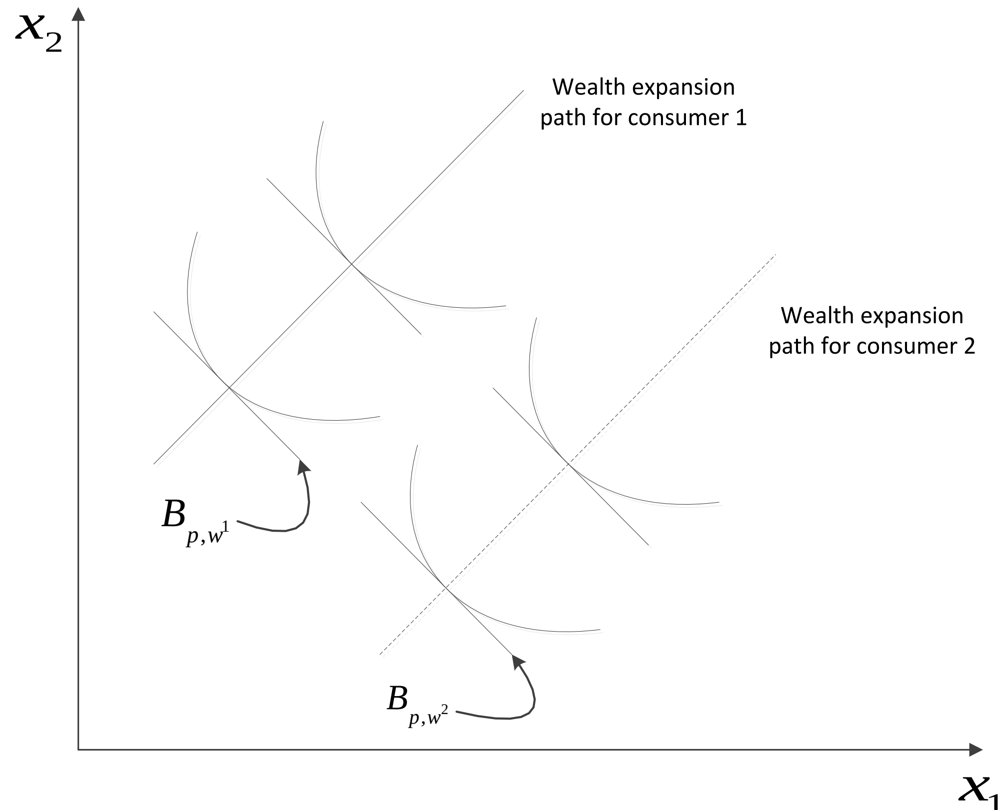
A given increase in wealth leads to changes in the consumption of good x_i that are *dependent* on the individual's wealth level





Aggregate Demand

- Individuals' wealth effects coincide.
- The wealth expansion path for consumers 1 and 2 are parallel to each other
  - both individuals' demands change similarly as they become richer.



Aggregate Demand

- Preference relations that yield *straight* wealth expansion paths:
  - Homothetic preferences
  - Quasilinear preferences
- Can we embody all these cases as special cases of a particular type of preferences?
  - Yes. We next present such cases.

Aggregate Demand: Gorman Form

- ***Gorman form.*** A necessary and sufficient condition for consumers to exhibit parallel, straight wealth expansion paths is that every consumer's indirect utility function can be expressed as:

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

  This indirect utility function is referred to as the *Gorman form*.

- Indeed, in case of quasilinear preferences

$$v_i(p, w_i) = a_i(p) + \frac{1}{p_k}w_i \quad \text{so that} \quad b(p) = \frac{1}{p_k}$$

Aggregate Demand: Gorman Form

- ***Example***:
  - Consider the Gorman form indirect utility function

$$v_i(p, w_i) = \underbrace{\gamma_i \frac{1}{\sqrt{p}}}_{a_i(p)} + \underbrace{\frac{1}{p}}_{b(p)} w_i$$

  - To depict the level sets of $v_i(p, w_i)$, first solve for p in the above expression

$$p(w_i) = \frac{2vw_i + \gamma_i\left(\gamma_i + \sqrt{4vw_i + \gamma_i^2}\right)}{2v^2}$$

  - For simplicity, we set $v = 10$ and $\gamma_i = 1$

$$p(w_i) = \frac{1 + 20w_i + \sqrt{1 + 40w_i}}{200}$$

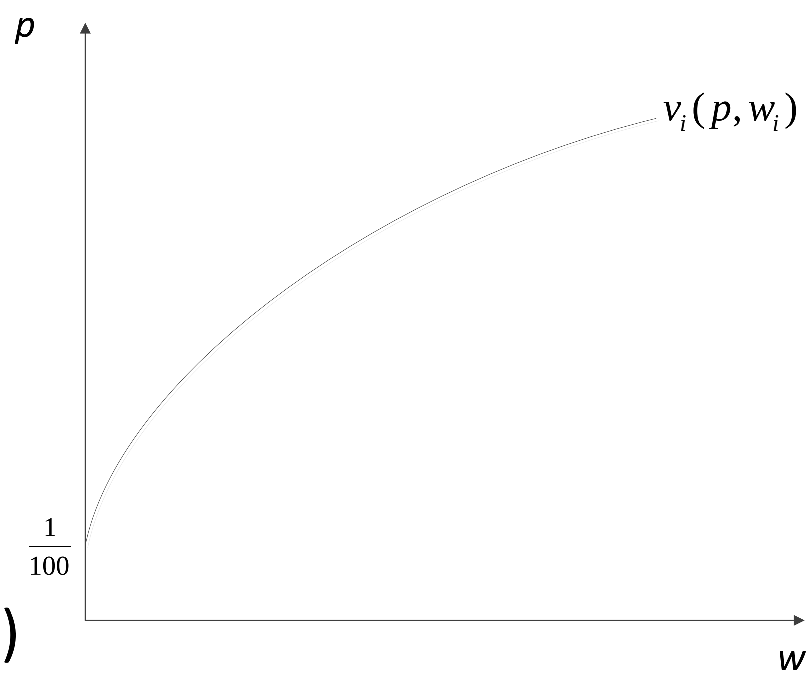
Aggregate Demand: Gorman Form

- *Example* (continued):
  - The vertical intercept of this function is $p(0) = \frac{1}{100}$.
  - The slope of this function is

$$\frac{\partial p(w_i)}{\partial w_i} = \frac{1}{10} + \frac{1}{10\sqrt{1 + 40w_i}} > 0$$

and it is decreasing in w_i (concavity)

$$\frac{\partial^2 p(w_i)}{\partial w_i^2} = \frac{2}{(1 + 40w_i)^{3/2}}$$



Aggregate Demand: Gorman Form

- Let's show that, for indirect utility functions of the Gorman form, we obtain

$$\sum_{i=1}^{I} x_i(p, w_i) = x(p, \sum_{i=1}^{I} w_i)$$

- First, use Roy's identity to find the Walrasian demand associated with this indirect utility function

$$-\frac{\frac{\partial v_i(p, w_i)}{\partial p}}{\frac{\partial v_i(p, w_i)}{\partial w}} = x_i(p, w_i)$$

Aggregate Demand: Gorman Form

- In particular, for good j,

$$-\frac{\frac{\partial v_i(p,w_i)}{\partial p_j}}{\frac{\partial v_i(p,w_i)}{\partial w}} = -\frac{\frac{\partial a_i(p)}{\partial p_j}}{b(p)} - \frac{\frac{\partial b(p)}{\partial p_j}}{b(p)} w_i = x_i^j(p,w_i)$$

- In matrix notation,

$$-\frac{\nabla_p v_i(p,w_i)}{\nabla_w v_i(p,w_i)} = -\frac{\nabla_p a_i(p)}{b(p)} - \frac{\nabla_p b(p)}{b(p)} w_i = x_i(p,w_i)$$

for all goods.

Aggregate Demand: Gorman Form

- We can compactly express $x_i(p, w_i)$ as follows

$$-\frac{\nabla_p v_i(p, w_i)}{\nabla_w v_i(p, w_i)} = \alpha_i(p) + \beta(p) w_i = x_i(p, w_i)$$

where $-\frac{\nabla_p a_i(p)}{b(p)} \equiv \alpha_i(p)$ and $-\frac{\nabla_p b(p)}{b(p)} \equiv \beta(p)$.

Aggregate Demand: Gorman Form

- Hence, aggregate demand can be obtained by summing individual demands

$$\alpha_i(p) + \beta(p)w_i = x_i(p, w_i)$$

across all I consumers, which yields

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} \alpha_i(p) + \beta(p) \sum_{i=1}^{I} w_i$$

$$= \sum_{i=1}^{I} \alpha_i(p) + \beta(p)w = x(p, \sum_{i=1}^{I} w_i)$$

where $\sum_{i=1}^{I} w_i = w$.