EconS 503 - Advanced Microeconomics II
Handout on Bayesian Nash Equilibrium

1. MWG 8.E.1

Consider the following strategic situation. Two opposed armies are poised to seize an island. Each army's general can choose either "attack" or "not attack." In addition, each army is either "strong" or "weak" with equal probability (the draws for each army are independent), and an army's type is known only to its general. Payoffs are as follows: The island is worth M if captured. An army can capture the island either by attacking when its opponent does not or by attacking when its rival does if it is strong and its rival is weak. If two armies of equal strength both attack, neither captures the island. An army also has a "cost" of fighting, which is s if it is strong and w if it is weak, where $s < w$. There is no cost of attacking if its rival does not.

Identify all pure strategy Bayesian Nash equilibria of this game.

Answer:

There are four pure strategies contingent on the type of player:

AA : Attack if either weak or strong type,
AN : Attack if strong and Not attack if weak,
NA : Not attack if strong and Attack if weak,
NN : Never attack.

We can determine the expected payoffs for each player by simple calculation. For example, the expected payoff for player 1 playing the strategy AA given that player 2 also plays the strategy AA is

$$\begin{aligned} EU_1(AA|AA) &= \overbrace{0.5}^{\text{Player 1 is Strong}} \left[\overbrace{0.5 \times (-s)}^{\text{Player 2 is Strong}} + \overbrace{0.5 \times (M - s)}^{\text{Player 2 is Weak}} \right] + 0.5 \left[0.5 \times (-w) + 0.5 \times (-w) \right] \\ &= \frac{M}{4} - \frac{s + w}{2} \end{aligned}$$

The remaining expected payoffs for each pair of strategies can be easily computed and are given in figure 1:

		Player 2			
		AA	AN	NA	NN
Player 1	AA	$\frac{M}{4}-\frac{s+w}{2},\frac{M}{4}-\frac{s+w}{2}$	$\frac{M}{4}-\frac{s}{2},\frac{M}{2}-\frac{s+w}{2}$	$\frac{3M}{4}-\frac{s+w}{4},-\frac{w}{2}$	$M,0$
	AN	$\frac{M}{4}-\frac{s}{2},\frac{M}{2}-\frac{s+w}{4}$	$\frac{M-s}{4},\frac{M-s}{4}$	$\frac{M}{2}-\frac{s}{4},\frac{M-w}{4}$	$\frac{M}{2},0$
	NA	$-\frac{w}{2},\frac{3M}{4}-\frac{s+w}{4}$	$\frac{M-w}{4},\frac{M}{2}-\frac{s}{4}$	$\frac{M-w}{4},\frac{M-w}{4}$	$\frac{M}{2},0$
	NN	$0,M$	$0,\frac{M}{2}$	$0,\frac{M}{2}$	$0,0$

Figure 1: Normal Form Representation

Any NE of this normal form game is a Bayesian NE of the original game.

[**Case 1**] $M > w > s$, and $w > \frac{M}{2} > s$
From the above payoffs we can see that (AA, AN) and (AN, AA) are both pure strategy Bayesian Nash equilibriua.

[**Case 2**] $M > w > s$, and $\frac{M}{2} < s$
From the above payoffs we can see (AA, NN) and (NN, AA) are both pure strategy Bayesian Nash equilibria.

[**Case 3**] $w > M > s$, and $\frac{M}{2} < s$
From the above payoffs we can see that (AN, AN), (AA, NN) and (NN, AA) are pure strategy Bayesian Nash equilibria.

[**Case 4**] $w > M > s$, and $\frac{M}{2} > s$
From the above payoffs we can see that (AN, AN) is the pure strategy Bayesian Nash equilibrium.

2. MWG 8.E.3

Consider the linear Cournot model described in Ex 8.B.5; two firms 1 and 2, simultaneously choose the quantities they will sell on the market, q_1 and q_2. The price each receives for each unit given these quantities is $P(q_1, q_2) = a - b(q_1 + q_2)$. Their costs are c per unit sold. Now, however, suppose that each firm has probability μ of having unit costs of c_L and $(1-\mu)$ of having unit costs of c_H, where $c_H > c_L$. Solve for the Bayesian Nash equilibrium.

Answer:

A firm of type $i = H$ or L will maximize its expected profit, taken as given that the other firm will supply q_H or q_L depending whether this firm is of type H or L. A type $i \in \{H, L\}$ firm 1 will maximize:

$$\underset{q_i^1 \geq 0}{Max}(1-\mu)\left[(a - b\left(q_i^1 + q_H^2\right) - c_i)q_i^1\right] + \mu\left[(a - b\left(q_i^1 + q_L^2\right) - c_i)q_i^1\right]$$

The FOC yields:

$$(1-\mu)(a - b\left(2q_i^1 + q_H^2\right) - c_i) + \mu(a - b\left(2q_i^1 + q_L^2\right) - c_i) = 0$$

In a symmetric Bayesian Nash equilibrium:

$$q_H^1 = q_H^2 = q_H \text{ and } q_L^1 = q_L^2 = q_L$$

Plugging this into the F.O.C we get the following two equations:

$$\begin{array}{c} (1-\mu)\left[a - 3bq_H - c_H\right] + \mu\left[a - b\left(2q_H + q_L\right) - c_H\right] = 0 \\ (1-\mu)\left[a - b\left(q_H + 2q_L\right) - c_L\right] + \mu\left[a - 3bq_L - c_L\right] = 0 \end{array}$$

Therefore, we obtain that

$$\begin{array}{c} q_H = \left[a - c_H + \frac{\mu}{2}\left(c_L - c_H\right)\right]\frac{1}{3b}, \\ q_L = \left[a - c_L + \frac{1-\mu}{2}\left(c_H - c_L\right)\right]\frac{1}{3b}. \end{array}$$

3. Auction Theory

Consider a first-price sealed-bid auction in which bidders simultaneously submit bids and the object goes to the highest bidder at a price equal to his/her bid. Suppose that there are $N \geq 2$ bidders, whose types θ_i are drawn from the cumulative distribution $F(.)$ and receive utility from the object from the function $u(\theta_i - b)$, where b is the amount of the bid paid by player i.

a) Solve for $s'(\theta)$, the derivative of the bidding function of each individual.

Answer:

We start by setting up the utility maximization problem for player i,

$$\max_{b \geq 0} \quad E_{-i}[v_i(b, s_{-i}(\theta_{-i}); \theta_i)|\theta_i] = [F(s^{-1}(b))]^{N-1} u(\theta_i - b)$$

Taking first-order conditions,

$$(N-1)[F(s^{-1}(b))]^{N-2} f(s^{-1}(b)) \frac{\partial s^{-1}(b)}{\partial b} u(\theta_i - b) - [F(s^{-1}(b))]^{N-1} u'(\theta_i - b) = 0$$

for $s(\theta)$ to be an optimal bidding function, it should be optimal for the bidder not to pretend to have a valuation different from his real one, θ_i. Hence $s(\theta) = b$ is the optimal solution for the above first-order condition, and we have that $s^{-1}(b) = \theta$, implying

$$\frac{(N-1)[F(\theta)]^{N-2} f(\theta) u(\theta - s(\theta))}{s'(\theta)} - [F(\theta)]^{N-1} u'(\theta - s(\theta)) = 0$$

Simplifying,

$$\frac{(N-1) f(\theta) u(\theta - s(\theta))}{s'(\theta)} = F(\theta) u'(\theta - s(\theta))$$

and solving for $s'(\theta)$ yields

$$s'(\theta) = \frac{u(\theta - s(\theta))}{u'(\theta - s(\theta))} \times \frac{f(\theta)}{F(\theta)} \times (N-1)$$

b) Assume that consumers are risk neutral, i.e. $u(x) = x$. Derive the bidding function $s(\theta)$.

Answer:

Using what we derived in part (a), we have

$$s'(\theta) = (\theta - s(\theta)) \times \frac{f(\theta)}{F(\theta)} \times (N-1)$$

rearranging some terms,

$$f(\theta)s(\theta)(N-1) + F(\theta)s'(\theta) = \theta f(\theta)(N-1)$$

we can multiply both sides by $[F(\theta)]^{N-2}$ to obtain

$$[F(\theta)]^{N-2} f(\theta)s(\theta)(N-1) + [F(\theta)]^{N-1} s'(\theta) = [F(\theta)]^{N-2} \theta f(\theta)(N-1)$$

Note that the left side is now just $\frac{d([F(x)]^{N-1}s(\theta))}{d\theta}$. Substituting,

$$\frac{d([F(x)]^{N-1}s(\theta))}{d\theta} = [F(\theta)]^{N-2} \theta f(\theta)(N-1)$$

and integrating both sides

$$[F(x)]^{N-1}s(\theta) = \int_{\underline{\theta}}^{\theta} [F(x)]^{N-2} x f(x)(N-1)dx$$

let

$$\begin{aligned} h(x) &= x \quad g'(x) = [F(x)]^{N-2} f(x)(N-1)dx \\ h'(x) &= dx \quad g(x) = [F(x)]^{N-1} \end{aligned}$$

and applying integration by parts, we have

$$[F(x)]^{N-1}s(\theta) = \theta[F(x)]^{N-1} - \int_{\underline{\theta}}^{\theta} [F(x)]^{N-1}dx$$

and solving for $s(\theta)$ yields the bidding function

$$s(\theta) = \theta - \frac{1}{[F(x)]^{N-1}} \int_{\underline{\theta}}^{\theta} [F(x)]^{N-1}dx$$

c) Now assume that consumers are risk averse, i.e. $u(x) = x^{\alpha}$ where $0 < \alpha < 1$. Derive the bidding function $s(\theta)$.

Answer:

Using what we derived in part (a), we have

$$s'(\theta) = \frac{1}{\alpha}(\theta - s(\theta)) \times \frac{f(\theta)}{F(\theta)} \times (N-1)$$

rearranging some terms,

$$f(\theta)s(\theta)\frac{1}{\alpha}(N-1) + F(\theta)s'(\theta) = \theta f(\theta)\frac{1}{\alpha}(N-1)$$

we can multiply both sides by $[F(\theta)]^{\frac{1}{\alpha}(N-1)-1}$ to obtain

$$[F(\theta)]^{\frac{1}{\alpha}(N-1)-1} f(\theta)s(\theta)\frac{1}{\alpha}(N-1) + [F(\theta)]^{\frac{1}{\alpha}(N-1)}s'(\theta) = [F(\theta)]^{\frac{1}{\alpha}(N-1)-1}\theta f(\theta)\frac{1}{\alpha}(N-1)$$

Note that the left side is now just $\frac{d([F(x)]^{\frac{1}{\alpha}(N-1)}s(\theta))}{d\theta}$. Substituting,

$$\frac{d([F(x)]^{\frac{1}{\alpha}(N-1)}s(\theta))}{d\theta} = [F(\theta)]^{\frac{1}{\alpha}(N-1)-1}\theta f(\theta)\frac{1}{\alpha}(N-1)$$

and integrating both sides

$$[F(x)]^{\frac{1}{\alpha}(N-1)}s(\theta) = \int_{\underline{\theta}}^{\theta} [F(x)]^{\frac{1}{\alpha}(N-1)-1} x f(x)\frac{1}{\alpha}(N-1)dx$$

let

$$\begin{aligned} h(x) &= x \quad g'(x) = [F(x)]^{\frac{1}{\alpha}(N-1)-1} f(x)\frac{1}{\alpha}(N-1)dx \\ h'(x) &= dx \quad g(x) = [F(x)]^{\frac{1}{\alpha}(N-1)} \end{aligned}$$

and applying integration by parts, we have

$$[F(x)]^{\frac{1}{\alpha}(N-1)} s(\theta) = \theta[F(x)]^{\frac{1}{\alpha}(N-1)} - \int_{\underline{\theta}}^{\theta} [F(x)]^{\frac{1}{\alpha}(N-1)} dx$$

and solving for $s(\theta)$ yields the bidding function

$$s(\theta) = \theta - \frac{1}{[F(x)]^{\frac{1}{\alpha}(N-1)}} \int_{\underline{\theta}}^{\theta} [F(x)]^{\frac{1}{\alpha}(N-1)} dx$$