EconS 503 - Advanced Microeconomics II
Handout on Subgame Perfect Equilibrium (SPNE)

1. Based on MWG 9.B.3

Consider the three-player finite game of perfect information depicted in figure 1.

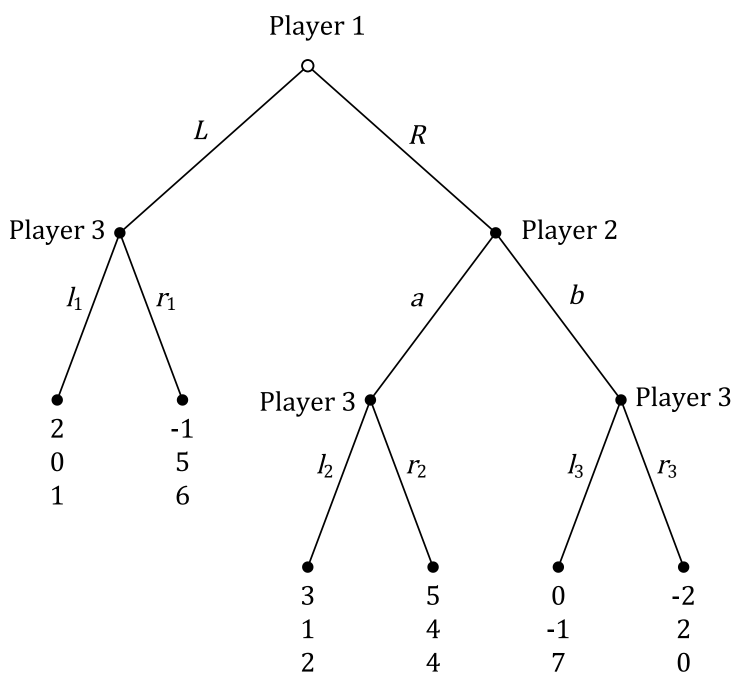

Figure 1: Extensive Form Game

Identify all pure strategy Nash Equilibria for this game. Verify which pure strategy Nash Equilibria are also subgame perfect. Argue that each of the non-subgame perfect Nash Equilibria do not satisfy sequential rationality.

Answer:

We can start our analysis by determining the strategy spaces for each player. Since both players 1 and 2 have single information sets, all of their strategies will be singletons, while Player 3, having three information sets in which to make actions, will have strategies that consist of triples. All of the possible strategies for each player are

$$\begin{aligned} s_1 &= \{(L), (R)\} \\ s_2 &= \{(a), (b)\} \\ s_3 &= \left\{ \begin{array}{c} (l_1 l_2 l_3), (l_1 l_2 r_3), (l_1 r_2 l_3), (l_1 r_2 r_3), \\ (r_1 l_2 l_3), (r_1 l_2 r_3), (r_1 r_2 l_3), (r_1 r_2 r_3) \end{array} \right\} \end{aligned}$$

From these strategies, we can construct the normal form of the game below. The best responses for each player are underlined:

Player 3

Player 2		$l_1l_2l_3$	$l_1l_2r_3$	$l_1r_2l_3$	$l_1r_2r_3$	$r_1l_2l_3$	$r_1l_2r_3$	$r_1r_2l_3$	$r_1r_2r_3$
	a	$2,\underline{0},1$	$2,\underline{0},1$	$2,\underline{0},1$	$2,\underline{0},1$	$-1,\underline{5},\underline{6}$	$-1,\underline{5},\underline{6}$	$-1,\underline{5},\underline{6}$	$-1,\underline{5},\underline{6}$
	b	$\underline{2},\underline{0},1$	$\underline{2},\underline{0},1$	$\underline{2},\underline{0},1$	$\underline{2},\underline{0},1$	$-1,\underline{5},\underline{6}$	$-\underline{1},\underline{5},\underline{6}$	$-1,\underline{5},\underline{6}$	$-\underline{1},\underline{5},\underline{6}$

Player 1: L

Player 3

Player 2		$l_1l_2l_3$	$l_1l_2r_3$	$l_1r_2l_3$	$l_1r_2r_3$	$r_1l_2l_3$	$r_1l_2r_3$	$r_1r_2l_3$	$r_1r_2r_3$
	a	$\underline{3},\underline{1},2$	$\underline{3},1,2$	$\underline{5},\underline{4},\underline{4}$	$\underline{5},\underline{4},\underline{4}$	$\underline{3},\underline{1},2$	$\underline{3},1,2$	$\underline{5},\underline{4},\underline{4}$	$\underline{5},\underline{4},\underline{4}$
	b	$0,-1,\underline{7}$	$-2,\underline{2},0$	$0,-1,\underline{7}$	$-2,2,0$	$\underline{0},-1,\underline{7}$	$-2,\underline{2},0$	$\underline{0},-1,\underline{7}$	$-2,2,0$

Player 1: R

From this normal form representation, we can identify six pure strategy Nash Equilibria:

$$(L, b, (r_1l_2r_3)), (L, b, (r_1r_2r_3)), (R, a, (l_1r_2l_3)), (R, a, (l_1r_2r_3)), (R, a, (r_1r_2l_3)), (R, a, (r_1r_2r_3))$$

Even though we found there to be six psNE in this game, we know from Zermelo's Theorem (Proposition 9.B.1 in MWG) that there will only be one subgame perfect Nash Equilibria due to no player having the same payoffs at any two terminal nodes. We also know that all subgame perfect Nash Equilibria must be a subset of the Nash Equilibria, and hence, one of these six strategy profiles will be the subgame perfect Nash Equilibrium that we are looking for. To find the SPNE, we must perform backwards induction on our extensive form game.

To do this, we must first identify all proper subgames, which can bee seen below in figure 2.

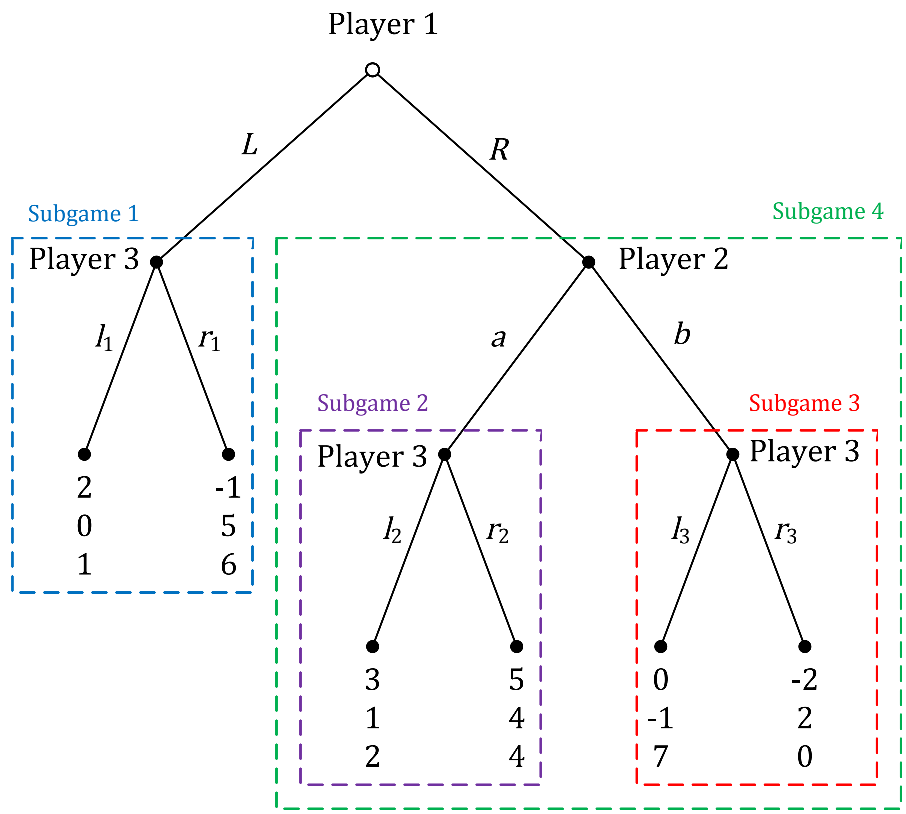

Figure 2: Proper Subgames of the Extensive Form

Starting with subgames 1, 2 and 3, we can evaluate player 3's decisions at each subgame as shown in figure 3.

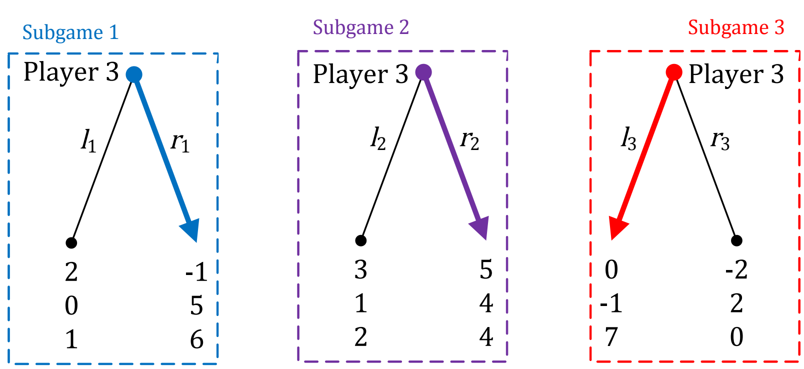

Figure 3: Evaluating Terminal Subgames.

It is clear that in subgame 1, player 3 will choose strategy r_1 since his payoff of 6 from selecting r_1 is greater than his payoff of 1 from selecting l_1. Likewise, in subgame 2, player 3 will select r_2 since $4 > 2$ and in subgame 3, player 3 will select l_3 since $7 > 0$. We can substitute the results from subgames 2 and 3 into subgame 4 in order to evaluate player 2's backward induction as shown in figure 4.

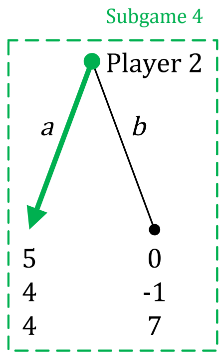

Figure 4: Evaluating Subgame 4.

Lastly, we can substitute the results from subgames 1 and 4 into the root game to evaluate player 1's backward induction as shown in figure 5.

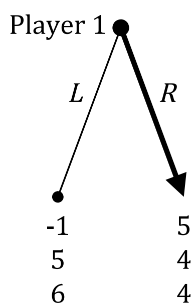

Figure 5: Evaluating Reduced Form.

This allows us to identify the subgame perfect Nash Equilibrium of this game as $(R, a, (r_1 r_2 l_3))$. Regarding the other five pure strategy Nash Equilibria that are not subgame perfect, we can identify why they are not by finding violations of sequential rationality for each of them. For every one of the non-subgame perfect Nash Equilibria, player 3 is not being sequentially rational for one or more of his subgames. For example, in the Nash Equilibrium $(L, b, (r_1 l_2 r_3))$, player 3 is not acting rational in either subgame 2 or 3. In both cases, he is choosing a payoff that is strictly worse for him, given the information he has. We would consider these strategies as incredible threats, meaning that they violate sequential rationality and are not subgame perfect.

2. MWG 12.B.8

Consider the following two-period model: A firm is a monopolist in a market with an inverse demand function (in each period) of $p(q) = a - bq$. The cost per unit in period 1 is $c_1 = c$.

In period 2, however, the monopolist has "learning by doing," and so its constant cost per unit of output is $c_2 = c - mq_1$, where q_1 is the monopolist's period 1 output level. Assume $a > c$ and $b > m$. Also assume that the monopolist does not discount future earnings.

a. What is the monopolist's level of output in each of the periods?

Answer:

The monopolist's intertemporal profit maximization problem is

$$\underset{q_1,q_2}{Max}\pi = (a - bq_1 - c)\, q_1 + (a - bq_2 - (c - mq_1))\, q_2,$$

with FOCs

$$\begin{aligned} \frac{\partial \pi}{\partial q_1} &= a - 2bq_1 - c + mq_2 = 0 \\ \frac{\partial \pi}{\partial q_2} &= a - 2bq_2 - c + mq_1 = 0 \end{aligned}$$

Solving this system of two equations and two unknowns will yield $q_1^m = q_2^m = \frac{a-c}{2b-m} > 0$ (by the excercise's assumptions).

b. What outcome would be implemented by a benevolent social planner who fully controlled the monopolist? Is there any sense in which the planner's period 1 output is selected so that "price equals marginal cost"?

Answer:

A benevolent social planner maximizes total social welfare (assuming no discounting of the future, we just add up both periods' consumer surplus, to both periods' firm's profits, and subtract both periods' costs),

$$\begin{aligned} \underset{q_1,q_2}{Max}SW &= \int_0^{q_1} p(x)\, dx + \int_0^{q_2} p(x)\, dx - cq_1 - (c - mq_1)\, q_2 \\ &= a(q_1 + q_2) - \frac{1}{2}b\left(q_1^2 + q_2^2\right) - cq_1 - (c - mq_1)\, q_2 \end{aligned}$$

and the FOCs are,

$$\begin{aligned} &(i) \quad (a - bq_1) + mq_2 = c, \\ &(ii) \quad (a - bq_2) = c - mq_1, \end{aligned}$$

which yield $q_1^{SP} = q_2^{SP} = \frac{a-c}{b-m} > 0$. Comparing this with the monopoly quantities we see that $q_i^m < q_i^{SP}$. The way we wrote down the FOCs shows that in fact there is a sense of "price equals marginal cost". Recall that price is marginal surplus, and the left hand side of both

FOCs is exactly the effective marginal surplus from each period's good. In the first period, aside from marginal consumer surplus, given q_2, any additional unit of q_1 reduces marginal cost next period by mq_2. The right hand side is the effective marginal cost in each period.

c. Given that the monopolist will be selecting the period 2 output level, would the planner like the monopolist to slightly increase the level of period 1 output above that identified in (a)? Can you give any intuition for this?

Answer:

As we have seen, the social planner would want to produce more in every period. By increasing the output in the first period above q_i^m, welfare in the first period will be higher, and this will lead to a lower second period marginal cost. This lower second period marginal cost will induce the monopolist to produce more in the second period and will therefore further increase welfare.

3. Ultimatum Bargaining Game

In the ultimatum bargaining game, a proposer is given a pie of size \$1, and he is asked to make a monetary offer, x, to the responder, who only has the option to accept or reject it (as if he was offered an "ultimatum" from the proposer). If the offer is accepted, then the responder receives it while the proposer keeps the remainder of the pie. However, if he rejects it, both players receive a zero payoff. Operating by backward induction, the responder should accept any offer x from the proposer (even if it is low) since the alternative (reject the offer) yields an even lower payoff (zero). Anticipating such as a response, the proposer should then offer one cent (or the smallest monetary amount) to the responder, since by doing so the proposer guarantees acceptance and maximizes his own payoff. Therefore, according to the subgame perfect equilibrium prediction in the ultimatum bargaining game, the proposer should make a tiny offer (one cent or, if possible, an amount approaching zero), and the responder should accept it, since his alternative (reject the offer) would give him a zero payoff.

However, in experimental tests of the ultimatum bargaining game, subjects who are assigned the role of proposer rarely make offers close to zero to the subject who plays as a responder. Furthermore, sometimes subjects in the role of the responder reject positive offers, which seems to contradict our equilibrium predictions. In order to explain this contradiction, many scholars have suggested that players' payoff functions are not as selfish as that specified in standard models (where players only care about the monetary payoff they receive). Instead, the payoff function should also include social preferences, measured by the difference between the payoff a player obtains and that of his opponent, which gives rise to spite (when the monetary amount he receives is lower than that of his opponent) or gratitude feelings (when the monetart amount he receives is higher than that of his opponent). In particular, suppose that the responder's payoff is given by

$$u_R(x, y) = x + \alpha(x - y),$$

where x is the responder's monetary payoff, y is the proposer's monetary payoff, and α is a positive constant. That is, the responder not only cares about how much money he receives, x, but also about the payoff inequality that emerges at the end of the game, $\alpha(x - y)$, which gives rise to either spite, if $x < y$, or gratitude, if $x > y$. For simplicity, assume that the proposer is selfish, i.e., his utility function only considers his own monetary payoffs $u_P(x, y) = y$, as in the basic model.

a. Use a game tree to represent this game in its extensive form, writing the payoffs in terms of m, the monetary offer of the proposer, and parameter α.

Answer:

First, player 1 offers a division of the pie, m, to player 2, who either accepts or rejects it. However, payoffs are not the same as in the standard ultimatum bargaining game. While the payoff of player 1 (proposer) is just the remaining share of the pie that he does not offer to player 2, i.e., $1 - m$, the payoff of player 2 (responder) is

$$m + \alpha(x - y) = m + \alpha[m - (1 - m)] = m + \alpha(2m - 1),$$

where x was defined as the payoff of the responder, and y as that of the proposer. We depict this modified ultimatum bargaining game in figure 6 below.

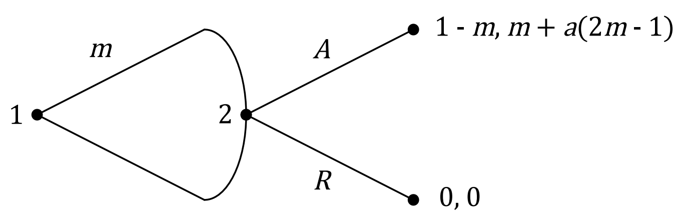

Figure 6: Ultimatum Bargaining Game.

b. Find the subgame perfect Nash Equilibrium. Describe how equilibrium payoffs are affected by changes in parameter α.

Answer:

Operating by backward induction, we first focus on the last mover (player 2, the responder). In particular, player 2 accepts any offer m from player 1 such that:

$$m + \alpha(2m - 1) \geq 0,$$

since the payoff he obtains from rejecting the offer is zero. Solving for m, this implies that player 2 accepts any offer m that satisfies $m \geq \frac{\alpha}{1+2\alpha}$. Anticipating such a response from player 2, player 1 offers the minimal m that generates acceptance, i. e., $m^* = \frac{\alpha}{1+2\alpha}$, since by doing so player 1 can maximize the share of the pie he keeps. This implies that equilibrium payoffs are:

$$(1-m^*, m^*) = \left(1 - \frac{a}{1+2a}, \frac{a}{1+2a} + a\left[\frac{2a}{1+2a} - 1\right]\right) = \left(\frac{1+a}{1+2a}, 0\right)$$

The proposer's share and equilibrium payoff is decreasing in α since its derivative with respect to α is

$$\frac{\partial\left(1-m^*\right)}{\partial \alpha} = -\frac{1}{(1+2a)^2}$$

which is negative for all $\alpha > 0$. In contrast, the responder's share is increasing in α given that its derivative with respect to α is

$$\frac{\partial m^*}{\partial \alpha} = \frac{1}{(1+2\alpha)^2},$$

which is positive for all $\alpha > 0$. however, the equilibrium payoff for player 2 does not change with respect to a (It will always be 0).

c. Depict the equilibrium monetary amount that the proposer keeps, and the payoff that the responder receives, as a function of parameter α.

Answer:

Taking the limit of our split as a approaches infinity,

$$\lim_{a\to\infty} m = \frac{a}{1+2a} = \frac{1}{2}$$

which intuitively makes sense, as the more effect that the inequality has on the responder, the closer the payoffs will have to be in order for him to accept. At the extreme, the payoffs will have to be identical and the inequality eliminated completely for the responder to accept. Interestingly, $a = \infty$ is actually the only point where the responder will have a non-zero payoff. Everywhere else, the proposer will guarantee that the responder's payoff is zero. The case where the responder will not accept anything other than an equal split of the pie is the only exception to this. In figure 7, we represent how the proposer's equilibrium share and payoff (in red) decreases in a, and how the share of the responder (in blue) increases in a.

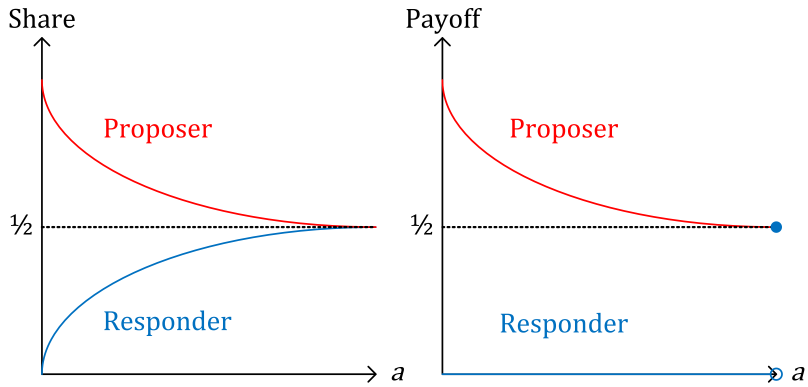

Figure 7: Shares and Payoffs as a function of a.

4. Aliprantis 4.18

[The Monitoring Game]

An employer has hired a worker who has been given a certain task to complete. The employer can choose to monitor (M) the worker or choose not to do so (DM). We will assume that monitoring is costly and that the worker knows whether or not he is being monitored.

The worker can choose to put in high effort in completing the task or be lazy. In case the employer monitors the worker, the effort level of the worker is observed by the employer; otherwise the employer does not observe the effort level. The employer then pays a high wage or a low wage when the worker reports that the task is complete. The value of the project to the employer when the worker puts in a high level of effort (h) is $v(h) = 90$, and the value of the project when the effort level is low (l) is $v(l) = 30$. The high wage is $w_h = 30$ and the low wage is $w_l = 10$. If the employer monitors, he gets to observe the effort level of the employee. He pays the employee the wage w_h if he observes h' and the low wage w_l if he observes l'. The situation that we have just described can be cast as a sequential game with imperfect information, as shown in figure 8.

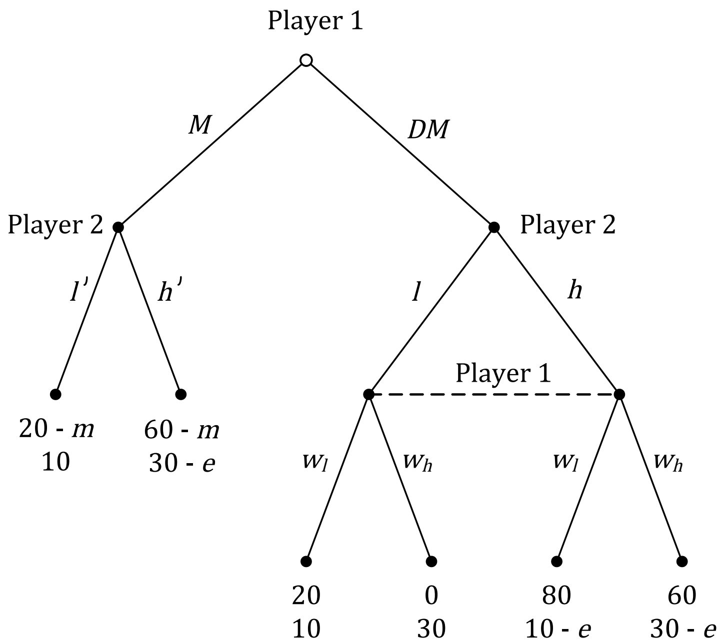

Figure 8: Extensive Form of the Monitoring Game.

In stage 1, the employer (player 1) chooses whether to monitor or not to monitor. In stage 2, the employee (player 2) chooses whether to work hard $[h, h']$ or be lazy $[l, l']$. In the following stage, player 1 (the employer) decides whether to pay a high wage w_h or a low wage w_l.

We denote by m the cost to the employer of monitoring and by e the cost of the effort put in by the employee. We assume that $0 < m < 20$ and $0 < e < 10$. (These conditions imply the following inequalities that will be used in finding the Nash equilibria of the subgames from their matrix forms: $60 - m > 40$ and $30 - e > 20$.)

Find all pure strategy Nash Equilibria for this game as well as all Subgame Perfect Nash Equilibria.

Answer:

Like before, we start by defining the strategy spaces for each player. Since both players have two information sets for which they get to take action, their strategy sets will consist of doubles, namely

$$\begin{aligned} s_1 &= \{(M, w_l), (M, w_h), (DM, w_l), (DM, w_h)\} \\ s_2 &= \{(l, l'), (l, h'), (h, l'), (h, h')\} \end{aligned}$$

and we can use these strategy spaces to construct the normal form representation of this game (best responses for both players are underlined).

		Player 2			
		l, l'	l, h'	h, l'	h, h'
Player 1	M, w_l	$20 - m, 10$	$\underline{60 - m}, \underline{30 - e}$	$20 - m, 10$	$60 - m, \underline{30 - e}$
	M, w_h	$20 - m, 10$	$\underline{60 - m}, \underline{30 - e}$	$20 - m, 10$	$60 - m, \underline{30 - e}$
	DM, w_l	$\underline{20}, \underline{10}$	$20, \underline{10}$	$\underline{80}, 10 - e$	$\underline{80}, 10 - e$
	DM, w_h	$0, \underline{30}$	$0, \underline{30}$	$60, 30 - e$	$60, 30 - e$

From this normal form representation, we can identify three pure strategy Nash Equilibria:

$$((M, w_l), (l, h')), ((M, w_h), (l, h')), ((DM, w_l), (l, l'))$$

(Note: We could evaluate this game for mixed strategies, but almost all of the mixing probabilities will be degenerate, and terribly uninteresting. This can be done by the reader as an exercise). With the Nash Equilibria identified, we will now move on to identify which of them are subgame perfect. To begin, we will identify all of the proper subgames from our extensive form game as shown in figure 9 and then perform backward induction:

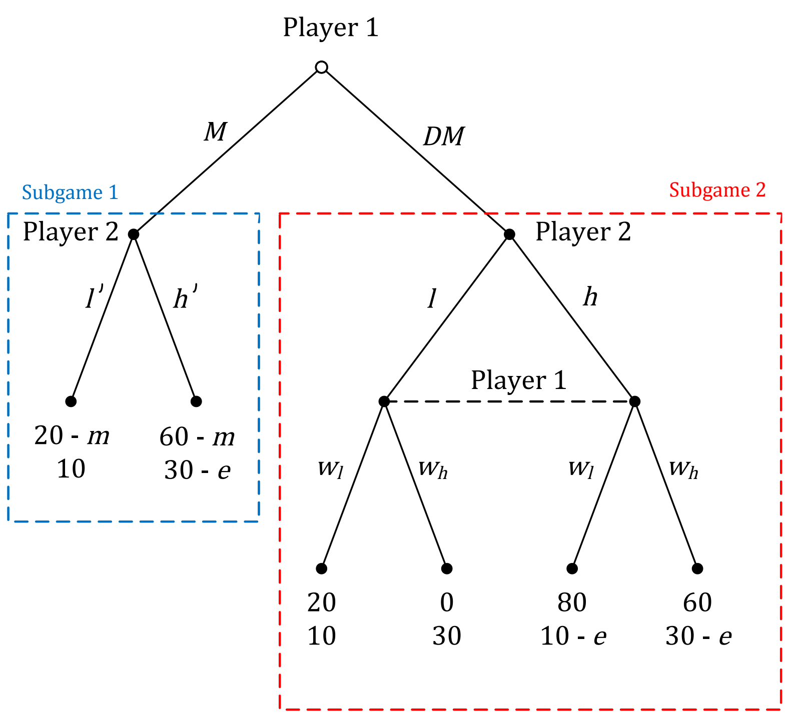

Figure 9: Proper Subgames in the Extensive Form.

Starting with subgame 1, we can identify player 2's best response, as shown in figure 10. It is clear that $30 - e > 10$ since $0 < e < 10$. Hence, player 2 will always prefer to exert a high effort level when he is being monitored by player 1.

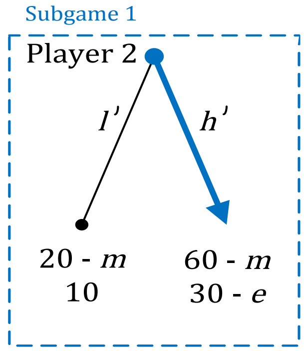

Figure 10: Subgame 1.

For subgame 2, we must put it into a normal form representation for it to be evaluated (Note that in the case of imperfect information, the solution to the extensive form of subgame 2 and the solution to the normal form of subgame 2 will be equivalent). Setting up the normal form representation, we have

		Player 2	
		l	h
Player 1	w_l	$\underline{20},\underline{10}$	$\underline{80}, 10-e$
	w_h	$0,\underline{30}$	$60, 30-e$

In this subgame, there is only one Nash Equilibrium, (w_l, l) (Note that strategy w_h is strictly dominated by strategy w_l for player 1, and strategy h is strictly dominated by strategy l for player 2). We can now substitute the results from subgames 1 and 2 into the root of the game, giving us a reduced form representation as depicted in figure 11.

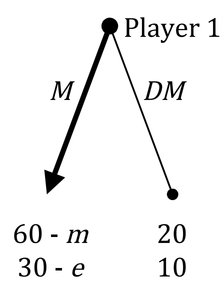

Figure 11: Reduced Form Representation.

As can be seen in the reduced form of the game, player 1's backward induction solution will choose M since $60 - m > 20$ $(0 < m < 20)$. Hence, we have our subgame perfect Nash Equilibrium of $((M, w_l), (l, h'))$.