EconS 503 - Advanced Microeconomics II
Handout on Repeated Games

1. MWG 9.B.9

Consider the game in which the following simultaneous-move game as depicted in figure 1 is played twice:

		Player 2		
		b_1	b_2	b_3
	a_1	$10, 10$	$2, 12$	$0, 13$
Player 1	a_2	$12, 2$	$5, 5$	$0, 0$
	a_3	$13, 0$	$0, 0$	$1, 1$

Figure 1: Stage Game Normal Form.

The players observe the actions chosen in the first stage of the game prior to the second stage. What are the pure strategy subgame perfect Nash equilibria of this game?

Answer:

To begin, we must first determine the pure strategy Nash Equilibria of the stage game. Performing a simple best response procedure for each player will show that (a_2, b_2) and (a_3, b_3) are the two Nash Equilibria of the stage game. We know that any strategy where a Nash Equilibrium of the stage game is played in every period is subgame perfect, and hence, we already have 4 strategies that are subgame perfect. They are

Player 1 - Play a_i in stage 1 and a_j in stage 2
Player 2 - Play b_i in stage 1 and b_j in stage 2

where $i, j \in \{2, 3\}$. We are not done, however, as the presence of more than one Nash Equilibrium enables us to design carrot and stick strategies conditional on the discount factor for each player. For example, consider the following strategy:

Player 1 - Play a_1 in stage 1. Play a_2 in stage 2 if player 2 played b_1 in stage 1, a_3 otherwise

In this case, if player 2 cooperates and chooses b_1, he will receive a payoff of 10 in the first stage and 5 in the second stage for a total payoff of

$$10 + \delta_2 5$$

Where δ_2 is player 2's discount factor. If player 2 were to deviate, his best deviation would be to play b_3 for a payoff of 13 in the first stage, and then be punished with a payoff of 1 in the second stage. Hence, his total payoff would be

$$13 + \delta_2 1$$

Therefore, to support cooperation, player 2's total payoff must be weakly better when cooperating than it is while deviating, i.e.,

$$\begin{aligned} 10 + \delta_2 5 &\geq 13 + \delta_2 1 \\ \implies & \delta_2 \geq \frac{3}{4} \end{aligned}$$

Hence, if $\delta_2 \geq \frac{3}{4}$, player 2 would prefer to cooperate with player 1's strategy, and this strategy can be supported as a subgame perfect Nash Equilibrium (Note that a symmetric strategy profile of player 2 selecting the same actions as player 1 can also be supported as long as $\delta_1 \geq \frac{3}{4}$. All other analyses in this problem will be assumed to be symmetric among players 1 and 2).

There are two more strategies for player 1 to consider, as well. Player 1 may not be limited to solely the pareto optimal outcome of (a_1, b_1) in stage 1 of the game. Let's consider the following strategy:

Player 1 - Play a_2 in stage 1. Play a_2 in stage 2 if player 2 played b_1 in stage 1, a_3 otherwise

In this case, if player 2 cooperates and chooses b_1, he will receive a payoff of 2 in the first stage and 5 in the second stage for a total payoff of

$$2 + \delta_2 5$$

whereas his optimal deviation would be to play b_2 in the first stage (The Nash Equilibrium becomes the outcome) for a payoff of 5 and then be punished with a payoff of 1 in the second stage. His total payoff becomes

$$5 + \delta_2 1$$

Therefore, to support cooperating, we must have

$$\begin{aligned} 2 + \delta_2 5 &\geq 5 + \delta_2 1 \\ \implies & \delta_2 \geq \frac{3}{4} \end{aligned}$$

Lastly, we must consider the following strategy profile:

Player 1 - Play a_3 in stage 1. Play a_2 in stage 2 if player 2 played b_1 in stage 1, a_3 otherwise

In this final case, if player 2 cooperates and chooses b_1, he will receive a payoff of 0 in the first stage and 5 in the second stage for a total payoff of

$$0 + \delta_2 5$$

whereas his optimal deviation would be to play b_3 in the first stage for a payoff of 1 and then be punished with a payoff of 1 in the second stage. His total payoff becomes

$$1 + \delta_2 1$$

Therefore, to support cooperating, we must have

$$\begin{aligned} \delta_2 5 &\geq 1 + \delta_2 1 \\ &\implies \delta_2 \geq \frac{1}{4} \end{aligned}$$

Hence, conditional on the discount value for player 2, there could be anywhere from three to six subgame perfect Nash equilibria for player 1. (Likewise, due to symmetry, there could be three to six subgame perfect Nash Equilibria for player 2 conditional on player 1's discount vlaue). Interestingly, the most greedy strategy is the strategy that can be supported with the lowest discount value. This is due to the fact that the suffering player's very low payoff from deviating in stage one makes the carrot of getting 5 times that in stage two all the much sweeter.

2. MWG 9.B.14

At time $t = 0$, an incumbent firm ($firm\ I$) is already in the widget market, and a potential entrant ($firm\ E$) is considering entry. In order to enter, $firm\ E$ must incur a cost of $K > 0$. $Firm\ E$'s only opportunity to enter is at time 0. There are three production periods. In any period in which both firms are active in the market, the game in figure 2 is played. $Firm\ E$ moves first, deciding whether to stay in or exit the market. If it stays in, $firm\ I$ decides whether to fight or not. If $firm\ I$ fights, $firm\ E$ receives a payoff of -1 while $firm\ E$ receives a payoff of y. Whereas, if $firm\ I$ accommodates, $firm\ E$ receives a payoff of 1 while $firm\ I$ receives a payoff of z. Once $firm\ E$ plays "out," it is out of the market forever; $firm\ E$ earns zero in any period during which it is out of the market, and $firm\ I$ earns x. The discount factor for both firms is δ.

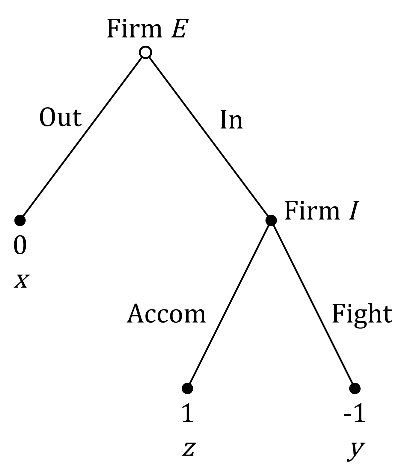

Figure 2: Extensive Form Stage Game

Assume that:

$$\begin{aligned} x &> z > y \\ y + \delta x &> (1+\delta)z \\ 1 + \delta &> K \end{aligned}$$

a. What is the (unique) subgame perfect Nash Equilibrium of this game?

Answer:

The extensive form of the game is depicted in figure 3 below. Simple backward induction (using the assumptions) leads to the unique SPNE which is shown by arrows in the figure: $firm\ E$ enters at $t = 0$, and always plays In thereafter. $Firm\ I$ accommodates for all $t = 1, 2, 3$.

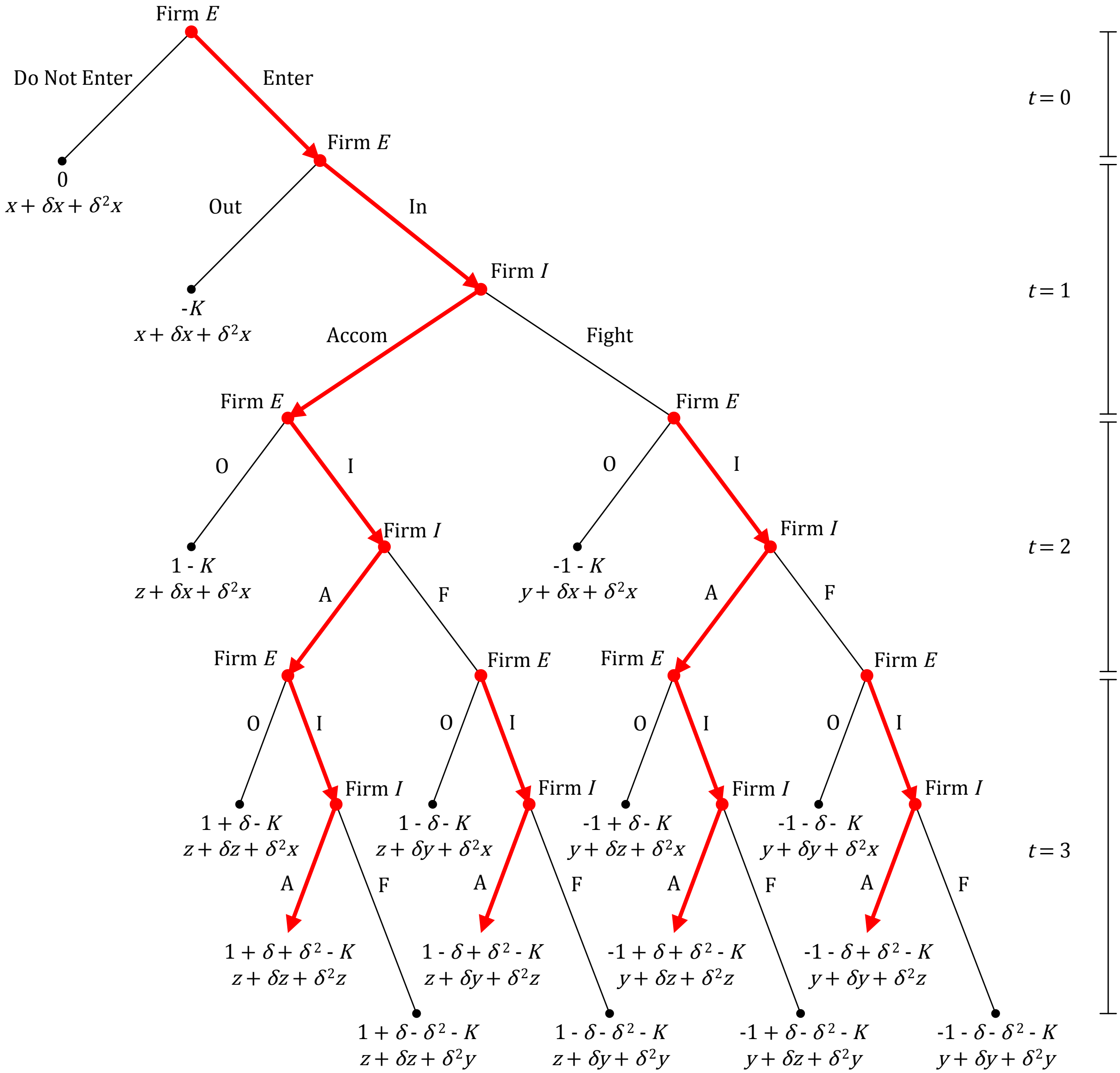

Figure 3: Extensive Form of the Repeated Game.

b. Suppose now that *firm E* faces a financial constraint. In particular, if *firm I* fights *once* against *firm E* (in any period), *firm E* will be forced out of the market from that point on. Now what is the (unique) subgame perfect Nash equilibrium of this game? (If the answer depends on the values of parameters beyond the three assumptions, indicate how.)

Answer:

The extensive form of the modified game is depicted in figure 4. Using backward induction, $firm\ I$ will always accommodate in period $t = 3$, and therefore if $t = 3$ is reached, $firm\ E$ will play In. This causes $firm\ I$ to choose Fight in $t = 2$ since $y + \delta x > (1 + \delta)z$ by our second assumption. This causes $firm\ E$ to exit the market forcibly at the beginning of period 3,which causes $firm\ E$ to choose Out in $t = 2$.

Working backward we get that at $t = 1$, $firm\ I$ chooses to accommodate and $firm\ E$ choose In. However, the choice of $firm\ E$ at $t = 0$ depends on the value of K. If $K > 1$ then $firm\ E$ will choose not to enter, and if $K < 1$ then $firm\ E$ will enter. For $K = 1$ both are part of the (unique) continuation subgame perfect Nash Equilibrium, so there are up to two SPNE in this case.

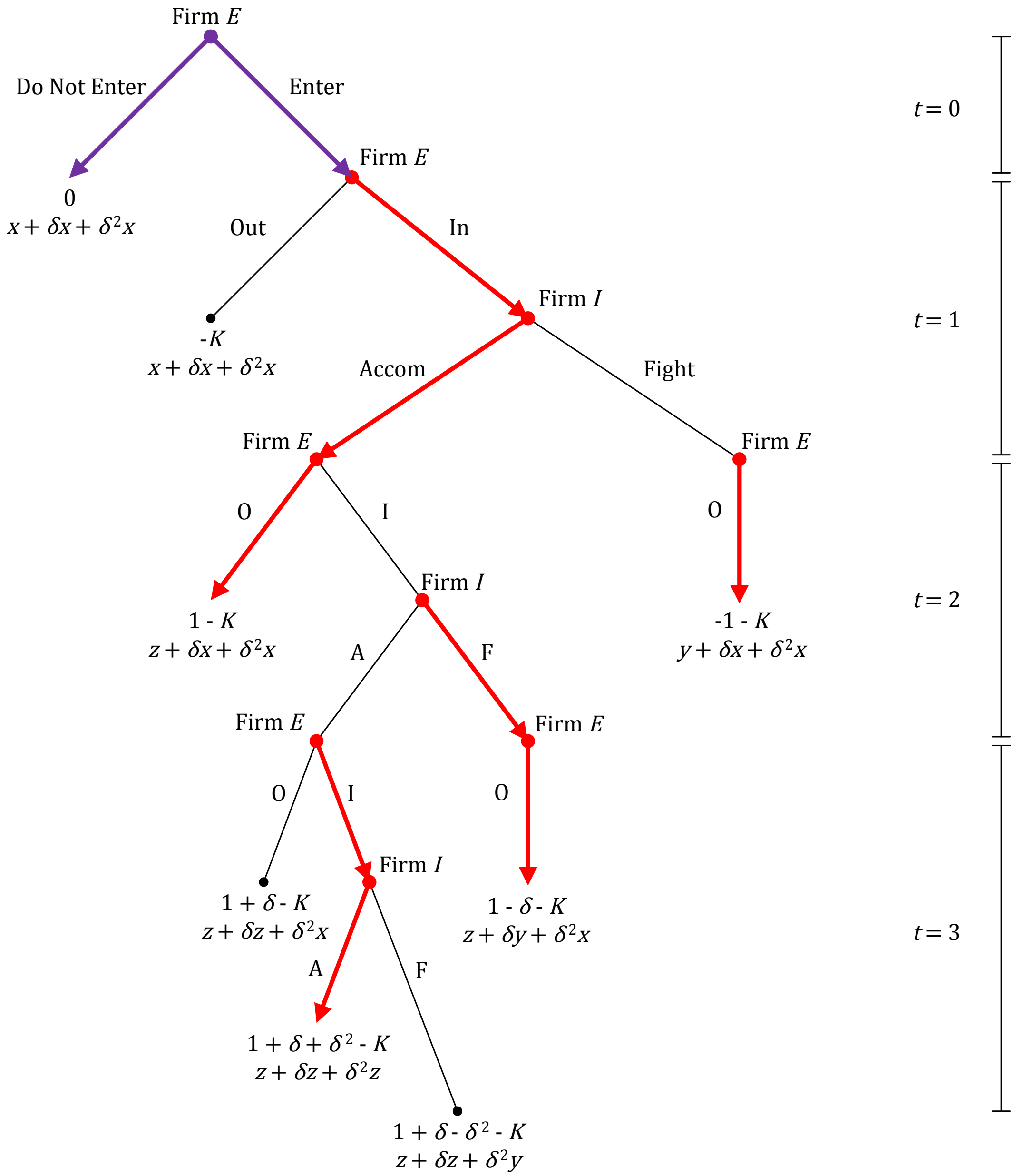

Figure 4: Extensive Form of the Repeated Game

3. MWG 12.D.1

Consider an infinitely repeated Bertrand duopoly with discount factor $\delta < 1$. Both players use the following strategies:

$$p_{jt}(H_{t-1}) = \left\{ \begin{array}{l} p^m \text{ if all elements of } H_{t-1} \text{ equal } (p^m, p^m) \text{ or } t = 1 \\ c \text{ otherwise} \end{array} \right\}$$

Intuitively, this strategy has both players playing the monopoly price and splitting the monopoly profits evenly until one of them deviates (by charging a price of $p^m - \varepsilon$ and claiming the whole market for himself). After a deviation is detected, both players will revert to the Nash Equilibrium of the stage game (A Grim-Trigger strategy) and charge $p_j = c$ forever after, earning zero profits. Determine the conditions under which strategies of the form above sustain the monopoly price in each of the following cases:

a. Market demand in period t is $x_t(p) = \gamma^t x(p)$ where $\gamma > 0$.

Answer:

This demand function is characterized by growth (either positive or negative) of γ each period. Monopoly profit in period t is

$$\max_p \gamma^t x(p)(p-c) = \gamma^t \max_p x(p)(p-c) = \gamma^t \pi^m$$

Of which each player gets half, or

$$\pi_{jt} = \gamma^t \frac{\pi^m}{2}$$

Summing up each player's lifetime profits, we have

$$\frac{\pi^m}{2} + \gamma\delta\frac{\pi^m}{2} + (\gamma\delta)^2\frac{\pi^m}{2} + \cdots = \sum_{t=o}^{\infty}(\gamma\delta)^t\frac{\pi^m}{2} = \frac{1}{1-\gamma\delta}\frac{\pi^m}{2}$$

If a player chooses to deviate, they will charge $p_j = p^m - \varepsilon$ (Intuitively, the lowest possible deviation for the monopoly price). For simplicity, we assume that ε is so small that it will not affect the actual amount of profits obtained. Hence, the deviating player will receive the entire market (π^m), and the cooperating player will receive nothing. Once the deviation is detected, both players will charge marginal cost for all future periods, wiping out all future profits. Therefore, to sustain cooperation, we must have the lifetime discounted profits be weakly higher than seizing the market for one period and receiving nothing afterwards, i.e.,

$$\frac{1}{(1-\gamma\delta)}\frac{\pi^m}{2} \geq \pi^m \implies \delta \geq \frac{1}{2\gamma}$$

Hence, the minimal discount factor supporting cooperation decreases in the rate of growth of demand, i.e., cooperation can be sustained under a larger set of discount factors as demand grows faster across periods. See figure 5 below.

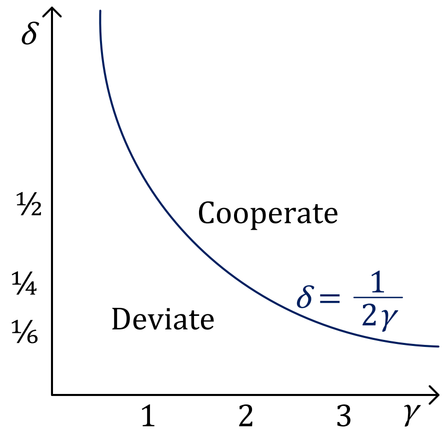

Figure 5: Cooperation Regions
as a Function of γ.

b. At the end of each period, the market ceases to exist with probability $1-\gamma \in [0,1]$.

Answer:

If a firm deviates, it can obtain π^m in that period, and it will get zero forever after. If it does not deviate, its expected payoff is

$$\frac{\pi^m}{2}+\delta\left(\gamma\left[\frac{\pi^m}{2}+\delta\left(\gamma\left[\frac{\pi^m}{2}+\cdots\right]\right)+(1-\gamma)0\right]+(1-\gamma)0\right)=\sum_{t=0}^{\infty}(\gamma\delta)^t\,\frac{\pi^m}{2}=\frac{1}{(1-\gamma\delta)}\frac{\pi^m}{2}$$

Intuitively, note that the interpretation of the discount factor is very similar to how we are treating γ in this case. What we essentially have are two discount factors that enter into this equation multiplicatively. Like before, deviation is not profitable if and only if the lifetime expected payoff is weakly higher than deviating for one period and claiming the whole market, i.e.,

$$\frac{1}{(1-\gamma\delta)}\frac{\pi^m}{2}\geq\pi^m \Longrightarrow \delta\geq\frac{1}{2\gamma}$$

Since $\gamma,\delta\in[0,1]$, we have a few interesting features of this relationship. For any $\gamma<\frac{1}{2}$, there does not exist a value for δ in which cooperation can be sustatined since the necessary value for δ would be greater than 1, outside of its domain. Figure 6 below illustrates this relationship.

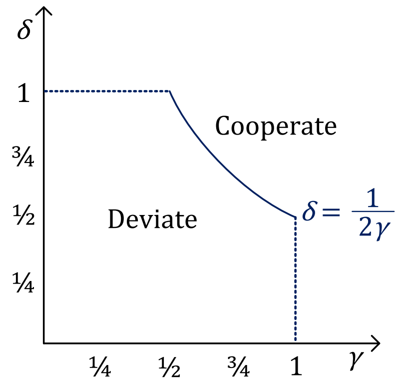

Figure 6:Cooperation Regions
as a function of γ.

c. It takes $K \geq 1$ periods to detect and respond to a deviation from the collusive agreement.

Answer:

In this case, a firm will receive $\frac{\pi^m}{2}$ each period for cooperating, yielding a lifetime payoff of

$$\frac{\pi^m}{2} + \delta\frac{\pi^m}{2} + \delta^2\frac{\pi^m}{2} + \cdots = \sum_{t=0}^{\infty} \delta^t \frac{\pi^m}{2} = \frac{1}{1-\delta}\frac{\pi^m}{2}$$

If, in contrast, the firm wanted to deviate, it would be able to deviate for several periods before being caught deviating and then receiving zero profits from then on. Hence, we can express the deviation payoff as

$$\pi^m + \delta\pi^m + \delta^2\pi^m + \cdots \delta^{K-1}\pi^m = \sum_{t=0}^{K-1} \delta^t \pi^m = \frac{1-\delta^K}{1-\delta}\pi^m$$

It may be helpful to recall geometric series here to derive the second term of this equation, let

$$\begin{aligned} a &= \pi^m + \delta\pi^m + \delta^2\pi^m + \cdots \delta^{K-1}\pi^m \\ \delta a &= \delta\pi^m + \delta^2\pi^m + \delta^3\pi^m + \cdots \delta^K\pi^m \\ a - \delta a &= \pi^m - \delta^K\pi^m \implies a = \frac{1-\delta^K}{1-\delta}\pi^m \end{aligned}$$

Therefore, cooperation can be sustained if and only if the payoff from cooperating is weakly higher than that from deviating, i.e.,

$$\frac{1}{(1-\delta)}\frac{\pi^m}{2} \geq \frac{\left(1-\delta^K\right)}{(1-\delta)}\pi^m \implies \delta \geq \left(\frac{1}{2}\right)^{\frac{1}{K}}$$

Hence, the more periods of time K that a cheating firm remains undetected by its colluding partners, the more attractive cheating becomes. Cooperation therefore can only be sustained under more restrictive sets of parameter values. Figure 7 demonstrates this (Note that this figure portrays K as a continuous variable, when in actuality it can only take on integer values).

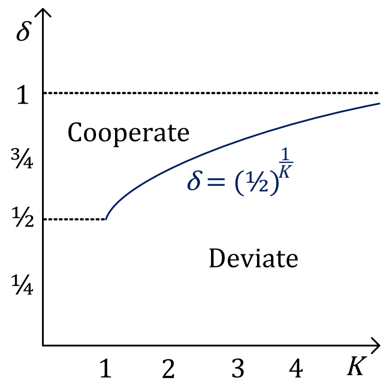

Figure 7: Cooperation Regions as a Function of K.

4. Collusion when N firms compete in prices

Consider a homogenous industry where n firms produce at zero cost and play the Bertrand game of price competition for an infinite number of periods. Assume that:

- When firms choose the same price, they earn a per-period profit $\pi(p) = p\alpha\frac{D(p)}{n}$, where parameter α represents the state of demand and $D(p)$ represents the quantity demand for the good.
- When a firm i charges a price of p_i lower than the price of all of the other firms, it earns a profit $\pi(p_i) = p_i\alpha D(p_i)$, and all of the other firms obtain zero profits.

Imagine that in the current period demand is characterized by $\alpha = 1$, but starting from the following period demand will be characterized by $\alpha = \theta$ in each of the following periods. All the players know exactly the evolution of the demand state at the beginning of the game, and firms have the same common discount factor, δ.

Assume that $\theta > 1$ and consider the following trigger strategies. Each firm plays the monopoly price P_m in the first period of the game, and continues to charge such a price until a profit equal to zero is observed. When this occurs, each firm charges a price equal to zero (the marginal cost) forever. Under which conditions does this trigger strategy represent an equilibrium? [Hint: In particular, show how θ and n affects such a condition, and give an economic intuition for this result.]

Answer:

[*Cooperation* :] Let us denote the collusive price by $P^c \in (c, P_m]$. At time $t = 0$, parameter α takes the value of $\alpha = 1$, whereas at any subsequent time period $t = \{1, 2, \ldots\}$, $\alpha = \theta$. Hence, by colluding, firm i obtains a discounted stream of profits of

$$\begin{aligned} & \frac{\pi\left(P^c\right)}{n} + \delta\theta\frac{\pi\left(P^c\right)}{n} + \delta^2\theta\frac{\pi\left(P^c\right)}{n} + \delta^3\theta\frac{\pi\left(P^c\right)}{n} + \cdots \\ = & \frac{\pi\left(P^c\right)}{n}\left(1 + \delta\theta + \delta^2\theta + \delta^3\theta + \cdots\right) = \left(1 + \frac{\delta\theta}{1-\delta}\right)\frac{\pi(P^c)}{n} \end{aligned}$$

[*Deviation* :] Deviating firm i charges a price marginally lower than the collusive price, i.e., $P^d = P^c - \varepsilon$ and captures all the market, thus obtaining a profit level of $\pi\left(P^c\right)$ (We assume that the price difference is so small that profits are unchanged). However, after that deviation, all firms revert to the Nash equilibrium of the unrepeated Bertrand game, which yields a profit of zero thereafter. Therefore, the payoff stream from deviating is $\pi\left(P^c\right) + 0 + \delta 0 + \cdots = \pi\left(P^c\right)$.
Incentives to collude: Hence, every firm i colludes as long as

$$\begin{aligned} \left(1 + \frac{\delta\theta}{1-\delta}\right)\frac{\pi(P^c)}{n} &\geq \pi\left(P^c\right) \\ 1 + \frac{\delta\theta}{1-\delta} \geq n &\implies \delta \geq \frac{n-1}{n-1+\theta} \end{aligned}$$

For compactness, we hereafter denote the previous ratio as $\frac{n-1}{n-1+\theta} \equiv \tilde{\delta}\left(n, \theta\right)$. Figure 8 depicts this critical threshold of the discount factor, evaluated at $\theta = 1.2$, i.e., demand increases 20% after the first year.

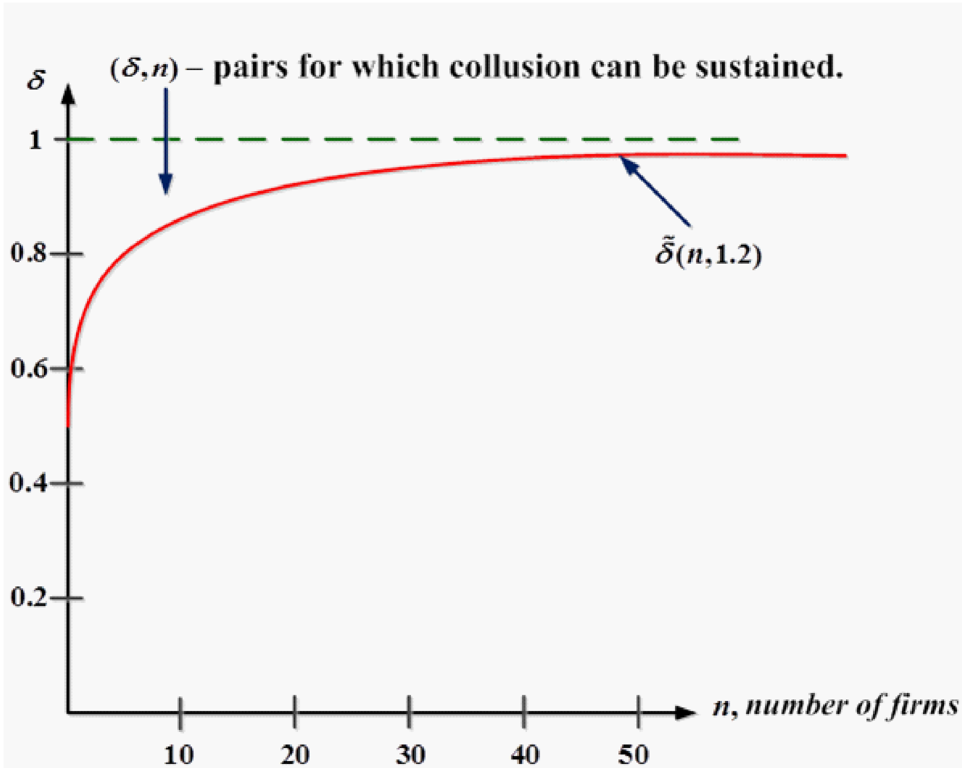

Figure 8: Critical Threshold $\tilde{\delta}(n, \theta)$.

In particular, if firms seek to collude on a collusive price equal to that under monopoly, i.e., $P^c = P_m$, such equilibrium can be sustained as long as $\delta \geq \tilde{\delta}(n, \theta)$.

Comparative statics : We can next examine how the critical discount factor $\tilde{\delta}(n, \theta)$ is affected by changes in demand, θ, and in the number of firms, n. In particular,

$$\frac{\partial \tilde{\delta}(n, \theta)}{\partial \theta} = \frac{-(n-1)}{(n-1+\theta)^2} < 0,$$

whereas

$$\frac{\partial \tilde{\delta}(n, \theta)}{\partial n} = \frac{\theta}{(n-1+\theta)^2} > 0.$$

In words, the higher the increments in demand, θ, the higher the present value of the stream of profits received from $t = 1$ onwards. That is, the opportunity cost of deviation increases as demand becomes stronger. Graphically, the critical discount factor $\tilde{\delta}(n, \theta)$ shifts downwards, thus expanding the region of (δ, n)-pairs for which collusion can be sustained. Figure 9 provides an example of this comparative statics result, whereby θ is evaluated at $\theta = 1.2$ and at $\theta = 1.8$. On the other hand, when n increases, the collusion is more difficult to sustain in equilibrium.

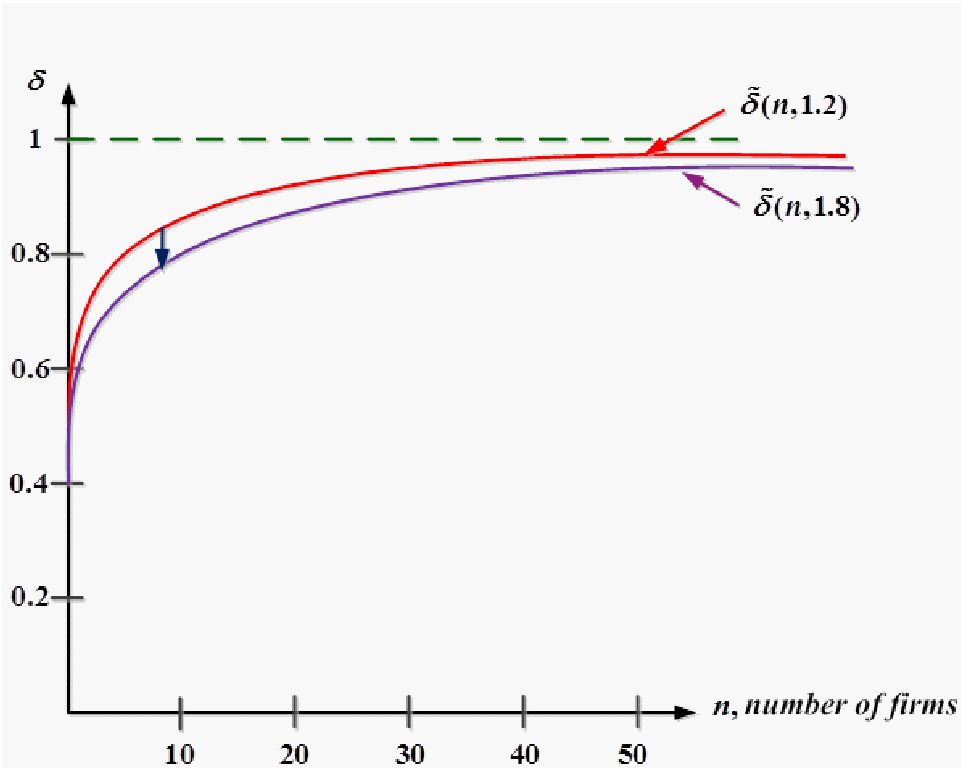

Figure 9: Cutoff $\tilde{\delta}(n,\theta)$ for $\theta = 1.2$ and $\theta = 1.8$.

5. Aliprantis 7.22

[A Subgame Perfect Equilibrium of an Infinite-Horizon Repeated Game with a Stage Game That Does Not Have a Pure-Strategy Nash Equilibrium]

We revisit the infinite-horizon repeated game, with the stage game given in figure 11. Consider the strategy profile $\sigma = (\sigma_1, \sigma_2)$ of this infinite-horizon game, in which

1. For $t = 1$, we let

$$\sigma_{1,1}\left(h^0\right) = U \text{ and } \sigma_{1,2}\left(h^0\right) = L.$$

2. For $t > 1$ and each

$$h^{t-1} = \begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \\ \vdots & \vdots \\ s_{t-1,1} & s_{t-1,2} \end{bmatrix} \in H^{t-1},$$

we define

$$\sigma_{t,1}\left(h^{t-1}\right)=\left\{\begin{array}{l} U \text{ if } \left(s_{t-1,1},s_{t-1,2}\right) \text{ is either } (U,L) \text{ or } (D,R) \\ D \text{ otherwise.}\end{array}\right\}$$

and

$$\sigma_{t,2}\left(h^{t-1}\right)=\left\{\begin{array}{l} L \text{ if } \left(s_{t-1,1},s_{t-1,2}\right) \text{ is either } (U,L) \text{ or } (D,R) \\ R \text{ otherwise.}\end{array}\right\}$$

We claim that σ is a subgame perfect equilibrium for this infinite-horizon repeated game, and will show that it is. Consider the subgame starting at $t+1$. We distinguish two cases.

[Case 1:] $(s_{t,1},s_{t,2})$ =either (U,L) or (D,R)

In this case the outcome of the strategy profile σ from period $t+1$ onward is as depicted in figure 11 below.

$$((U,L),(U,L),(U,L),\cdots).$$

		Player 2	
		L	R
Player 1	U	$3,3$	$1,2$
	D	$4,0$	$0,1$

Figure 11: Normal Form of the Stage Game

This implies that the discounted utility of both players for the subgame starting at $t+1$ are

$$v_1^{t+1}(\sigma)=v_2^{t+1}(\sigma)=\sum_{s=t+1}^{\infty}\delta^{s-t-1}3=\frac{3}{1-\delta}.$$

Notice first that since 3 is the highest payoff player 2 can receive in each period, it follows that $v_2^{t+1}(\sigma)$ is the highest discounted payoff that player 2 can get starting at $t+1$, irrespective of the strategies of the two players. In particular, player 2 cannot improve his payoff in the subgame starting at period $t+1$ by changing his strategy when player 1 continues to play σ_1.

Now assume that player 2 continues to play σ_2 while player 1 deviates. We have two cases to check.

1) *The action in period t is (U,L)* : Because player 2 does not change his strategy, player 2 plays R in period $t+1$. If player 1 deviates in period $t+1$ and then plays σ_1 from period $t+2$ onward, the outcome is

$$((D,L),(D,R),(U,L),(U,L),...),$$

which gives player 1 the discounted payoff

$$4 + 0 \times \delta + 3\delta^2 + 3\delta^3 + \cdots \qquad (1)$$

versus his discounted payoff from cooperating of

$$3 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots = v_1^{t+1}(\sigma)$$

We can verify that player 1 will cooperate as long as

$$3 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots \geq 4 + 0 \times \delta + 3\delta^2 + 3\delta^3 + \cdots$$

which holds for $\delta \geq \frac{1}{3}$. If player 1 deviates in periods $t+1$ and $t+2$ and the plays σ_1, then his discounted payoff is

$$\begin{aligned} & 4 + 0 \times \delta + 0 \times \delta^2 + 3\delta^3 + \cdots \\ < \; & 4 + 0 \times \delta + 3\delta^2 + 3\delta^3 + \cdots < v_1^{t+1}(\sigma) \end{aligned}$$

for all values of δ. In fact, it can be checked that the larger the number of periods that player 1 deviates from playing σ_1, the lower the discounted payoff.

2) *The action in period t is* (D,R) : In this case if player 1 deviates in period $t+1$ and then plays σ_1, the outcome is

$$((D,L),(D,R),(U,L),(U,L),...),$$

which the discounted payoff

$$4 + 0 \times \delta + 3\delta^2 + 3\delta^3 + \cdots ,$$

which, as in (1), is no more than $v_1^{t+1}(\sigma)$ for $\delta \geq \frac{1}{3}$. If player 1 deviates in both periods $t+1$ and $t+2$ and the plays σ_1, the outcome is

$$((D,L),(U,R),(D,R),(U,L),(U,L),...),$$

whose discounted payoff is

$$4 + \delta + 0 \times \delta^2 + 3\delta^3 \leq 4 + 0 \times \delta + 3\delta^2 + 3\delta^3 + \cdots < v_1^{t+1}(\sigma)$$

which, again holds for $\delta \geq \frac{1}{3}$.

Again as in (1), it can be checked that the larger the number of periods for which player 1 deviates, the lower the discounted payoff of player 1.

[Case 2:] $(s_{t,1}, s_{t,2})$ =**either** (D, L) **or** (U, R)

1) *The action in period t is (D, L)* : If the players play σ from period $t+1$ onward, then the outcome in the subgame is

$$((D, R), (U, L), (U, L), (U, L), ...),$$

The discounted payoff of player 1 is

$$0 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots = v_1^{t+1}(\sigma),$$

and the discounted payoff of player 2 is

$$1 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots = v_2^{t+1}(\sigma).$$

If player 1 deviates in period $t+1$ and then plays σ_1, the outcome is

$$((U, R), (D, R), (U, L), (U, L), ...),$$

with the discounted payoff

$$1 + 0.\delta + 3\delta^2 + 3\delta^3 + \cdots.$$

For $\delta \geq \frac{1}{3}$, we have our cooperation condition holding, i.e.,

$$1 + 0 \times \delta + 3\delta^2 + 3\delta^3 + \cdots \leq 0 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots.$$

Thus, player 1 does not gain by deviating in period $t+1$ if $\delta \geq \frac{1}{3}$.

If player 1 deviates in periods $t+1$ and $t+2$ and the plays σ_1, then the outcome is

$$((U, R), (U, R), (D, R), (U, L), (U, L), ...).$$

Its payoff is

$$1 + \delta + 0 \times \delta^2 + 3\delta^3 + \cdots \leq 1 + 0 \times \delta + 3\delta^2 + 3\delta^3 + \cdots < v_1^{t+1}(\sigma)$$

for any value of $\delta \geq \frac{1}{3}$. Therefore, player 1 cannot gain by deviating in $t+1$ and $t+2$. In fact, it can be checked that if $\delta \geq \frac{1}{3}$, then the larger the number of periods during which player 1 deviates, the lower is the discounted payoff.

Now consider deviations by player 2. If player 2 deviates in period $t+1$ and then plays σ_2 while player 1 now continues to play σ_1, the outcome is

$$((D, L), (D, R), (U, L), (U, L), ...),$$

and the discounted payoff of player 2 is

$$0 + \delta + 3\delta^2 + 3\delta^3 + \cdots < 1 + 3\delta + 3\delta^2 + 3\delta^3 + \cdots = v_2^{t+1}(\sigma).$$

for all values of δ. Therefore, player 2 cannot gain by deviating in period $t+1$.

If player 2 deviates in periods $t+1$ and $t+2$ and then plays σ_2, then the outcome is

$$((D,L),(D,L),(D,R),(U,L),(U,L),...).$$

The discounted payoff is

$$0+0\times\delta+3\delta^2+3\delta^3+\cdots\leq 1+3\delta+3\delta^2+3\delta^3+\cdots=v_2^{t+1}(\sigma).$$

which holds for all values of δ. Therefore, player 2 cannot gain by deviating in periods $t+1$ and $t+2$. In fact, exactly as in the case of player 1, it can be checked that the larger the number of periods during which player 2 deviates, the lower is the discounted payoff.

2) *The action in period t is* (U,R) : If the players play σ from period $t+1$ onward, then the outcome in the subgame is

$$((D,R),(U,L),(U,L),(U,L),...),$$

and the discounted payoff of player 1 is

$$0+3\delta+3\delta^2+3\delta^3+\cdots=v_1^{t+1}(\sigma).$$

The discounted payoff of player 2 is

$$1+3\delta+3\delta^2+3\delta^3+\cdots=v_2^{t+1}(\sigma).$$

If player 1 deviates in period $t+1$ and then plays σ_1, the outcome is

$$((U,R),(D,R),(U,L),(U,L),...),$$

and as in (1), above player 1 cannot gain by deviating if $\delta\geq\frac{1}{3}$.
If player 1 deviates in periods $t+1$ and $t+2$ and then plays σ_1, then the outcome is

$$((U,R),(U,R),(D,R),(U,L),(U,L),...).$$

And arguing exactly as in (1), it then follows that player 1 cannot gain by deviating if $\delta\geq\frac{1}{3}$.

The argument that player 2 cannot gain by deviating is now exactly as in Case 2 (1).

We have thus shown that if $\delta\geq\frac{1}{3}$, then neither player 1 nor player 2 can gain by deviating from the strategy profile $\sigma=(\sigma_1,\sigma_2)$ in the subgame starting from period $t+1$ given any period t outcome $(U,L),(D,R),(D,L)$, and (U,R). Because the strategy profile σ depends only on these outcomes in period t, we have shown that σ is an equilibrium strategy profile in any subgame starting from period $t+1$ given any history h^t. Because this is true for any $t\geq 0$, we have shown that the strategy profile σ is a subgame perfect equilibrium strategy profile.