EconS 503 - Advanced Microeconomics II
Review Session #2

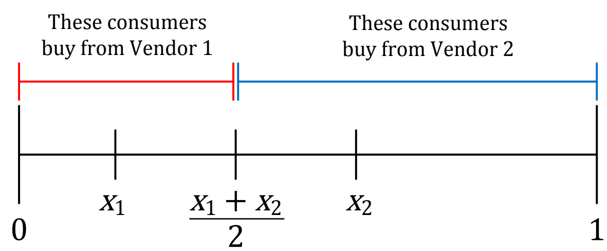
1. MWG 8.D.5 (NE in Continuous Action Spaces)

Consumers are uniformly distributed along a boardwalk that is 1 mile long. Ice-cream prices are regulated, so consumers go to the nearest vendor because they dislike walking (assume that at the regulated prices all consumers will purchase an ice cream even if they have to walk a full mile). If more than one vendor is at the same location, they split the business evenly.

a) Consider a game in which two ice cream vendors pick their locations simultaneously. Show that there exists a unique pure strategy Nash equilibrium and that involves both vendors locating at the midpoint of the boardwalk.

Answer:

Let x_1 be the location of Vendor 1 and x_2 be the location of Vendor 2. Thus, we can associate a strategy for Player i with $x_i \in [0,1]$. First, let us find out the payoff function for each of the ventors. Since the price of the ice cream is regulated (i.e., the vendors cannot lower their prices to attract people from farther away) we can identify the profit of each vendor with the number of customers they get. Suppose $x_1 < x_2$. In this case, all consumers located to the left of (below) $\frac{x_1+x_2}{2}$ will purchase from Vendor 1, while all customers located to the right of $\frac{x_1+x_2}{2}$ will buy ice cream from Vendor 2. Note that $\frac{x_1+x_2}{2}$ is the midpoint between points x_1 and x_2, as shown in the figure below.



Thus

$$
\begin{aligned}
u_1(x_1,x_2) &= \frac{x_1+x_2}{2} \qquad \left(\text{the length of } \left[0, \frac{x_1+x_2}{2}\right]\right) \\
u_2(x_1,x_2) &= 1-\frac{x_1+x_2}{2} \qquad \left(\text{the length of } \left[\frac{x_1+x_2}{2}, 1\right]\right)
\end{aligned}
$$

We can derive a similar result for $x_2 < x_1$:

$$
\begin{aligned}
u_1(x_1, x_2) &= 1 - \frac{x_1 + x_2}{2} \qquad \left(\text{the length of } \left[\frac{x_1 + x_2}{2}, 1\right]\right) \\
u_2(x_1, x_2) &= \frac{x_1 + x_2}{2} \qquad \left(\text{the length of } \left[0, \frac{x_1 + x_2}{2}\right]\right)
\end{aligned}
$$

Now, if $x_1 = x_2$, the vendors split the business so that $u_1(x_1, x_2) = u_2(x_1, x_2) = \frac{1}{2}$. Thus, summarizing:

$$
\begin{aligned}
u_1(x_1, x_2) &= \begin{cases} \frac{x_1+x_2}{2} \text{ if } x_1 < x_2 \\ \frac{1}{2} \text{ if } x_1 = x_2 \\ 1 - \frac{x_1+x_2}{2} \text{ if } x_1 > x_2 \end{cases} \\
u_2(x_1, x_2) &= \begin{cases} 1 - \frac{x_1+x_2}{2} \text{ if } x_1 < x_2 \\ \frac{1}{2} \text{ if } x_1 = x_2 \\ \frac{x_1+x_2}{2} \text{ if } x_1 > x_2 \end{cases}
\end{aligned}
$$

First, let's consider a few cases where the positions of the vendors can differ. Let $i, j \in \{1, 2\}$ and $i \neq j$.

- *Case 1*: $x_i < x_j < 0.5$. If $x_i < x_j < 0.5$, Vendor i could increase his profits by positioning himself ε to the right of x_j and increase his profits from $\frac{x_i+x_j}{2}$ to $1 - \frac{x_j+\varepsilon+x_j}{2}$. Thus, any case where $x_i < x_j < 0.5$ cannot be supported as a Nash equilibrium.

- *Case 2*: $0.5 < x_i < x_j$. If $0.5 < x_i < x_j$, Vendor j could increase his profits by positioning himself ε to the left of x_i and increase his profits from $1 - \frac{x_i+x_j}{2}$ to $\frac{x_i+x_i-\varepsilon}{2}$. Thus, any case where $0.5 < x_i < x_j$ cannot be supported as a Nash equilibrium.

- *Case 3*: $x_i < 0.5 < x_j$. If $x_i < 0.5 < x_j$, Vendor i could increase his profits by positioning himself ε to the left of x_j and increase his profits from $\frac{x_i+x_j}{2}$ to $\frac{x_j-\varepsilon+x_j}{2}$. Thus, any case where $x_i < 0.5 < x_j$ cannot be supported as a Nash equilibrium. (Note that Vendor j would also want to deviate to ε right of Vendor i)

Thus, there cannot exist a Nash equilibrium where x_i and x_j differ. Now, we consider a few cases where they are the same.

- *Case 1*: $x_1 = x_2 < 0.5$. If $x_1 = x_2 < 0.5$, Vendor 1 could increase his profits by positioning himself ε to the right of x_2 and increase his profits from $\frac{1}{2}$ to $1 - \frac{x_2+\varepsilon+x_2}{2}$. Thus, any case where $x_1 = x_2 < 0.5$ cannot be supported as a Nash equilibrium.

- *Case 2*: $0.5 < x_1 = x_2$. If $0.5 < x_1 = x_2$, Vendor 2 could increase his profits by positioning himself ε to the left of x_1 and increase his profits from $\frac{1}{2}$ to $\frac{x_1+x_1-\varepsilon}{2}$. Thus, any case where $0.5 < x_1 = x_2$ cannot be supported as a Nash equilibrium.

- *Case 3*: $x_1 = x_2 = 0.5$. If $x_1 = x_2 = 0.5$, neither Vendor 1 nor Vendor 2 have a profitable deviation. For example, if Vendor 1 were to deviate in either direction, it would receive $\frac{x_2+x_2-\varepsilon}{2} = 1 - \frac{x_2+x_2+\varepsilon}{2} = \frac{1}{2} - \frac{\varepsilon}{2} < \frac{1}{2}$.

Thus, there only exists one Nash equilibrium, and it exists at $x_1 = x_2 = 0.5$

b) Show that with three vendors, no pure strategy Nash equilibrium exists.

Answer:

Suppose that an equilibrium (x_1^*, x_2^*, x_3^*) exists. Suppose, first, that $x_1^* = x_2^* = x_3^*$. Then each firm will sell $\frac{1}{3}$. But any firm can increase its sales by moving to the right (if $x_1^* = x_2^* = x_3^* < \frac{2}{3}$) or to the left (if $x_1^* = x_2^* = x_3^* > \frac{1}{3}$), a contradiction. Suppose that two firms located at the same point, let's say $x_1^* = x_2^*$. If $x_1^* = x_2^* < x_3^*$, then firm 3 can do better by moving to $x_1^* + \varepsilon$. If $x_1^* = x_2^* > x_3^*$, then firm 3 can do better by moving to $x_1^* - \varepsilon$, a contradiction. Finally, suppose that all 3 firms are located at different points. But then the firm that is located the farthest on the right will be able to increase its sales by moving to ε to the right of its closest competitor, a contradiction (Note that the left could also move, or the center could even leap to the right or left of a competitor, too). Thus, there exists no pure strategy NE in this game.

2. Harrington Ch.7 #10 (An easy msNE)

Each of three players is deciding between the pure strategies *go* and *stop*. The payoff to *go* is $\frac{120}{m}$, where m is the number of players that choose *go*, and the payoff to *stop* is 55 (which is received regardless of what the other players do).

- Find all Nash Equilibria.

Answer:

There are the potential for at least *seven* Nash Equilibria in pure and mixed strategies.

- First, there are three asymmetric pure-strategy Nash Equilibria in which two players choose *go* and the other one chooses *stop*. Each player who chooses *go* earns a payoff of $\frac{120}{2} = 60$, which exceeds the payoff of 55 from choosing *stop*. The player who chooses *stop* earns 55, which exceeds the payoff from choosing *go*, which is $\frac{120}{3} = 40$.

- Now consider a strategy profile in which one player chooses the pure strategy *go* and the other two players symmetrically randomize, choosing *go* with probability p. The mixed strategy equilibrium condition for both mixing players is:

$$\begin{aligned} EU(go) &= EU(stop) \\ \underbrace{(1-p)}_{\substack{\text{I go and the} \\ \text{other player stops}}} *60 + \underbrace{p}_{\substack{\text{I go and the} \\ \text{other player goes}}} *40 &= 55 \\ &\implies p = \frac{1}{4} \end{aligned}$$

  The solution for p is the mixed strategy for the other player that makes this player indifferent between *stop* and *go*. It is a Nash Equilibrium for one player to use the pure strategy *go* and each of the other two players to choose *go* with probability $\frac{1}{4}$; this gives us another three Nash Equilibria.

- Finally, there is a symmetric mixed-strategy Nash Equilibrium in which each player chooses *go* with probability q. The mixed strategy equilibrium condition is defined by:

$$EU(go) = EU(stop)$$

$$\underbrace{(1-q)^2}_{\text{I go and the other players stop}} *120 + \underbrace{2q(1-q)}_{\text{I go, one player stops and the other player goes}} *60 + \underbrace{q^2}_{\text{I go and the other players go}} *40 = 55$$

$$40q^2 - 120q + 65 = 0$$

Using the quadratic formula, one finds that q is approximately 0.71. There is then a symmetric Nash Equilibrium in which each player chooses *go* with probability 0.71.

- It is possible that there are asymmetric Nash Equilibria in which two or more players randomize but with different probabilities.

3. Mixed Strategies (Something harder)

Consider the following game

		Player 2 L	Player 2 R
Player 1	U	2, 0	2, 0
	M	3, 0	0, 1
	D	0, 1	3, 0

a) Are there any strictly dominated strategies for either player (in either pure or mixed strategies)?

Answer:

To find our candidates for elimination, we first underline the best responses for each player, as shown below.

		Player 2 L	Player 2 R
Player 1	U	$2,\underline{0}$	$2,\underline{0}$
	M	$\underline{3},0$	$0,\underline{1}$
	D	$0,\underline{1}$	$\underline{3},0$

Since Player 1 does not select the strategy U as a best response to either action of Player 2, it is a candidate for elimination. It is clear that it cannot be dominated by a pure strategy, but let's see if it can be dominated by a mixed strategy of M and D. Let's assign p as the probability that Player 1 plays M and $1-p$ as the probability that Player 1 plays D. For simplicity, we can create a reduced form game where the bottom cells are linear combinations of the payoffs from M and D

		Player 2 L	Player 2 R
Player 1	U	$2, 0$	$2, 0$
Player 1	pM $+(1-p)$D	$3p, 1-p$	$3-3p, p$

In order for U to be strictly dominated by the mixed strategy pM $+(1-p)$D, we must find at least one value of p for which every payoff of pM $+(1-p)$D for Player 1 is strictly larger than the payoff of U, i.e.,

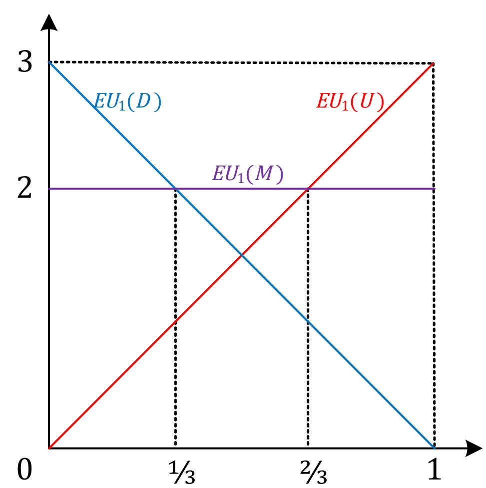
$$
\begin{aligned}
3p &> 2 \implies p > \frac{2}{3} \\
3-3p &> 2 \implies p < \frac{1}{3}
\end{aligned}
$$

and since there are no values of p for which both of these conditions hold at the same time, strategy U cannot be dominated by a mixed strategy. Hence, there are no strictly dominated strategies in this game.

- *Intuition*: Another way to show that strategy U is not dominated is to consider the best response function for Player 1 as a function of his beliefs on Player 2's action. Let q be the probability that Player 2 plays L and $(1-q)$ be the probability that Player 2 plays R. Player 1's expected payoff from each of his strategies then becomes

$$
\begin{aligned}
EU_1(U) &= 2 \\
EU_1(M) &= 3q \\
EU_1(D) &= 3(1-q) = 3-3q
\end{aligned}
$$

For convenience, all three functions are plotted below,



As can be seen in the figure, when Player 1 puts closer to even probabilities ($q \in \left(\frac{1}{3}, \frac{2}{3}\right)$) on Player 2 choosing L, he actually prefers to choose strategy U, as the certain payoff becomes better in that situation. Hence, for some of Player 1's beliefs, strategy U is his best response, and thus it cannot be dominated.

b) Find a mixed strategy where Player 1 mixes between strategies M and D, and Player 2 mixes between strategies L and R

Answer:

Considering a strategy where Player 1 mixes between strategies M and D, and Player 2 mixes between strategies L and R, with p and q behaving the same way as in part a, we can solve for the mixed strategy. Starting with Player 1, we have

$$\begin{aligned} EU_1(M) &= 3q \\ EU_1(D) &= 3(1-q) = 3 - 3q \end{aligned}$$

Setting these two equal to one another yields $q = \frac{1}{2}$. Doing the same for Player 1, we have

$$\begin{aligned} EU_2(L) &= 1-p \\ EU_2(R) &= p \end{aligned}$$

and setting these two equal and solving for p yields $p = \frac{1}{2}$. Hence, our mixed strategy is

$$\left(\frac{1}{2}M\frac{1}{2}D, \frac{1}{2}L\frac{1}{2}R\right)$$

c) Is the strategy you found in part (b) a Nash Equilibrium?

Answer:

No. Recall from part (a) that when $q \in \left(\frac{1}{3}, \frac{2}{3}\right)$, Player 1's best response is to play U. Since our mixed strategy gives $q = \frac{1}{2}$, Player 1 would be better off deviating from the expected payoff of 1.5 to the certain payoff of 2. Thus, the mixed strategy we found in part (b) cannot be a Nash Equilibrium.

d) Find all Nash Equilibria

Answer:

Since there are no pure strategy Nash equilibria and the mixed strategy we considered is part (b) is not a Nash equilibrium, we consider Player 1 playing a pure strategy U and Player 2 mixing between strategies L and R. We know that $q \in \left(\frac{1}{3}, \frac{2}{3}\right)$ for Player 1 to prefer to play U, and Player 2 would be completely indifferent between L and R when Player 1 plays U (since his payoff is 0 either way). Hence, a continuum of mixed strategies exist that satisfy

$$(U, qL(1-q)R), \text{ where } q \in \left(\frac{1}{3}, \frac{2}{3}\right)$$