EconS 503 - Microeconomic Theory II
Homework #1 - Answer Key

1. **[Strict dominant equilibrium and IDSDS]** A strategy profile $s^* = (s_1^*, \ldots, s_N^*)$ is a strict dominant equilibrium if $s_i \neq s_i^*$ is a strictly dominant strategy for every player $i \in N$, that is,

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all } s_i \neq s_i^* \text{ and for all } s_{-i} \in S_{-i}.$$

(a) Show that if strategy profile s^* is a strictly dominant equilibrium, then it must be the only strategy profile surviving the iterative deletion of strictly dominated strategies (IDSDS).

- If $s^* = (s_1^*, \ldots, s_N^*)$ is a strictly dominant strategy equilibrium then, by definition, for every player i all other strategies $s_i \neq s_i^*$ are strictly dominated by s_i^*. This implies that after one stage of elimination we will be left with a single profile of strategies, which is exactly s^*, and this concludes the proof (Tadelis, 2013, p.68).

(b) Show that if a strategy profile is the only one surviving IDSDS, it does not need to coincide with the strict dominant equilibrium. [*Hint*: An example suffices, where you provide a game where the strict dominant equilibrium does not exist.]

- Consider the following example,

Player 2

Player 1	L	C	R
U	3, 2	5, 1	4, 3
M	2, 8	8, 4	3, 6
D	4, 0	4, 6	6, 8

Note that there is no strictly dominant strategy for player 1 and similarly for player 2. However, the following steps show that there is only one strategy surviving IDSDS.

- Beginning with player 2, note that his strategy C is strictly dominated by strategy R, leaving us with the following reduced-form matrix:

Player 2

Player 1	L	R
U	3, 2	4, 3
M	2, 8	3, 6
D	4, 0	6, 8

Next, proceeding with player 1, note that his strategy M is strictly dominated by strategy D, leaving us with the following reduced-form matrix:

Player 2

Player 1	L	R
U	3, 2	4, 3
D	4, 0	6, 8

Returning to player 2, note that his strategy L is strictly dominated by strategy R, leaving us with the following reduced-form matrix:

		Player 2
		R
Player 1	U	4, 3
	D	6, 8

Lastly, continuing with player 1, note that his strategy U is strictly dominated by strategy D, leaving us with the following reduced-form matrix:

		Player 2
		R
Player 1	D	6, 8

such that strategy (D, R) is the only one surviving IDSDS.

2. **[Strict dominance and Rationalizability]** Consider the 3x3 matrix at the bottom of page 72 in Tadelis.

		Player 2		
		L	C	R
	U	3, 3	5, 1	6, 2
Player 1	M	4, 1	8, 4	3, 6
	D	4, 0	9, 6	6, 8

(a) Find the strict dominant equilibrium of this game.

- This game does not have a strict dominant equilibrium because none of the players has a strictly dominant strategy. For a strict dominant equilibrium to exist, we need all players to use a strictly dominant strategy (or strategies). In this context, we need that, every player i, strategy s_i satisfies

$$u_i(s_i, s_j) \geq u_i\left(s_i', s_j\right)$$

for every $s_i' \neq s_i$ and for all $s_j \in S_j$.

(b) Which strategy profile/s survive IDSDS?

- Let us start with player 1, who does not have strictly dominated strategy. To see this, note that:
  - $u_1(U, s_2) < u_1\left(M, s_2\right)$ when player 2 selects $s_2 = L$ (in the left-hand column) and when he selects $s_2 = C$ (in the center column), but
  - $u_1(U, s_2) > u_1\left(M, s_2\right)$ when player 2 selects $s_2 = R$ in the right-hand column.
- A similar argument applies when comparing player 1's payoffs from choosing M and D:
  - $u_1(M, L) = u_1\left(D, L\right)$ when player 2 chooses L,
  - $u_1(M, C) < u_1\left(D, C\right)$ when player 2 chooses C, and
  - $u_1(M, R) < u_1\left(D, R\right)$ when player 2 chooses R.

- We can now move to player 2, where C is strictly dominated by R since $u_2(C, s_1) < u_2(R, s_1)$ for every strategy s_1 chosen by player 1. We can then reproduce the remaining matrix after the first two rounds of IDSDS, i.e., after deleting nothing for player 1 and strategy C for player 2.

		Player 2	
		L	R
Player 1	U	3, 3	6, 2
	M	4, 1	3, 6
	D	4, 0	6, 8

- We cannot find any more strictly dominated strategies relying on pure strategies. (As a practice, check that allowing for player 1 to randomize would not help us to further reduce the set of strategy profiles surviving IDSDS.) Then, the set of strategies surviving IDSDS is the six strategy profiles in the reduced matrix:

$$\{(U, L), (U, R), (M, L), (M, R), (D, L), (D, R)\}$$

(c) Which strategy profile/s survive rationalizability?

- The following payoff matrix underlines the payoffs that each player obtains when playing a best response to his opponent's strategies.

		Player 2		
		L	C	R
Player 1	U	3, $\underline{3}$	5, 1	$\underline{6}$, 2
	M	$\underline{4}$, 1	8, 4	3, $\underline{6}$
	D	4, 0	$\underline{9}$, 6	$\underline{6}$, $\underline{8}$

For instance, when player 2 chooses strategy L, player 1's best response is $BR_1(L) = \{M, D\}$. Whereas, when player 2 chooses C, player 1's best response is $BR_1(C) = \{D\}$. Furthermore, when when player 2 chooses R, player 1's best response is $BR_1(R) = \{U, D\}$. In other words, player 1 deploys all of his available strategies as a best response to at least one of his opponent's strategies. Alternatively, player 1 finds that the set of strategies that are never a best response is nil, that is, $NBR_1 = \varnothing$.

- Operating similarly for player 2, we find that his best responses are

$$BR_2(U) = \{L\}, \; BR_2(M) = \{R\}, \text{ and } BR_2(D) = \{R\},$$

such that the strategy that player 2 never uses as a best response is $NBR_2 = \{C\}$. We can now delete the strategies that are NBR to obtain the following reduced matrix in the first round of the application of rationalizability,

		Player 2	
		L	R
Player 1	U	3, 3	6, 2
	M	4, 1	3, 6
	D	4, 0	6, 8

Move to player 1 again, we can find that his best responses are now

$$BR_1(L) = \{M, D\} \text{ and } BR_1(R) = \{U, D\}$$

implying that, again, $NBR_1 = \varnothing$. In the second round of applying rationalizability, we can therefore find no more strategies that are NBR to either player, leaving us with the following equilibria from the application of rationalizability:

$$\{(U, L), (U, R), (M, L), (M, R), (D, L), (D, R)\}$$

3. [**Equilibrium predictions from IDSDS vs. IDWDS**] While the order of deletion of dominated strategies does not affect the equilibrium outcome when applying IDSDS, it can affect the set of equilibrium outcomes when we delete weakly (rather than strictly) dominated strategies. Use the payoff matrix below to show that the order in which weakly dominated strategies are eliminated can affect equilibrium outcomes.

Player 2

Player 1		L	M	R
	U	$2, 1$	$1, 1$	$0, 0$
	C	$1, 2$	$3, 1$	$2, 1$
	D	$2, -2$	$1, -1$	$-1, -1$

- *First Route:* Taking the above payoff matrix, first note that, for player 1, strategy U weakly dominates D, since U yields a weakly larger payoff than D for any strategy (column) selected by player 2, that is

  $$u_1(U, s_2) \geq u_1(D, s_2) \text{ for all } s_2 \in \{L, M, R\}.$$

  In particular, U provides player 1 with the same payoff as D when player 2 selects L (a payoff of 2 for both U and D) and M (a payoff of 1 for both U and D), but a strictly higher payoff when player 2 chooses R (in the right-hand column) since $0 > -1$. Once we have deleted D because of being weakly dominated, we obtain the reduced-form matrix depicted below.

  Player 2

Player 1		L	M	R
	U	$2, 1$	$1, 1$	$0, 0$
	C	$1, 2$	$3, 1$	$2, 1$

  We can now turn to player 2, and detect that strategy L strictly dominates R, since it yields a strictly larger payoff than R, regardless of the strategy selected by player 1 (both when he chooses U in the top row, i.e., $1 > 0$ and when he chooses C in the bottom row, i.e., $2 > 1$), or more compactly

  $$u_2(s_1, L) \geq u_2(s_2, R) \text{ for all } s_1 \in \{U, C\}.$$

  Since R is strictly dominated for player 2, it is also weakly dominated. After deleting the column corresponding to the weakly dominated strategy R, we obtain the matrix below.

		Player 2	
		L	M
Player 1	U	$2,1$	$1,1$
	C	$1,2$	$3,1$

At this point, notice that we are not done examining player 2, since you can easily detect that M is weakly dominated by strategy L. Indeed, when player 1 selects U (in the top row of the above matrix), player 2 obtains the same payoff from L and M, but when player 1 chooses C (in the bottom row), player 2 is better off selecting L, which yields a payoff of 2, rather than M, which only produces a payoff of 1, i.e., $u_2(s_1, L) \geq u_2(s_1, M)$ for all $s_1 \in \{U, C\}$. Hence, we can delete M because of being weakly dominated for player 1, leaving us with the (further reduced) payoff matrix below.

		Player 2
		L
Player 1	U	$2,1$
	C	$1,2$

Now we can turn to player 1, and identify that U strictly dominates C (and thus it also weakly dominates C), since the payoffs that player 1 obtains from U, 2, is strictly larger than that from C, 1. Therefore, after deleting C, we are left with a single strategy profile, (U, L), as depicted in the matrix below. Hence, using this particular order in our iterative deletion of weakly dominated strategies (IDWDS) we obtain the unique equilibrium prediction (U, L).

		Player 2
		L
Player 1	U	$2,1$

- *Second Route*: Let us now consider the same initial 3×3 matrix, which we reproduce below, and check if the application of IDWDS, but using a different deletion order (i.e., different "route"), can lead to a different equilibrium result than that found above. i.e., (U, L).

		Player 2		
		L	M	R
	U	$2,1$	$1,1$	$0,0$
Player 1	C	$1,2$	$3,1$	$2,1$
	D	$2,-2$	$1,-1$	$-1,-1$

Unlike in our first route, let us now start identifying weakly dominated strategies for player 2. In particular, note that R is weakly dominated by M, since the former yields a weakly lower payoff than the latter (i.e., it provides a strictly higher payoff when player 1 chooses U in the top row, but the same payoff otherwise). That is,

$$u_2(s_1, M) \geq u_2(s_1, R) \text{ for all } s_1 \in \{U, C, D\}.$$

Once we have deleted R as being weakly dominated for player 2, the remaining matrix becomes that depicted below.

		Player 2	
		L	M
Player 1	U	$2, 1$	$1, 1$
	C	$1, 2$	$3, 1$
	D	$2, -2$	$1, -1$

Turning to player 1, note that we can no longer find weakly dominated strategies: U and D provide the same payoff under a given strategy of player 2, and such payoff is higher than that of C when player 2 selects L but lower when player 2 chooses M.

- Therefore, there are no more weakly dominated strategies for either player, and the equilibrium prediction after using IDWDS is the six remaining strategy profiles:

$$\{(U, L), (U, M), (C, L), (C, M), (D, L), (D, M)\}$$

This equilibrium prediction is, of course, different (and significantly less precise) than what found when we started the application of IDWDS from player 1. Hence, equilibrium outcomes that arise from applying IDWDS are sensitive to the order of deletion.

4. **Exercise 7.6 from JR** (see page 365 in new edition). Show that any strategy surviving iterative deletion of weakly dominated strategies (IDWDS) also survives iterative deletion of strictly dominated strategies (IDSDS).

- We start by reproducing the definition of IDWDS and IDSDS.
- *IDWDS.* Strategy profile $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_N)$ survives IDWDS if, at any round of deletion, player i's strategy $\hat{s}_i$ satisfies

$$u_i(\hat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$$

where $\hat{s}_i \neq s_i \in S_i$ and for all $s_{-i} \in S_{-i}$, and satisfies $u_i(\hat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for at least one strategy profile of player i's opponents s_{-i}.

- *IDSDS.* Strategy profile $s' = (s'_1, \ldots, s'_N)$ survives IDSDS if, at every round of deletion, player i's strategy s'_i satisfies

$$u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$$

where $\hat{s}_i \neq s_i \in S_i$ and for all $s_{-i} \in S_{-i}$.

- Operating by contradiction, assume that strategy $\hat{s}_i$ survives IDWDS but does not survive IDSDS. Therefore, there must be another strategy $\overline{s}_i \neq \hat{s}_i$ that survives IDSDS, yielding a strictly higher payoff than $\hat{s}_i$ in at least one round of deletion, that is,

$$u_i(\overline{s}_i, s_{-i}) > u_i(\hat{s}_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$, which contradicts the above definition of IDWDS. Then, if a strategy profile survives IDWDS, it must also survive IDSDS.

5. **Exercises from Tadelis:**

   (a) Chapter 4: Exercises 4.3, and 4.7.

   (b) Chapter 5: Exercises 5.1, and 5.9.

      - See answer key in the following pages.

p_i. If $p_i < p_j$, then firm i gets all of the market while no one demands the good of firm j. If the prices are the same then both firms equally split the market demand. Imagine that there are no costs to produce any quantity of the good. (These are two large dairy farms, and the product is manure.) Write down the normal form of this game.

Answer: The players are $N = \{1,2\}$ and the strategy sets are $S_i = [0,\infty]$ for $i \in \{1,2\}$ and firms choose prices $p_i \in S_i$. To calculate payoffs, we need to know what the quantities will be for each firm given prices (p_1, p_2). Given the assumption on ties, the quantities are given by,

$$q_i(p_i,p_j) = \begin{cases} 100 - p_i \text{ if } p_i < p_j \\ 0 \text{ if } p_i > p_j \\ \frac{100-p_i}{2} \text{ if } p_i = p_j \end{cases}$$

which in turn means that the payoff function is given by quantity times price (there are no costs):

$$v_i(p_i,p_j) = \begin{cases} (100 - p_i)p_i & \text{if } p_i < p_j \\ 0 & \text{if } p_i > p_j \\ \frac{100-p_i}{2}p_i & \text{if } p_i = p_j \end{cases}$$

■

3.7

7. **Public Good Contribution:** Three players live in a town and each can choose to contribute to fund a street lamp. The value of having the street lamp is 3 for each player and the value of not having one is 0. The Mayor asks each player to either contribute 1 or nothing. If at least two players contribute then the lamp will be erected. If one or less people contribute then the lamp will not be erected, in which case any person who contributed will not get their money back. Write down the normal form of this game.

Answer: The set of players is $N = \{1,2,3\}$ and each has an strategy set $S_i = \{0,1\}$ where 0 is not to contribute and 1 is to contribute. The payoffs of player i from a profile of strategies (s_1, s_2, s_3) is given by,

$$v_i(s_1, s_2, s_3) = \begin{cases} 0 & \text{if } s_i = 0 \text{ and } s_j = 0 \text{ for some } j \neq i \\ 3 & \text{if } s_i = 0 \text{ and } s_j = 1 \text{ for both } j \neq i \\ -1 & \text{if } s_i = 1 \text{ and } s_j = 0 \text{ for both } j \neq i \\ 2 & \text{if } s_i = 1 \text{ and } s_j = 1 \text{ for some } j \neq i \end{cases}$$

■

4
Rationality and Common Knowledge

4.1

1. Prove Proposition ??: If the game $\Gamma = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$ has a strictly dominant strategy equilibrium s^D, then s^D is the unique dominant strategy equilibrium.

   Answer: Assume not. That is, there is some other strategy profile $s^* \neq s^D$ that is also a strictly dominant strategy equilibrium. But this implies that for every i, $s_i^* > s_i^D$, which contradicts that s^D is a strictly dominant strategy equilibrium. ■

2. **Weak dominance.** We call the strategy profile $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$.

   (a) Provide an example of a game in which there is no weakly dominant strategy equilibrium.

   Answer:

Player 2

		H	T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

■

(b) Provide an example of a game in which there is more than one weakly dominant strategy equilibrium.

Answer: In the following game each player is indifferent between his strategies and so each one is weakly dominated by the other. This means that any outcome is a weakly dominant strategy equilibrium.

Player 1 \ Player 2	H	T
H	1, 1	1, 1
T	1, 1	1, 1

■

4.3

3. **Discrete first-price auction:** An item is up for auction. Player 1 values the item at 3 while player 2 values the item at 5. Each player can bid either 0, 1 or 2. If player i bids more than player j then i win's the good and pays his bid, while the loser does not pay. If both players bid the same amount then a coin is tossed to determine who the winner is, who gets the good and pays his bid while the loser pays nothing.

(a) Write down the game in matrix form.

Answer: We need to determine what the payoffs are if the bidders tie. The one who wins the coin toss bids his bid and the loser gets and pays nothing. Hence, we can just calculate the expected payoff as a 50:50 lottery between getting nothing and winning. For example, if both players bid 2 then player 1 gets $3-2=1$ unit of payoff with probability $\frac{1}{2}$ and player 2 gets $5-2=3$ units of payoff with probability $\frac{1}{2}$, so the pair of payoffs is $(\frac{1}{2}, \frac{3}{2})$.

		Player 2: 0	1	2
Player 1	0	$\frac{3}{2}, \frac{5}{2}$	0, 4	0, 3
	1	2, 0	1, 2	0, 3
	2	1, 0	1, 0	$\frac{1}{2}, \frac{3}{2}$

(b) Does any player have a strictly dominated strategy?

Answer: Yes - for player 2 bidding 0 is strictly dominated by bidding 2. ■

(c) Which strategies survive IESDS?

Answer: After removing the strategy 0 of player 2, player 1's strategy of 0 is dominated by 2, so we can remove that too. But then, in the remaining 2×2 game where both players can choose 1 or 2, bidding 1 is strictly dominated by bidding 2 for player 2, and after this round, bidding 1 is strictly dominated by bidding 2 for player 1. Hence, the unique strategy that survives IESDS is (2, 2) yielding expected payoffs of $(\frac{1}{2}, \frac{3}{2})$. ■

4. **eBay's recommendation:** It is hard to imagine that anyone is not familiar with eBay©, the most popular auction website by far. The way a typical eBay auction works is that a good is placed for sale, and each bidder places a "proxy bid", which eBay keeps in memory. If you enter a proxy bid that is lower than the current highest bid, then your bid is ignored. If, however, it is higher, then the current bid increases up to one increment (say, 1 cent) above the *second highest* proxy bid. For example, imagine that three people placed bids on a used laptop of \$55, \$98 and \$112. The current price will be at \$98.01, and if the auction ended the player who bid \$112 would win at a price of \$98.01. If you were to place a bid of \$103.45 then the who bid \$112 would still win, but at a price of \$103.46, while if your bid was \$123.12 then

campaign, call this a balanced campaign (or B), and finally, focus only on attacking one's opponent, call this a negative campaign (or N). All a candidate cares about is the probability of winning, so assume that if a candidate expects to win with probability $\pi \in [0,1]$, then his payoff is π. The probability that a candidate wins depends on his choice of campaign and his opponent's choice. The probabilities of winning are given as follows:

- – If both choose the same campaign, each wins with probability 0.5.
  - If candidate i uses a positive campaign while $j \neq i$ uses a balanced one, then i loses for sure.
  - If candidate i uses a positive campaign while $j \neq i$ uses a negative one, then i wins with probability 0.3.
  - If candidate i uses a negative campaign while $j \neq i$ uses a balanced one, then i wins with probability 0.6.

4.7

(a) Model this story as a normal form game. (It suffices to be specific about the payoff function of one player, and explaining how the other player's payoff function is different and why.)

Answer: There are two players $i \in \{1,2\}$, each has three strategies $S_i = \{P, B, N\}$ and the payoffs are $v_i(P,P) = v_i(B,B) = v_i(N,N) = 0.5$; $v_1(B,P) = v_2(P,B) = 1$; $v_2(B,P) = v_1(P,B) = 0$; $v_1(P,N) = v_2(N,P) = 0.3$; $v_2(P,N) = v_1(N,P) = 0.7$; $v_1(N,B) = v_2(B,N) = 0.6$; and $v_2(N,B) = v_1(B,N) = 0.4$. ■

(b) Write the game in matrix form.

Answer:

		Player 2		
		P	B	N
Player 1	P	$0.5, 0.5$	$0, 1$	$0.3, 0.7$
	B	$1, 0$	$0.5, 0.5$	$0.4, 0.6$
	N	$0.7, 0.3$	$0.6, 0.4$	$0.5, 0.5$

■

(c) What happens at each stage of elimination of strictly dominated strategies? Will this procedure lead to a clear prediction?

Answer: Notice that for each player B strictly dominates P. In the remaining 2×2 game without the strategies P, N strictly dominates B for each player. Hence, the unique clear prediction is that both candidates will engage in negative campaigns. ■

8. Consider the p-Beauty contest presented in section 4.3.5.

(a) Show that if player i believes that everyone else is choosing 20 then 19 is not the only best response for any number of players n.

Answer: If everyone else is choosing 20 and if player i chooses 19 then $\frac{3}{4}$ of the average will be somewhere below 15, and 19 is closer to that number, and therefore is a best response. But the same argument holds for any choice of player i that is between 15 and and 20 regardless of the number of players. (In fact, you should be able to convince yourself that this will be true for any choice of i between 10 and 20.) ■

(b) Show that the set of best response strategies to everyone else choosing the number 20 depends on the number of players n.

Answer: Imagine that $n = 2$. If one player j is choosing 20, then any number s_i between 0 and 19 will beat 20. This follows because the target number ($\frac{3}{4}$ of the average) is equal to $\frac{3}{4} \times \frac{20+s_i}{2} = \frac{15}{2} + \frac{3}{8}s_i$, the distance between 20 and the target number is $\frac{25}{2} - \frac{3}{8}s_i$ (this will always be positive because the target number is less than 20) while the distance between s_i and the target number is $\left|\frac{5}{8}s_i - \frac{15}{2}\right|$. The latter will be smaller than the former if and only if $\left|\frac{5}{8}s_i - \frac{15}{2}\right| < \frac{25}{2} - \frac{3}{8}s_i$, or $-20 < s_i < 20$. Given the constraints on the choices, $BR_i \in \{0, 1, ...19\}$. Now imagine that $n = 5$. The target number is equal to $\frac{3}{4} \times \frac{80+s_i}{5} = 12 + \frac{3}{20}s_i$, the distance between 20 and the target number is $8 - \frac{3}{20}s_i$ while the distance between s_i and the target number is $\left|\frac{17}{20}s_i - 12\right|$. The latter will be smaller than

5

Pinning Down Beliefs: Nash Equilibrium

5.1

1. Prove Proposition ??.

   Answer: (1) Assume that s^* is a strict dominant strategy equilibrium. This implies that for any player i, s_i^* is a best response to any choice of his opponents including s_{-i}^*, which in turn implies that s^* is a Nash equilibrium.
   (2) Assume that s^* is the unique survivor of IESDS. By construction of the IESDS procedure, there is no round in which s_i^* is strictly dominated against the surviving strategies of i's opponents, an in particular, against s_{-i}^*, implying that s_i^* is a best response to s_{-i}^*, which in turn implies that s^* is a Nash equilibrium.
   (3) Assume that s^* is the unique Rationalizable strategy profile. By construction of the Rationalizability procedure, any strategy of player i that survives a round of rationalizability can be a best response to some strategy of i's opponents that survives that round. Hence, by definition, s_i^* is a best response to s_{-i}^*, which in turn implies that s^* is a Nash equilibrium. ■

2. A strategy $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i^W, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$. Provide an example of a game

for which there is a weakly dominant strategy equilibrium, as well as another Nash equilibrium.

Answer: Consider the following game:

		Player 2	
		L	R
Player 1	U	$1,1$	$1,1$
	D	$1,1$	$2,2$

In this game, (D, R) is a weakly dominant strategy equilibrium (and of course, a Nash equilibrium), yet (U, L) is a Nash equilibrium that is not a weakly dominant strategy equilibrium. ■

5.3

3. Consider a 2 player game with m pure strategies for each player that can be represented by a $m \times m$ matrix. Assume that for each player no two payoffs in the matrix are the same.

   (a) Show that if $m = 2$ and the game has a unique pure strategy Nash equilibrium then this is the unique strategy profile that survives IESDS.

   Answer: Consider a general 2×2 game as follows,

		Player 2	
		s_{2a}	s_{2b}
Player 1	s_{1a}	v_1^{aa}, v_2^{aa}	v_1^{ab}, v_2^{ab}
	s_{1b}	v_1^{ba}, v_2^{ba}	v_1^{bb}, v_2^{bb}

   and assume without loss of generality that (s_{1a}, s_{2a}) is the unique pure strategy Nash equilibrium.[1] Two statements are true: first, because (s_{1a}, s_{2a}) is a Nash equilibrium and no two payoffs are the same for each player then $v_1^{aa} > v_1^{ba}$ and $v_2^{aa} > v_2^{ab}$. Second, because (s_{1b}, s_{2b}) is *not* a Nash equilibrium then $v_1^{bb} > v_1^{ab}$ and $v_2^{bb} > v_2^{ba}$ cannot hold together (otherwise it would have been another Nash equilibrium). These

[1] The term "without loss of generality" means that we are choosing one particular strategy profile but there is nothing special about it and we could have chosen any one of the others using the same argument.

—P

two statements imply that either (i) $v_1^{aa} > v_1^{ba}$ *and* $v_1^{ab} > v_1^{bb}\}$ in which case s_{1b} is dominated by s_{1a}, or (ii) $v_2^{aa} > v_2^{ab}$ *and* $v_2^{ba} > v_2^{bb}$ in which case s_{2b} is strictly dominated by s_{2a}. This implies that either s_{1b} or s_{2b} (or both) will be eliminated in the first round of IESDS, and from the fact that $v_1^{aa} > v_1^{ba}$ and $v_2^{aa} > v_2^{ab}$ it follows that if only one of the strategies was removed in the first round of IESDS then the remaining one will be removed in the second and final round, leaving (s_{1a}, s_{2a}) as the unique strategy that survives IESDS. ■

(b) Show that if $m = 3$ and the game has a unique pure strategy equilibrium then it may not be the only strategy profile that survives IESDS.

Answer: Consider this following game:

		Player 2		
		L	M	R
	U	7, 6	3, 0	6, 5
Player 1	C	1, 3	4, 4	0, 2
	D	8, 7	2, 1	5, 8

Notice that for both players none of the strategies are strictly dominated implying that IESDS does not restrict any strategy profile survives IESDS. However, this game has a unique Nash equilibrium: (C, M). ■

4. **Splitting Pizza:** You and a friend are in an Italian restaurant, and the owner offers both of you an 8-slice pizza for free under the following condition. Each of you must simultaneously announce how many slices you would like; that is, each player $i \in \{1, 2\}$ names his desired amount of pizza, $0 \leq s_i \leq 8$. If $s_1 + s_2 \leq 8$ then the players get their demands (and the owner eats any leftover slices). If $s_1 + s_2 > 8$, then the players get nothing. Assume that you each care only about how much pizza you individually consume, and the more the better.

(a) Write out or graph each player's best-response correspondence.

total hours spent (by everyone) cleaning, minus a number c times the hours spent (individually) cleaning. That is,

$$v_i(s_1, s_2, \ldots, s_n) = -c \cdot s_i + \sum_{j=1}^{n} s_j$$

Assume everyone chooses simultaneously how much time to spend cleaning.

5.9

(a) Find the Nash equilibrium if $c < 1$.

Answer: The payoff function is linear in one's own time spent s_i and in the time spent by the other roommates s_j, and we can rewrite the payoff function as

$$v_i(s_i, s_{-i}) = s_i - cs_i + \sum_{j \neq i} s_j \ .$$

Considering this payoff function, if $c < 1$ then every additional amount ε of time that i spends cleaning gives him an extra payoff of $(1-c)\varepsilon > 0$ so that each player i would choose to spend all the 5 hours cleaning. Note that using a first-order condition would not work here because taking the derivative of $v_i(s_i, s_{-i})$ with respect to s_i will just yield $1 - c = 0$ which is not true for $c < 1$. This implies that there is a "corner" solution in the range $s_i \in [0, 5]$, in this case the Nash equilibrium is at the corner $s_i^* = 5$ for all $i = 1, 2, ..., n$. ■

(b) Find the Nash equilibrium if $c > 1$.

Answer: Similarly to (a) above, every additional amount ε of time that i spends cleaning gives him an extra payoff of $(1 - c)\varepsilon < 0$, so that each player i would choose to spend no time cleaning and the Nash equilibrium is $s_i^* = 0$ for all $i = 1, 2, ..., n$. ■

(c) Set $n = 5$ and $c = 2$. Is the Nash equilibrium Pareto efficient? If not, can you find an outcome where everyone is better off than at the Nash equilibrium outcome?

Answer: Following the analysis in part (b), the unique Nash equilibrium is where everyone chooses to spend no time cleaning and everyone's payoff is equal to zero. Consider the case where everyone is somehow forced to choose $s_i = 1$. Each player's payoff will be

$$\begin{aligned} v_i(s_i, s_{-i}) &= s_i - cs_i + \sum_{j \neq i} s_j \\ &= 1 - 2 \times 1 + 4 \times 1 = 3 > 0 . \end{aligned}$$

so that all the players will be better off if they all chose $s_i = 1$. In fact, each amount of time $\varepsilon > 0$ that player i chooses to clean cause him a personal loss of $\varepsilon - 2\varepsilon = \varepsilon$, but increases the payoff of each of the other players by ε. Hence, if we can get each player to increase his time cleaning by ε, this yields an increase of value for each player that equals his own loss, but the former is multiplied by the number of players. Hence, the best symmetric outcome is when each player chooses $s_i = 5$. ■

10. **Synergies:** Two division managers can invest time and effort in creating a better working relationship. Each invests $e_i \geq 0$, and if both invest more then both are better off, but it is costly for each manager to invest. In particular, the payoff function for player i from effort levels (e_i, e_j) is $v_i(e_i, e_j) = (a + e_j)e_i - e_i^2$.

   (a) What is the best response correspondence of each player?

   Answer: If player i believes that player j chooses e_j then i's first order optimality condition for maximizing his payoff is,

$$a + e_j - 2e_i = 0 ,$$

   yielding the best response function,

$$BR_i(e_j) = \frac{a + e_j}{2} \text{ for all } e_j \geq 0.$$

   ■