EconS 424 - Games with Incomplete Information I

Félix Muñoz-García

Washington State University

fmunoz@wsu.edu

April 28, 2014

Harrington, Ch. 10 Exercise 6

- Two U.S. senators are considering entering the race for the Democratic nomination for U.S. president. Each candidate has a privately known personal cost to entering the race. Assume that the probability of having a low entry cost, f_L, is p and the probability of having a high entry cost, f_H, is $1 - p$. Thus, the type space has just two values.

Harrington, Ch. 10 Exercise 6

- A candidate's payoff depends on whether he enters the race and whether the other senator enters as well. Let v_2 be a candidate's payoff when he enters and the other senator does as well (so that there are two candidates), v_1 be a candidate's payoff when he enters and the other senator does not (so there is one candidate), and 0 be the payoff when he does not enter. Assume that

$$v_1 > v_2 > 0$$
$$f_H > f_L > 0$$
$$v_2 - f_L > 0 > v_2 - f_H$$
$$v_1 - f_H > 0$$

Harrington, Ch. 10 Exercise 6

- Derive the conditions whereby it is a symmetric Bayes-Nash equilibrium for a candidate to enter only when she has a low personal cost from doing so.

- It is a symmetric Bayes-Nash equilibrium for a senator to enter only when she has low personal cost if and only if

$$\text{Low Type: } p(v_2 - f_L) + (1-p)(v_1 - f_L) \geq 0$$

where the first term, $p(v_2 - f_L)$, indicates this senator's payoff when the other senator is also low type (and thus he enters the race), which occurs with probability p. In which case, his gross payoff is v_2 since the other senator also entered the race.

- In contrast, when the other senator is high type (which occurs with probability $1 - p$), he does not enter the race leaving a gross payoff for the low type senator we analyze of v_1. Rearranging, we have:

$$pv_2 + (1 - p)v_1 \geq f_L \tag{1}$$

Harrington, Ch. 10 Exercise 6

- And for the senator with a high personal cost (the high type), we have:

$$\text{High type: } 0 \geq p(v_2 - f_H) + (1-p)(v_1 - f_H)$$

- Similarly as our previous analysis, when the other senator is low type (with probability p) he enters, having a gross payoff of v_2 for the high type senator we analyze, whereas when the other senator is high type (with probability $1-p$) he doesn't enter, leaving a gross payoff of v_1 for the high type senator we analyze.

- Rearranging we obtain:

$$f_H \geq pv_2 + (1-p)v_1 \tag{2}$$

Combining (1) and (2) yields:

$$f_H \geq pv_2 + (1-p)v_1 \geq f_L$$

This condition holds when p is close to 1.

Harrington, Ch. 10 Exercise 6

- Derive the conditions whereby it is a symmetric Bayes-Nash equilibrium for a candidate to enter for sure when she has a low personal cost and to enter with some probability between 0 and 1 when she has a high personal cost.

Harrington, Ch. 10 Exercise 6

- Consider a symmetric strategy profile such that (1) if the candidate has low cost, she enters and (2) if she has high cost, she enters with probability q. The equilibrium conditions are

$$\text{Low type: } [p + (1-p)q](v_2 - f_L) + (1-p)(1-q)(v_1 - f_L) \geq 0$$

Harrington, Ch. 10 Exercise 6

- Notice that the low-type senator is not alone in the race (obtaining $v_2 - f_L$) if either:
  1. The other senator is also low type (which occurs with probability p) since that type of senator always enters; or when the other senator is high type (with probability $1 - p$) and, in addition, chooses to enter (which he does with probability q).
  2. Similarly the low type is alone in the race (obtaining $v_2 - f_L$) if the other senator is high type and, in addition, he chooses not to enter the race; which occurs with probability $(1 - p)(1 - q)$.

Harrington, Ch. 10 Exercise 6

- Rearranging the above expression, we obtain:

$$\implies [p + (1 - p)q]v_2 + (1 - p)(1 - q)v_1 \geq f_L \qquad (3)$$

Harrington, Ch. 10 Exercise 6

- And for the senator with a high personal cost (the "high type"), we have High type:

$$0 = [p + (1-p)q](v_2 - f_H) + (1-p)(1-q)(v_1 - f_H)$$

- A similar intuition as above applies, however, notice that the high type senator must be indifferent between entering and not entering the race, since otherwise he would not be randomizing. Solving for probability q, we obtain:

$$\Longrightarrow \quad q = \frac{pv_2 + (1-p)v_1 - f_H}{(1-p)(v_1 - v_2)}$$

$$\Longleftrightarrow \quad q = \frac{(1-p)(v_1 - v_2) + v_2 - f_H}{(1-p)(v_1 - v_2)} \qquad (4)$$

Harrington, Ch. 10 Exercise 6

- For this to be an equilibrium, the derived value for q must lie between 0 and 1:

$$0 < \frac{(1-p)(v_1 - v_2) + v_2 - f_H}{(1-p)(v_1 - v_2)} < 1$$

$$\implies (1-p)(v_1 - v_2) + v_2 - f_H > 0 \implies pv_2 + (1-p)v_1 > f_H \quad (5)$$

- Let us now check if the condition on f_L that we found in (3) holds given our initial assumptions.

Harrington, Ch. 10 Exercise 6

- From (3) we have:

$$[p + (1-p)q]v_2 + (1-p)(1-q)v_1 \geq f_L$$

Rearranging,

$$pv_2 + v_1 - pv_1 + q[(1-p)v_2 - (1-p)v_1] \geq f_L$$

$$pv_2 + v_1 - pv_1 - q(1-p)(v_1 - v_2) \geq f_L$$

Harrington, Ch. 10 Exercise 6

- Plugging in q from (4):

$$pv_2 + v_1 - pv_1 - \left(\frac{(1-p)(v_1 - v_2) + v_2 - f_H}{(1-p)(v_1 - v_2)} \right) (1-p)(v_1 - v_2) \geq f_L$$

Rearranging,

$$pv_2 + v_1 - pv_1 - ((1-p)(v_1 - v_2) + v_2 - f_H) \geq f_L$$

$$(pv_2 - v_2) + (v_1 - pv_1) - (1-p)v_1 + (1-p)v_2 + f_H \geq f_L$$

$$-(1-p)v_2 + (1-p)v_1 - (1-p)v_1 + (1-p)v_2 + f_H \geq f_L$$

Harrington, Ch. 10 Exercise 6

- Cancelling out $(1-p)v_1$ and $(1-p)v_2$, we obtain:

$$f_H \geq f_L$$

which always holds given the initial conditions. This means (4) implies (3), so the only condition we need is (4).

  - If $[pv_2 + (1-p)v_1] > f_H$, (the condition for probability q to be positive), then there is an equilibrium in which a low-cost senator enters and a high-cost one randomizes.
  - If in contrast, $f_H \geq pv_2 + (1-p)v_1 \geq f_L$, then, as we know from part (a), there is an equilibrium in which a low-cost senator enters and a high-cost one does not.

Harrington, Ch. 10 Exercise 6

- Find some other Bayes-Nash equilibrium distinct from those described in (a) and (b).

- There is also an asymmetric equilibrium in which Senator 1 always enters and Senator 2 enters only when he has low cost. Given Senator 2' sstrategy, it is always optimal for Senator 1 to enter, regardless of her type, if and only if:

$$\begin{aligned} \text{Low type:} & \quad p(v_2 - f_L) + (1-p)(v_1 - f_L) \geq 0 \\ \Longrightarrow & \quad pv_2 + (1-p)v_1 \geq f_L \end{aligned}$$

$$\begin{aligned} \text{High type:} & \quad p(v_2 - f_H) + (1-p)(v_1 - f_H) \geq 0 \\ \Longrightarrow & \quad pv_2 + (1-p)v_1 \geq f_H \end{aligned}$$

which can be reduced to:

$$[pv_2 + (1-p)v_1] \geq f_H > f_L$$

Harrington, Ch. 10 Exercise 6

- Given Senator 1's strategy, it is optimal for Senator 2 to enter only when he has low personal cost if and only if

$$\text{Low type: } v_2 - f_L \geq 0$$

$$\text{High type: } v_2 - f_H \leq 0$$

where he is always accompanied by Senator 1 (no probabilities are involved in these expressions) given that Senator 1 enters regardless of her type.

- The last two inequalities obviously hold given our initial conditions. Hence, this equilibrium exists if:

$$pv_2 + (1 - p)v_1 \geq f_H$$

Harrington, Ch. 10 Exercise 7

- Assume that two countries are on the verge of war and are simultaneously deciding whether or not to attack. A country's military resources are its type, and their relevance is summarized in a parameter which influences the likelihood that they would win a war. Suppose the type space is made up of two values: p' and p'', where $0 < p' < p'' < 1$. A country is type p'' with probability q and type p' with probability $1 - q$.

Harrington, Ch. 10 Exercise 7

- Consider a country of type p (which equals p' or p''). If it chooses to attack and it attacks first, then it believes it will win the war with probability xp, where x takes a value such that $p < xp < 1$. If the two countries both attack, then the probability that a type p country wins is p. If a type p country does not attack and the other country does attack, then the probability of victory for the type p country is yp, where y takes a value such that $0 < yp < p$.

- Finally, if neither country attacks, then there is no war. A country is then more likely to win the war the higher its type is and if it attacks before the other country.
  A country's payoff when there is no war is 0, from winning a war is W, and from losing a war is L. Assume that $W > 0 > L$.

Harrington, Ch. 10 Exercise 7

- Derive the conditions for it to be a symmetric Bayes-Nash equilibrium for a country to attack regardless of its type.

Harrington, Ch. 10 Exercise 7

- If one country expects the other to attack for sure (that is, both when its type p' and p''), then it is always optimal to attack. Doing so results in an expected payoff of $pW + (1 - p)L$, where p is the probability of wining the war when both countries attack, and not doing so results in $pyW + (1 - py)L$, where py is the probability of wining the war when the country does not attack but its rival attacks.

- Comparing these expected payoffs, we obtain:

$$pW + (1 - p)L > pyW + (1 - py)L$$

since rearranging we have:

$$W(p - py) > L(p - py)$$

then $W > L$, which holds given our initial assumptions. Therefore, attacking regardless of one's type is a symmetric Bayesian-Nash equilibrium (BNE).

- Derive the conditions for it to be a symmetric Bayes-Nash equilibrium for a country to attack only if its type is p''.

Harrington, Ch. 10 Exercise 7

- Consider a symmetric strategy profile in which a country attacks only if its type is p''. The equilibrium conditions (for each country) are

$$\text{Type } p'' : q[p''W + (1-p'')L] + (1-q)[xp''W + (1-xp'')L]$$

$$\geq q[yp''W + (1-yp'')L] + (1-q)0$$

- Let's start analyzing the *left hand side* of the inequality:
    - The first term represents the expected payoff that a type p'' country obtains when facing a country which is also type p'' (which occurs with probability q). In such case, this equilibrium prescribes that both countries attack (since they are both p''), and the country we analyze wins the war with probability p'' (and hence loses with probability $1 - p''$).
    - The second term represents the expected payoff that a type p'' country obtains when facing a country which is, instead, type p' (which occurs with probability $1 - q$). In this case his rival does not attack, increasing the probability of wining the war for the country we analyze to xp'', where $xp'' > p''$.

- Let us now examine the *right hand side* of the inequality, which illustrates the expected payoff that country p'' obtains when it deviates from his equilibrium strategy, i.e., it does not attack.
  - The first term represents the expected payoff that a type p'' country obtain when facing a country which is also type p'' (which occurs with probability q). In such case, country p'' does not attack while country p'' attacks, lowering the chances that country p'' wins the war to yp'', where $yp'' < p''$.
  - If in contrast, the type of his rival is p', then no country attacks and his payoff is zero.

Harrington, Ch. 10 Exercise 7

- Rearranging, we obtain:

$$q[p''W + (1-p'')L] + (1-q)[xp''W + (1-xp'')L]$$
$$\geq q[yp''W + (1-yp'')L] + (1-q)0$$

$$qp''W + q(1-p'')L + (1-q)xp''W + (1-q)(1-xp'')L$$
$$\geq qyp''W + q(1-yp'')L$$

$$qp''W + (1-q)xp''W - qyp''W$$
$$\geq q(1-yp'')L - q(1-p'')L - (1-q)(1-xp'')L$$

$$W[qp'' + (1-q)xp'' - qyp'']$$
$$\geq L[q(1-yp'') - q(1-p'') - (1-q)(1-xp'')]$$

Harrington, Ch. 10 Exercise 7

$$W[qp'' + xp'' - qxp'' - qyp''] \geq L[qp'' + xp'' - qxp'' - qyp'' - (1-q)]$$

$$W \geq L\left(1 - \frac{(1-q)}{p''[q(1-y) + (1-q)x]}\right)$$

- Since $(1-q) > 0$ and $p''[q(1-y) + (1-q)x] > 0$, thus

$$\left(1 - \frac{(1-q)}{p''[q(1-y) + (1-q)x]}\right) < 1$$

and this condition holds since by the initial condition we know that $W > L$.

Harrington, Ch. 10 Exercise 7

- Let us now examine the country of type p'. Recall that, according to the equilibrium we analyze, this country prefers to not attack, that is

$$\text{Type } p' : q[yp'W + (1 - yp')L] + (1 - q)0$$

$$\geq q[p'W + (1 - p')L] + (1 - q)[xp'W + (1 - xp')L]$$

An opposite intuition as above can be constructed for this inequality.

Harrington, Ch. 10 Exercise 7

- In this case rearranging we obtain:

$$q[yp'W + (1 - yp')L] + (1 - q)0$$
$$\geq q[p'W + (1 - p')L] + (1 - q)[xp'W + (1 - xp')L]$$

$$qyp'W + q(1 - yp')L$$
$$\geq qp'W + q(1 - p')L + (1 - q)xp'W + (1 - q)(1 - xp')L$$

$$qyp'W - (1 - q)xp'W - qp'W$$
$$\geq q(1 - p')L + (1 - q)(1 - xp')L - q(1 - yp')L$$

$$W[qyp' - (1 - q)xp' - qp']$$
$$\geq L[q(1 - p') + (1 - q)(1 - xp') - q(1 - yp')]$$

Harrington, Ch. 10 Exercise 7

$$W[qyp' - xp' + qxp' - qp'] \geq L[qyp' - xp' + qxp' - qp' + (1-q)]$$
$$W \geq L\left(1 - \frac{(1-q)}{p'[qy - x + qx - q]}\right)$$

- Since $(1-q) > 0$ and $p'[qy - x + qx - q] > 0$, thus

$$\left(1 - \frac{(1-q)}{p'[qy - x + qx - q]}\right) < 1$$

and this condition is also implied by the initial condition $W > L$.

Harrington, Ch. 10 Exercise 7

- The first condition assures that a country prefers to attack when it is type p''. The second condition assures that it prefers not to attack when it is type p'.