EconS 424 - Backward Induction and Subgame Perfection

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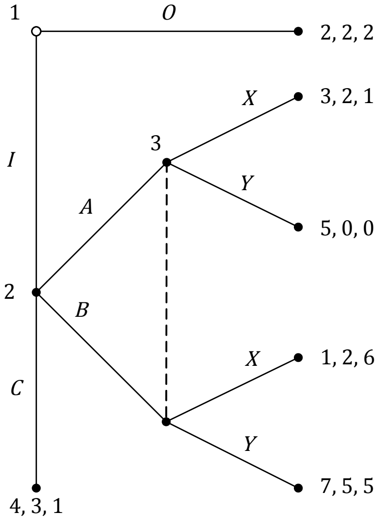
Watson, Ch. 15 # 1

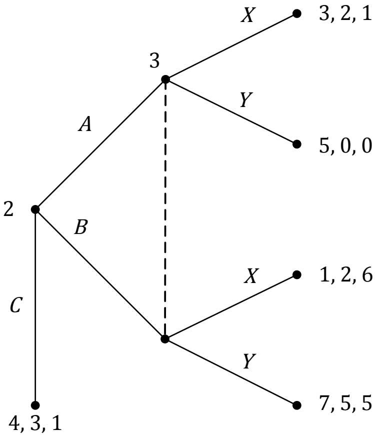
- Consider the extensive-form game on the next slide. Solve the game using backward induction.

Watson, Ch. 15 # 1



Watson, Ch. 15 # 1

- Starting from the terminal nodes, the smallest proper subgame we can identify is depicted below:



Watson, Ch. 15 # 1

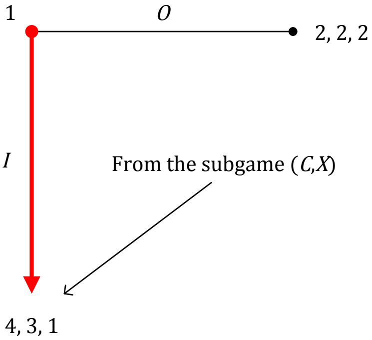
- In this subgame, player 3 chooses his action without observing player 2's choice. In order to find the NE of this subgame, we must represent it in its normal (matrix) form.

		Player 3: X	Player 3: Y
Player 2	A	$3, 2, \underline{1}$	$5, 0, 0$
	B	$1, 2, \underline{6}$	$7, \underline{5}, 5$
	C	$4, \underline{3}, \underline{1}$	$4, 3, \underline{1}$

- Note that player 1's payoffs are only included for completeness and have no bearing on the decisions made by players 2 and 3.

Watson, Ch. 15 # 1

- Hence, the NE of this subgame predicts that players 2 and 3 choose strategy profile (C, X). We can now plug the payoff triple resulting from the NE of this subgame, $(4, 3, 1)$, at the end of the branch indicating that player 1 chooses action I, as follows.



- Then the SPNE is (I, C, X).

Watson, Ch. 15 # 5

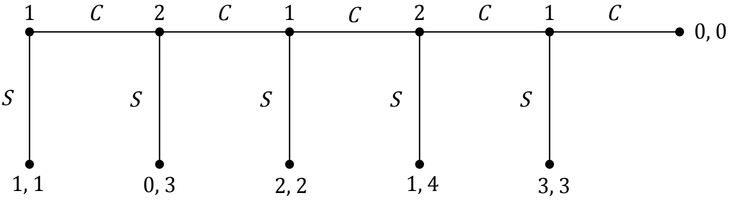
- In the Envelope Game, there are two players and two envelopes. One of the envelopes is marked "player 1," and the other is marked "player 2." At the beginning of the game, each envelope contains one dollar.
- Player 1 is given the choice between stopping the game and continuing. If he chooses to stop, then each player receives the money in his own envelope and the game ends. If player 1 chooses to continue, then a dollar is removed from his envelope and two dollars are added to player 2's envelope.
- Player 2 then gets to make the same choices with the same outcomes.

Watson, Ch. 15 # 5

- Play continues like this, alternating between the players, until either one of them decides to stop or k rounds of play have elapsed. If neither player chooses to stop by the end of the kth round, then both players obtain zero. Assume players want to maximize the amount of money they earn.
- Draw this game's extensive-form tree for $k = 5$.

Watson, Ch. 15 # 5

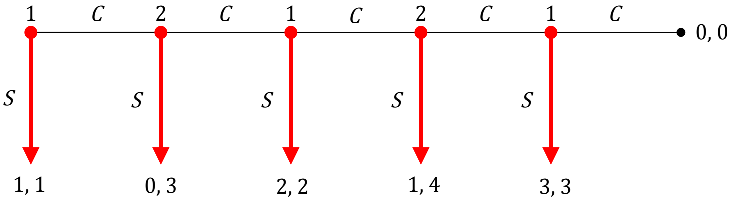
- This game is similar to all of the centipede games we have done in the past and follows the same form.



Watson, Ch. 15 # 5

- Use backward induction to find the subgame perfect equilibrium.

- Working backward, it is easy to see that in round 5, player 1 will choose S $(3 > 0)$. Thus, in round 4, player 2 will choose S $(4 > 3$ and $4 > 0)$. Continuing in this fashion, we find that, in any equilibrium, each player will choose S and time he is able to move.



Watson, Ch. 15 # 5

- Describe the backward induction outcome of this game for any finite integer k.

Watson, Ch. 15 # 5

- For any finite k, the backward induction outcome is that player 1 chooses S in the first round and each player receives one dollar. This is because if neither player chooses to stop by the end of the kth round, then both players obtain zero $(1 > 0)$.

Watson, Ch. 15 # 8

- Imagine a game in which players 1 and 2 simultaneously and independently select A or B. If they both select A, then the game ends and the payoff vector is $(5, 5)$. If they both select B, then the game ends with the payoff vector $(-1, -1)$.
- If one of the players chooses A while the other selects B, then the game continues and the players are required to simultaneously and independently sleect positive numbers. After these decisions, the game ends and each player receives the payoff

$$\frac{x_1 + x_2}{1 + x_1 + x_2},$$

where x_1 is the positive number chosen by player 1 and x_2 is the positive number chosen by player 2.

Watson, Ch. 15 # 8

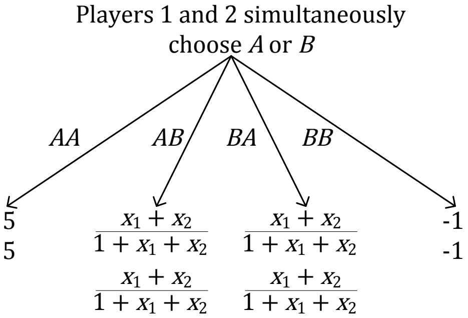
- Describe the strategy spaces of the players.

Watson, Ch. 15 # 8

- Each player has to choose an initial move (A or B), and potentially a positive number. We can thus describe each of their strategy spaces as
$$S_i = \{A, B\} \times (0, \infty) \times (0, \infty)$$
- Why is the positive interval included in there *twice*? Because the outcomes AB and BA are not considered the same, so each player has to choose a positive number for each *possible* outcome!

Watson, Ch. 15 # 8

- We can depict the outcomes of this game similar to that of an extensive form. Note, however, that this is *not* an extensive form representation of this game, rather just a simple visualization tool.



Watson, Ch. 15 # 8

- Compute the NE of this game.

- It is easy to see that when one of the players chooses A and the other selects B, then

$$0 < \frac{\overbrace{x_1 + x_2}^{\text{Both } x_1 \text{ and } x_2 \text{ are positive}}}{\underbrace{1 + x_1 + x_2}_{\substack{\text{Donominator is always larger} \\ \text{than the numerator}}}} < 1$$

and that

$$\frac{x_1 + x_2}{1 + x_1 + x_2} \rightarrow 1 \text{ as } (x_1 + x_2) \rightarrow \infty$$

Watson, Ch. 15 # 8

- Thus, each has a higher payoff when both choose A. Further, B $(-1,-1)$ will never be selected in equilibrium. The Nash Equilibria of this game are given by (Ax_1, Ax_2) where x_1 and x_2 are any positive numbers.

- Determine the subgame perfect equilibria

Watson, Ch. 15 # 8

- If the game proceeds through AB or BA, every player i maximizes his payoff $\frac{x_1+x_2}{1+x_1+x_2}$ by optimally selecting x_i.

$$\max_{x_i} \frac{x_1+x_2}{1+x_1+x_2}$$

- Taking FOCs with respect to x_i,

$$\frac{1}{(1+x_1+x_2)^2} = 0$$

- Unfortunately, we can't use this to develop best response function for the players since we have a corner solution.
  - **Intuition**: Both players are going to want to select the highest value of x_i possible in order to maximize their payoffs. We can then assume that $x_i \to \infty$.

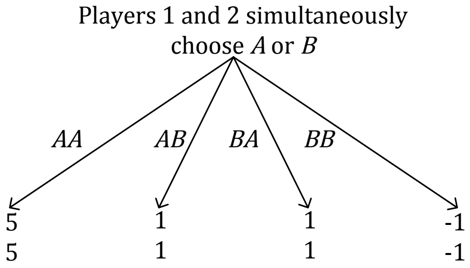
Watson, Ch. 15 # 8

- Taking the limit of our payoff function, we find

$$\lim_{x_i \to \infty} \frac{x_1 + x_2}{1 + x_1 + x_2} = 1$$

- Implying that both players will recieve a payoff of $(1, 1)$ at either of those nodes after selecting $x_1 = x_2 = \infty$.

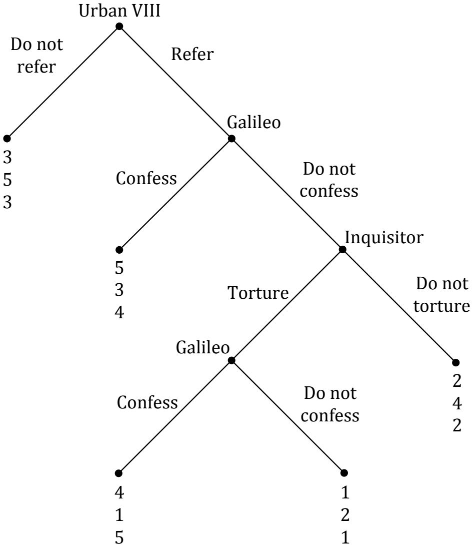
- Substituting these payoffs into our above figure



- It is clear that the SPNE of this game is where both players select A in the first round, and ∞ if they reach the second round, or $(A\infty, A\infty)$.

Harrington, Ch. 8 # 1

- Consider the game "Galileo and the Inquisition" on the next slide. Find all Nash equilibria.


Urban VIII
Do not refer
Refer
3
5
3
Galileo
Confess
Do not confess
5
3
4
Inquisitor
Torture
Do not torture
Galileo
Confess
Do not confess
2
4
2
4
1
5
1
2
1

Harrington, Ch. 8 # 1

- The strategic form games are shown below:

Galileo

Urban VIII	C/C	C/DNC	DNC/C	DNC/DNC
DNR	$3, \underline{5}, \underline{3}$	$3, \underline{5}, \underline{3}$	$3, \underline{5}, \underline{3}$	$\underline{3}, \underline{5}, \underline{3}$
R	$\underline{5}, \underline{3}, \underline{4}$	$\underline{5}, \underline{3}, \underline{4}$	$\underline{4}, 1, \underline{5}$	$1, 2, 1$

Inquisitor: T

Galileo

Urban VIII	C/C	C/DNC	DNC/C	DNC/DNC
DNR	$3, \underline{5}, \underline{3}$	$3, \underline{5}, \underline{3}$	$\underline{3}, \underline{5}, \underline{3}$	$\underline{3}, \underline{5}, \underline{3}$
R	$\underline{5}, 3, \underline{4}$	$\underline{5}, 3, \underline{4}$	$2, \underline{4}, 2$	$2, \underline{4}, \underline{2}$

Inquisitor: DNT

- There are five psNE for this game: $(DNR, DNC/DNC, T)$, $(R, C/C, T)$, $(R, C/DNC, T)$, $(DNR, DNC/C, DNT)$, and $(DNR/DNC/DNC/DNT)$.
    - Note: Do all of these equilibria make sense? Look at the third one: Galileo confesses before torture, but does not confess after. While this would end the game early, this is actually the opposite result we would expect.

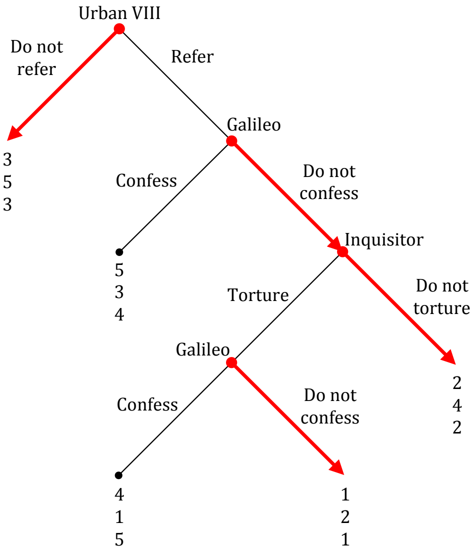
Harrington, Ch. 8 # 1

- Find all of the subgame perfect Nash equilibria.

Harrington, Ch. 8 # 1

- In his last decision node (which is associated with the path Refer, Do not confess, Torture), Galileo chooses Do not confess ($2 > 1$).
- Given this choice, the Inquisitor chooses Do not torture ($2 > 1$).
- At his first decision node (associated with Urban VIII having chosen Refer), Galileo chooses Do not confess ($4 > 3$).
- Finally, Urban VIII chooses Do not refer ($3 > 2$).
- Hence, the unique subgame perfect Nash equilibrium is $(DNR, DNC/DNC, DNT)$.

Harrington, Ch. 8 # 1



Harrington, Ch. 8 # 1

- For each Nash equilibrium that is not a subgame perfect Nash equilibrium, explain why it is not a subgame perfect Nash equilibrium.

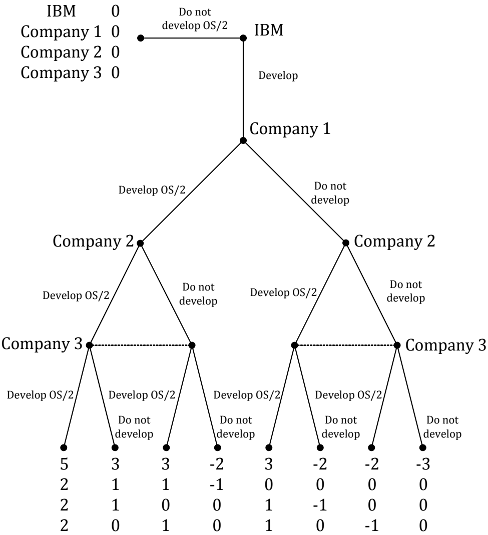
Harrington, Ch. 8 # 1

- There are four Nash equilibria that are no subgame perfect Nash equilibriua.
- In Nash equilibria $(DNR, DNC/DNC.T)$ and $(R, C/DNC/T)$, the Inquisitor is making a nonoptimal decision by choosing to torture Galileo given Galileo plays Do not confess in his last decision node.
- In Nash equilibria $(R, C/C, T)$ and $(DNR, DNC/C, DNT)$, Galileo is making a nonoptimal decision at his last decision node. He should play Do not confess instead.

Harrington, Ch. 9 # 2

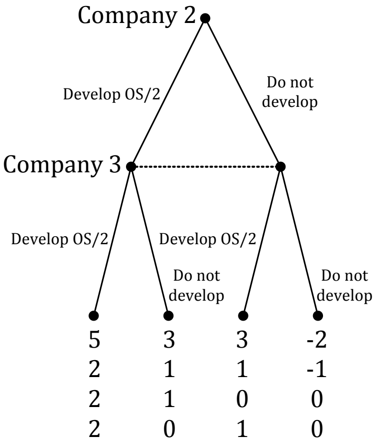
- Consider the Revised OS/2 game on the next slide. Derive all subgame perfect Nash Equilibria.

Harrington, Ch. 9 # 2



Harrington, Ch. 9 # 2

- Consider the proper subgame between companies 2 and 3 associated with IBM having developed OS/2 and company 1 having developed an application.



Harrington, Ch. 9 # 2

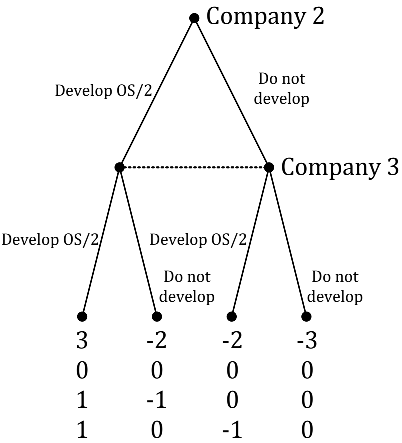
- The strategic form of the game is shown in the figure below:

		Company 3	
		D	DND
Company 2	D	5, 2, $\underline{2}$, $\underline{2}$	3, 1, $\underline{1}$, 0
	DND	3, 1, 0, $\underline{1}$	-2, -1, 0, 0

- *Develop* is a dominant strategy for each company (Remember we're only looking at company 2 and 3's payoffs), so there is a unique Nash equilibrium of (*Develop*, *Develop*) for this subgame.

Harrington, Ch. 9 # 2

- Next, consider the subgame associated with IBM having developed OS/2 and company 1 not having developed an application.



Harrington, Ch. 9 # 2

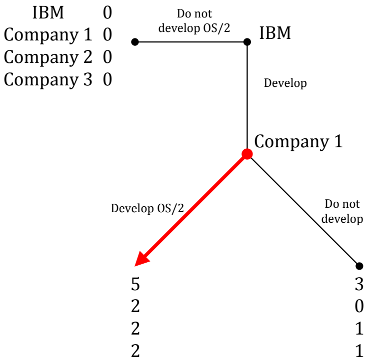
- The strategic form of the game is shown below:

		Company 3: D	Company 3: DND
Company 2	D	3, 0, <u>1</u>, <u>1</u>	-2, 0, -1, 0
	DND	-2, 0, 0, -1	-3, 0, <u>0</u>, <u>0</u>

- There are two psNE of this game: (*Develop*, *Develop*) and (*Do not develop*, *Do not develop*).

- Move up the tree to the subgame initiated by IBM having developed OS/2, where company 1 has to decide whether or not to develop an application.
- Suppose that the Nash equilibrium for the subgame in which company 1 does not develop an application is (*Develop*, *Develop*) (One of our two choices).
- Replacing the two final subgames with the Nash equilibrium payoffs, the situation is as depicted on the next slide.

Harrington, Ch. 9 # 2

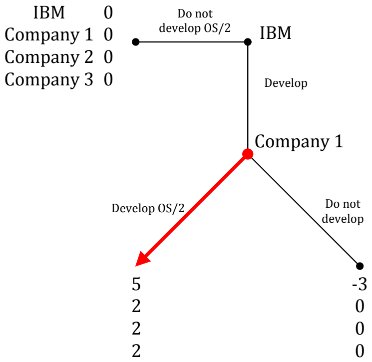


- As we can see, if company 1 develops an application, then its payoff is 2, while its payoff is 0 from not doing so. Hence, it chooses *Develop*.

Harrington, Ch. 9 # 2

- Now suppose the Nash equilibrium when company 1 does not develop an application is (*Do not develop*, *Do not develop*).
- Replacing the two final subgames with the Nash equilibrium payoffs, the situation is as depicted in the figure on the next slide.

Harrington, Ch. 9 # 2



- Again, if company 1 develops an application, then its payoff is 2, while its payoff is 0 from not doing so. Hence, it chooses *Develop*.

Harrington, Ch. 9 # 2

- Thus, regardless of which Nash equilibrium is used in the subgame in which company 1 chooses *Do not develop*, company 1 optimally chooses *Develop*.
- Now we go to the subgame that is the game itself. If IBM chooses to develop OS/2 $(5 > 0)$, then, as previously derived, company 1 develops an application and this induces both companies 2 and 3 to do so as well.
- Hence, IBM' spayoff is 5. It is then optimal for IBM to develop OS/2.

Harrington, Ch. 9 # 2

- There are then three subgame perfect Nash equilibria (where a strategy for company 2, as well as for company 3, is an action in response to company 1 choosing *Develop* and an action in response to company 1 choosing *Do not develop*):
  - "Wait? *Three*? We only talked about two!" There is a third Nash equilibrium of the second subgame we looked at using mixed strategies. I will leave that for you to calculate on your own.
  - The two SPNE that we calculated are (*Develop OS*/2, *Develop*, *Develop*/*Develop*, *Develop*/*Develop*), and (*Develop OS*/2, *Develop*, *Develop*/*Do not develop*, *Develop*/*Do not develop*)
  - Both equilibria result in the same outcome path of $(5, 2, 2, 2)$.

- Derive a Nash equilibrium that is not a subgame perfect Nash equilibrium, and explain why it is not a subgame perfect Nash equilibrium.

Harrington, Ch. 9 # 2

- Consider any strategy profile in which IBM chooses *Do not develop* $OS/2$ and the other three companies' strategies are such that at most one of them develops an application if $OS/2$ were to be developed. Given the latter, it is optimal not to develop OS/2 and, given that OS/2 is not developed, a company's payoff is 0 regardless of its strategy.
  - We can show all of these by creating the normal form of the entire game. We are not going to do that, but leave it as a challenge for you.

Harrington, Ch. 9 # 2

- Thus, these are Nash equilibria, but they are not subgame perfect Nash equilibria. There are 16 Nash equilibria (4 of each kind):
  - $(\textit{Do not develop OS}/2, \textit{Do not develop}, */\textit{Do not develop}, */\textit{Do not develop})$
  - $(\textit{Do not develop OS}/2, \textit{Do not develop}, */\textit{Do not develop}, */\textit{Develop})$
  - $(\textit{Do not develop OS}/2, \textit{Do not develop}, */\textit{Develop}, */\textit{Do not develop})$
  - $(\textit{Do not develop OS}/2, \textit{Do not develop}, \textit{Do not develop}/*, \textit{Do not develop}/*)$
    - In the strategy profiles just shown, you can put either *Do not develop* or *Develop* as the placeholder $*$.