EconS 424 - Nash Equilibrium in Games with Continuous Action Spaces.

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Watson, Ch. 10 # 1

- Consider a more general Cournot model. Suppose there are n firms. The firms simultaneously and independently select quantities to bring to the market.
- Firm i's quantity is denoted $q_i \geq 0$. All of the units of the good are sold, but the prevailing market price depends on the total quantity in the industry, which is $Q = \sum_{i=1}^{n} q_i$.
- Suppose the price is given by $p = a - bQ$ and suppose each firm produces with marginal cost c. There are no fixed costs for the firms. Assume $a > c > 0$ and $b > 0$.
- Note that firm i's profit is given by $u_i = pq_i - cq_i = (a - bQ)q_i - cq_i$. Defining Q_{-i} as the sum of the quantities produced by all firms except firm i, we have $u_i = (a - bq_i - bQ_{-i})q_i - cq_i$. Each firm maximizes its own profit.

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- Represent this game in the normal form by describing the strategy spaces and payoff functions.

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- We know that the strategy spaces consist of every strategy that is possible, regardless of whether it makes sense or not. Since each firm gets to choose q_i and the only restriction we have been given is $q_i \geq 0$, we can define the strategy space for firm i as

$$S_i = [0, \infty)$$

- For each firm's payoff function, we will use what the typical firm maximizes: it's utility

$$u_i(q_i, Q_{-i}) = (a - bq_i - bQ_{-i})q_i - cq_i$$

where $Q_{-i} \equiv \sum_{i \neq j} q_j$.

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- Find firm i's best-response function as a function of Q_{-i}. Graph this function.

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- Firm i solves their profit maximization problem

$$\max_{q_i} \left[(a - bq_i - bQ_{-i}) q_i - cq_i \right]$$

which, taking first-order conditions with respect to q_i yields

$$a - 2bq_i - bQ_{-i} - c = 0$$

and solving for q_i, we obtain firm i's best-response function:

$$q_i = \frac{a - c - bQ_{-i}}{2b}$$

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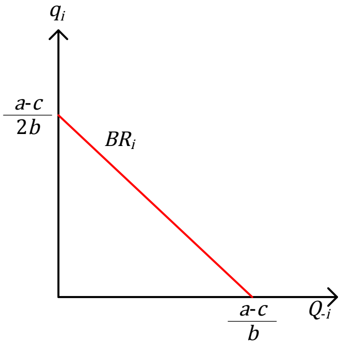
- Alternatively, we can express firm i's best response function as

$$q_i(Q_{-i}) = BR_i = \frac{a-c}{2b} - \frac{Q_{-i}}{2}$$

- This function can be represented by the following graph:



- Compute the Nash Equilibrium of this game. Report the equilibrium quantities, price, and total output (**Hint:** Summing the best-response functions over the different players will help.)

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- By symmetry, total equilibrium output is $Q^* = nq^*$, where q^* is the equilibrium output of an individual firm. Thus, $Q^*_{-i} = (n-1)q^*$.
  - This is because $Q_{-i} = \sum_{i \neq j} q_i$, meaning that Q_{-i} is the sum of every firm *except* firm i. Therefore it is the sum of $(n-1)$ identical quantities by symmetry.
  - Why can we invoke symmetry? Since all firms have the same marginal cost, c, and demand function, they will all choose the same quantity to produce, i.e., $q_1 = q_2 = ... = q_n = q$.
- Substituting these new values into our best-response function:

$$q^* = \frac{a - c - b(n-1)q^*}{2b}$$

and solving for q^* yields our solution

$$q^* = \frac{a-c}{(n+1)b}$$

Watson, Ch. 10 # 1

- We can now find aggregate equilibrium output, $Q^* = nq^*$ by multiplying the individual quantity, q^* we found before by the number of firms, n.

$$Q^* = \frac{n(a-c)}{(n+1)b}$$

- We also have that equilibrium price, p^* is given by the inverse demand function $p^* = a - bQ^*$. We can plug our value of Q^* that we found before in to get

$$p^* = a - b\left(\frac{n(a-c)}{(n+1)b}\right)$$

and rearranging,

$$p^* = \frac{a+nc}{n+1}$$

Watson, Ch. 10 # 1

- Finally, equilibrium profits, u^*, are given by $u^* = p^* q^* - cq^*$. Plugging in the values of p^* and q^* we found before, we obtain:

$$u^* = \left(\frac{a + nc}{n + 1}\right)\left(\frac{a - c}{(n+1)b}\right) - c\left(\frac{a - c}{(n+1)b}\right)$$

and rearranging:

$$u^* = \frac{(a-c)^2}{b(n+1)^2}$$

Harrington, Ch. 6 # 2

- In some presidential elections, there is a serious third-party candidate. In 1992, Ross Perot, running as an independent against Democratic nominee Bill Clinton and Republican nominee George H.W. Bush, garnered a respectable 19% of the vote. More recent third-party presidential attempts were made by Ralph Nader as the Green Party nominee in 2000 and 2004.
- In light of such instances, consider a model of electoral competition with three candidates, denoted D, R, and I(ndependent). As in Section 6.2, each candidate chooses a position from the interval $[0, 1]$.
- A candidate receives a payoff of 1 if he receives more votes than the other two candidates (so that he wins for sure), a payoff of $\frac{1}{2}$ if he is tied with one other candidate for the most votes, a payoff of $\frac{1}{3}$ if all three candidates share the vote equally, and a payoff of zero if his share of the vote is less than another candidate's (so that he loses for sure). Assume that voters vote for the candidate whose position is nearest to their own.

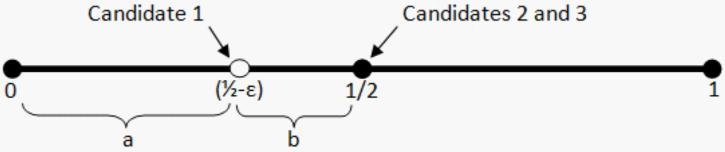
Harrington, Ch. 6 # 2

- Is there a Nash Equilibrium for all three candidates to locate at $\frac{1}{2}$?

Harrington, Ch. 6 # 2

- No. Consider Candidate 1 instead locating at $\frac{1}{2} - \varepsilon$ where $\varepsilon > 0$.
  - Remember that ε represents the smallest possible increment above zero.

Figure:



- where a represents voters who vote for candidate 1 plus half of the voters in b.

- That is,

$$a = \frac{1}{2} - \varepsilon \qquad \frac{b}{2} = \frac{\frac{1}{2} - \left(\frac{1}{2} - \varepsilon\right)}{2} = \frac{\varepsilon}{2}$$

- Therefore, Candidate 1 receives $\frac{1}{2} - \varepsilon + \frac{\varepsilon}{2} = \frac{1-\varepsilon}{2}$ votes
- Candidates 2 and 3 receive the remaining $1 - \frac{1-\varepsilon}{2} = \frac{1+\varepsilon}{2}$ votes.
  - Remember that they split these votes, so each candidate receives $\frac{1}{2}\left(\frac{1+\varepsilon}{2}\right) = \frac{1+\varepsilon}{4}$ votes.

Harrington, Ch. 6 # 2

- Hence, candidate 1 wins if:

$$\underbrace{\frac{1-\varepsilon}{2}}_{\substack{\text{Candidate} \\ \text{1 votes}}} > \underbrace{\frac{1+\varepsilon}{4}}_{\substack{\text{Candidate} \\ \text{2 (or 3) votes}}}$$

- Or, rearranging, $\varepsilon < \frac{1}{3}$.
  - He does not want to deviate too far away from the midpoint.

Harrington, Ch. 6 # 5

- Two manufacturers, denoted 1 and 2, are competing for 100 identical customers. Each manufacturer chooses both the price and quality of its product, where each variable can take any nonnegative real number.
- Let p_i and x_i denote, respectively, the price and quality of manufacturer i's product. The cost to manufacturer i of producing for one customer is $10 + 5x_i$. Note in this expression that cost is higher when the quality is higher. If manufacturer i sells to q_i customers, then its total cost is $q_i(10 + 5x_i)$.
- Each customer buys from the manufacturer who offers the greatest value, where the value of buying from manufacturer i is $1000 + x_i - p_i$; higher quality and lower prices mean more value.
- A manufacturer's payoff is its profit, which equals $q_i(p_i - 10 - 5x_i)$.

Harrington, Ch. 6 # 5

- If one manufacturer offers higher value, then all 100 customers buy from it. If both manufacturers offer the same value, then 50 customers buy from manufacturer 1 and the other 50 from manufacturer 2.
- Find all symmetric Nash equilibria.

- Since we are only looking for symmetric Nash equilibria, we can start by setting $p_1 = p_2 = p'$ and $x_1 = x_2 = x'$. Let's look at a situation where $p' > 10 + 5x'$. This is where the symmetric price chosen is above the marginal cost.
- In this case, both firms have 50 out of the 100 customers. One of the firms could lower his price by some very small $\varepsilon > 0$ and undercut his rival, capturing all 100 customers and increasing his profits from $50(p' - 10 - 5x')$ to $100(p' - \varepsilon - 10 - 5x')$.
- This undercutting would continue until $p' = 10 + 5x'$, which, for any further price decreases, those firms would receive negative profits as opposed to zero profits when pricing at marginal cost.

Harrington, Ch. 6 # 5

- Let's now look at our possible values for quality. Suppose $x' > 0$.
- If one manufacturer lowered his quality to zero, and instead offered a compensation in price of z (making his price $p' - z$), he could capture the entire market as long as $z > x'$.
  - Why? Because the manager offering the compensation would be perceived as offering the higher value.

$$1000 + 0 - (p' - z) > 1000 + x' - p'$$

- **Intuition:** It is very expensive to offer high quality (marginal cost $= 10 + 5x'$). As we showed in the last slide, if a manufacturer can reduce his marginal cost, he can respond by lowering his price, capturing the whole market while actually increasing the value offered to the customer (Since quality enters with a coefficient of 1 into the value equation but -5 in the marginal cost equation).

Harrington, Ch. 6 # 5

- If $z > x'$, then it sells to all 100 customers and its profit is

$$100(p' - z - 10) = 100(\underbrace{10 + 5x'}_{=p'} - z - 10) = 100(5x' - z)$$

which, for Firm 1 to offer the discount, it must still be making positive profits

$$\begin{aligned} 100(5x' - z) &> 0 \\ &\implies 5x' > z \end{aligned}$$

Thus, if $5x' > z > x'$, then all consumers buy from manufacturer 1, and manufacturer 1 is pricing above marginal cost and making a positive payoff.

Harrington, Ch. 6 # 5

- But then it cannot be optimal for manufacturer 2 to choose p' and x', as that pair results in a zero payoff.
  - He would respond by lowering his quality below that of manufacturer 1, then also compensating with an even lower price.
- As this analysis assumed that $x' > 0$, it follows that, at an equilibrium, quality must be zero.
- The final question is whether $p = 10$ and $x = 0$ is a symmetric Nash equilibrium, and it is straightforward to show that it is.
  - The game is just now a standard Bertrand price competition game with homogeneous products! We know that the equilibrium for this game is where $p = MC = 10$.

Harrington, Ch. 6 # 8

- An arms buildup is thought to have been a contributing factor to World War I. The naval arms race between Germany and Great Britain is particularly noteworthy. In 1889, the British adopted a policy for maintaing naval superiority whereby they required their navy to be at least two-and-a-half times as large as the next-largest navy.
- This aggressive stance induced Germany to increase the size of its navy, which according to Britain's policy, led to yet a bigger British navy, and so forth. In spite of attempts at disarmament in 1899 and 1907, this arms race fed on itself. By the start of World War I in 1914, the tonnage of Britain's navy was 2,205,000 pounds, not quite 2.5 times that of Germany's navy, which, as the second largest, weighed in at 1,019,000 pounds.

Harrington, Ch. 6 # 8

- With this scenario in mind, let us model the arms race between two countries, denoted 1 and 2. The arms expenditure of country i is denoted x_i and is restricted to the interval $[1, 25]$. The benefit to a country from investing in arms comes from security or war-making capability, both of which depend on relative arms expenditure.
- Thus, assume that the benefit to country 1 is $36\left(\frac{x_1}{x_1+x_2}\right)$, so it increases with country 1's expenditure relative to total expenditure. The cost is simply x_i, so country 1's payoff function is

$$V_1(x_1, x_2) = 36\left(\frac{x_1}{x_1+x_2}\right) - x_1,$$

and there is an analogous payoff for country 2:

$$V_2(x_1, x_2) = 36\left(\frac{x_2}{x_1+x_2}\right) - x_2.$$

These payoff functions are hill shaped.
- Derive each country's best-reply function.

Harrington, Ch. 6 # 8

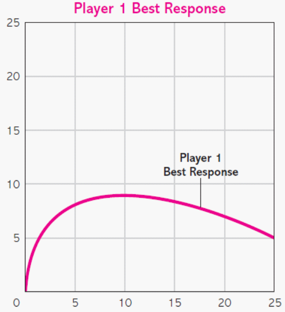
- Country 1's optimal arms expenditure is that value for x_1 which maximizes its payoff function. Taking first-order conditions with respect to x_1:

$$
\begin{aligned}
\frac{\partial V_1(x_1, x_2)}{\partial x_1} &= 36\left[\frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2}\right] - 1 = 0 \\
&= 36\left[\frac{x_2}{(x_1 + x_2)^2}\right] - 1 = 0 \\
&\implies 36x_2 = (x_1 + x_2)^2 \\
&\implies \sqrt{36x_2} = \sqrt{(x_1 + x_2)^2} \\
&\implies 6\sqrt{x_2} = x_1 + x_2 \\
&\implies x_1 = 6\sqrt{x_2} - x_2 \\
BR_1(x_2) &= 6\sqrt{x_2} - x_2
\end{aligned}
$$

Harrington, Ch. 6 # 8

- That country 1's best-reply function is $6\sqrt{x_2} - x_2$ presumes that whatever value it takes (after specifying a value for x_2) lies in the feasible set of $[1, 25]$. One can show that for all values of x_2 in $[1, 25]$, $6\sqrt{x_2} - x_2$ also lies in $[1, 25]$ so we're fine.
- Plotting this best-reply function,

FIGURE SOL 6.8.1



Harrington, Ch. 6 # 8

- Regarding the shape of the best-reply curve, we can prove its shape by taking the first and second derivatives of the function with respect to x_2. Starting with the first derivative,

$$\frac{\partial BR_1(x_2)}{\partial x_2} = \frac{\partial\left(6\sqrt{x_2} - x_2\right)}{\partial x_2} = 3(x_2)^{\frac{-1}{2}} - 1$$

setting it equal to zero, we can find the extrema,

$$3(x_2)^{\frac{-1}{2}} - 1 = 0 \implies x_2 = 9$$

and, using the second derivative

$$\frac{\partial^2 BR_1(x_2)}{\partial x_2^2} = -\frac{3}{2}(x_2)^{\frac{-3}{2}} - 1 < 0$$

since the second derivative is always less than zero, we know that $x_2 = 9$ is the maximum, rather than the minimum.

Harrington, Ch. 6 # 8

- The highest value for country 1's best reply occurs when country 2 spends 9, in which case country 1 spends $9 = \left(6\sqrt{9} - 9\right)$, which does lie in $[1, 25]$. The lowest value for country 1's best reply occurs either at 1 or 25 and the associated best replies are 5 and 5, coincidentally, which also lie in $[1, 25]$.
- We conclude that for all x_2 in $[1, 25]$, $6\sqrt{x_2} - x_2$ is in country 1's strategy set. Thus, $6\sqrt{x_2} - x_2$ is country 1's best reply. By symmetry, country 2's best reply function is $BR_2(x_1) = 6\sqrt{x_1} - x_1$.

Harrington, Ch. 6 # 8

- Define a symmetric Nash Equilibrium.

Harrington, Ch. 6 # 8

- We could do this two separate ways: First, we could substitute one best reply function into the other, then solve it out.
  - This will get ugly very fast. Is there a simpler way we can solve this? Yes!
  - Since both countries have identical payoff functions (just switch the 1's and the 2's), we can invoke symmetry.
- A symmetric Nash Equilibrium is an arms expenditure, denoted x^*, such that a country finds it optimal to spend x^* when the other country spends x^*. It is then a solution to

$$\begin{aligned} x^* &= 6\sqrt{x^*} - x^* \\ 2x^* &= 6\sqrt{x^*} \\ (x^*)^2 &= (3\sqrt{x^*})^2 = 9x^* \\ x^* &= 9 \end{aligned}$$

- The unique Nash Equilibrium then has each country spend 9. This equilibrium is depicted on the next slide.

Harrington, Ch. 6 # 8

FIGURE SOL 6.8.2