EconS 424 - Dominated Strategies and Nash Equilibrium

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Harrington, Ch. 4 # 5

- Consider the two-player game depicted below

		Player 2		
		x	y	z
Player 1	a	1, 2	1, 2	0, 3
	b	4, 0	1, 3	0, 2
	c	3, 1	2, 1	1, 2
	d	0, 2	0, 1	2, 4

a.) Derive those strategies which survive the iterated deletion of strictly dominated strategies.

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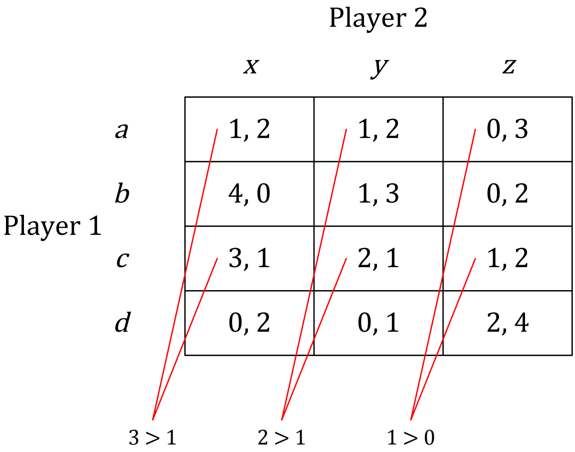
- Notice that for player 1, strategy *a* is always strictly dominated by strategy *c*, which allows us to rule out strategy *a*.



		Player 2		
		x	y	z
Player 1	a	1, 2	1, 2	0, 3
	b	4, 0	1, 3	0, 2
	c	3, 1	2, 1	1, 2
	d	0, 2	0, 1	2, 4

$3 > 1$ $2 > 1$ $1 > 0$

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- After deleting strategy *a* (represented in the matrix below), player 2 finds that strategy *z* strictly dominates strategy *x*, which allows us to rule out strategy *x*.

		Player 2		
		x	*y*	*z*
	b	4, 0	1, 3	0, 2
Player 1	*c*	3, 1	2, 1	1, 2
	d	0, 2	0, 1	2, 4

$2 > 0$ $2 > 1$ $4 > 2$

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- After deleting strategy x, (as represented in the smaller matrix below), notice that for player 1 strategy c strictly dominates strategy b.

		Player 2	
		y	z
	b	1, 3	0, 2
Player 1	c	2, 1	1, 2
	d	0, 1	2, 4

$2 > 1$ $\quad$ $1 > 0$

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- We can hence delete strategy *b*, obtaining the smaller matrix:

		Player 2	
		y	*z*
Player 1	*c*	2, 1	1, 2
	d	0, 1	2, 4

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- In the previous matrix, note that for player 2 strategy z strictly dominates strategy y. We can hence delete strategy y obtaining the following matrix:

$$\begin{array}{ccc} & & \text{Player 2} \\ & & z \\ \text{Player 1} & c & 1, 2 \\ & d & 2, 4 \end{array}$$

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- Finally, we can see how for player 1 it is optimal to play strategy d instead of c, so the only strategy profile that survives IDSDS is (d, z).

		Player 2
		z
Player 1	d	2, 4

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b.) Derive all strategy pairs that are Nash equilbria.

- To find the Nash equilibria, we have to evaluate which strategy for player 2 is the best in response to every possible strategy player 1 could play, and vice versa.
- We show this by underlining the highest payoff that player 2 can obtain given a particular strategy of player 1.

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- For instance, player 2 can obtain the highest payoff of 3 by selecting z in the case that player 1 chooses a in the first row. We can hence say that $BR_2(a) = z$.
  - Similarly, when player 1 selects b, $BR_2(b) = y$; when player 1 chooses c, $BR_2(c) = z$; and when player 1 selects d, $BR_2(d) = z$.

		Player 2		
		x	y	z
Player 1	a	$1, 2$	$1, 2$	$0, \underline{3}$
	b	$4, 0$	$1, \underline{3}$	$0, 2$
	c	$3, 1$	$2, 1$	$1, \underline{2}$
	d	$0, 2$	$0, 1$	$2, \underline{4}$

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- For player 1, fixing a particular strategy for player 2 (a given column), we can see that:
  - $BR_1(x) = b$ with a payoff of 4, $BR_1(y) = c$ with a payoff of 2, and $BR_1(z) = d$ with a payoff of 2.

		Player 2		
		x	y	z
Player 1	a	1, 2	1, 2	0, 3
	b	$\underline{4}$, 0	1, 3	0, 2
	c	3, 1	$\underline{2}$, 1	1, 2
	d	0, 2	0, 1	$\underline{2}$, 4

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- The Nash Equilibrium is the box where the two player's best responses coincide, i.e., where both payoffs in the cell have been underlined. In this case, the unique Nash Equilibrium is (d, z).
  - Why? Because $BR_1(z) = d$ and $BR_2(d) = z$. Try picking another best response function and follow the path.

		Player 2		
		x	y	z
Player 1	a	$1, 2$	$1, 2$	$0, \underline{3}$
	b	$\underline{4}, 0$	$1, \underline{3}$	$0, 2$
	c	$3, 1$	$\underline{2}, 1$	$1, \underline{2}$
	d	$0, 2$	$0, 1$	$\underline{2}, \underline{4}$

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- Find all of the Nash equilibria for the three-player game shown below:

Player 2

		x	*y*	*z*
	a	2, 0, 4	1, 1, 1	1, 2, 3
Player 1	*b*	3, 2, 3	0, 1, 0	2, 1, 0
	c	1, 0, 2	0, 0, 3	3, 1, 1

Player 3: *A*

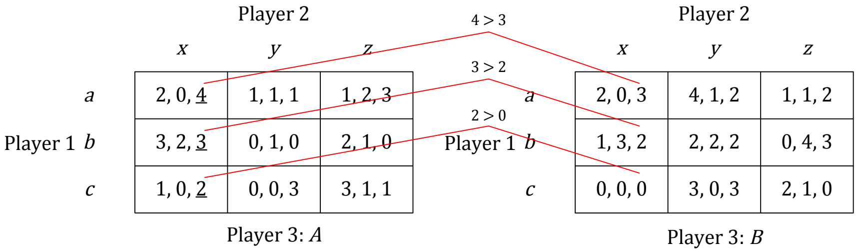
Player 2

		x	*y*	*z*
	a	2, 0, 3	4, 1, 2	1, 1, 2
Player 1	*b*	1, 3, 2	2, 2, 2	0, 4, 3
	c	0, 0, 0	3, 0, 3	2, 1, 0

Player 3: *B*

Harrington, Ch. 4 # 13

- Let's start by evaluating the payoffs for player 3 if player 2 selects x (first column)



Player 2

		x	y	z
	a	2, 0, $\underline{4}$	1, 1, 1	1, 2, 3
Player 1	b	3, 2, $\underline{3}$	0, 1, 0	2, 1, 0
	c	1, 0, $\underline{2}$	0, 0, 3	3, 1, 1

Player 3: A

Player 2

		x	y	z
	a	2, 0, 3	4, 1, 2	1, 1, 2
Player 1	b	1, 3, 2	2, 2, 2	0, 4, 3
	c	0, 0, 0	3, 0, 3	2, 1, 0

Player 3: B

- In terns of best responses, $BR_3(x, a) = A$, $BR_3(x, b) = A$, and $BR_3(x, c) = A$.

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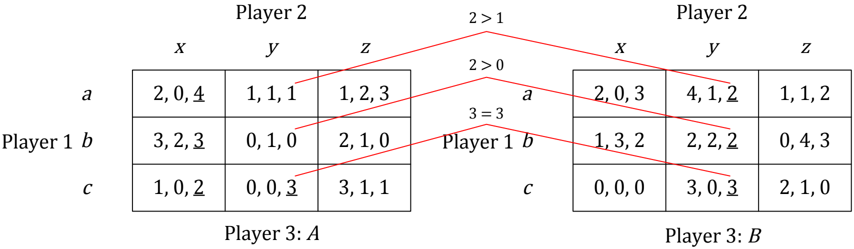
- If player 2 selects y (in the second column):



Player 2

Player 1	x	y	z
a	$2, 0, \underline{4}$	$1, 1, 1$	$1, 2, 3$
b	$3, 2, \underline{3}$	$0, 1, 0$	$2, 1, 0$
c	$1, 0, \underline{2}$	$0, 0, \underline{3}$	$3, 1, 1$

Player 3: A

Player 2

Player 1	x	y	z
a	$2, 0, 3$	$4, 1, \underline{2}$	$1, 1, 2$
b	$1, 3, 2$	$2, 2, \underline{2}$	$0, 4, 3$
c	$0, 0, 0$	$3, 0, \underline{3}$	$2, 1, 0$

Player 3: B

- In terms of best responses, $BR_3(y, a) = B$, $BR_3(y, b) = B$, and $BR_3(y, c) = \{A, B\}$.

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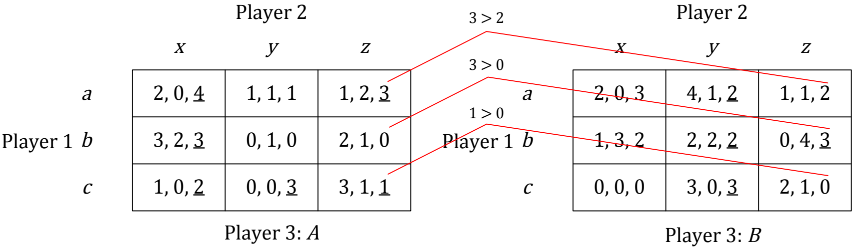
- If player 2 selects z (in the third column):



Player 2

Player 1	x	y	z
a	2, 0, $\underline{4}$	1, 1, 1	1, 2, $\underline{3}$
b	3, 2, $\underline{3}$	0, 1, 0	2, 1, 0
c	1, 0, $\underline{2}$	0, 0, $\underline{3}$	3, 1, $\underline{1}$

Player 3: A

Player 2

Player 1	x	y	z
a	2, 0, 3	4, 1, $\underline{2}$	1, 1, 2
b	1, 3, 2	2, 2, $\underline{2}$	0, 4, $\underline{3}$
c	0, 0, 0	3, 0, $\underline{3}$	2, 1, 0

Player 3: B

- In terms of best responses, $BR_3(z, a) = A$, $BR_3(z, b) = B$, and $BR_3(z, c) = A$.

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- Now let's evaluate player 2's payoffs given player 1's strategy (fixing the row) in both matrices:

$$
\begin{array}{rc|c|c|c|}
 & & \multicolumn{3}{c}{\text{Player 2}} \\
 & & x & y & z \\ \cline{3-5}
 & \xrightarrow{a} & 2, 0, \underline{4} & 1, 1, 1 & 1, \underline{2}, \underline{3} \\ \cline{3-5}
\text{Player 1} & \xrightarrow{b} & 3, \underline{2}, \underline{3} & 0, 1, 0 & 2, 1, 0 \\ \cline{3-5}
 & \xrightarrow{c} & 1, 0, \underline{2} & 0, 0, \underline{3} & 3, \underline{1}, \underline{1} \\ \cline{3-5}
 & & \multicolumn{3}{c}{\text{Player 3: } A}
\end{array}
\qquad
\begin{array}{rc|c|c|c|}
 & & \multicolumn{3}{c}{\text{Player 2}} \\
 & & x & y & z \\ \cline{3-5}
 & \xrightarrow{a} & 2, 0, 3 & 4, \underline{1}, \underline{2} & 1, \underline{1}, 2 \\ \cline{3-5}
\text{Player 1} & \xrightarrow{b} & 1, 3, 2 & 2, 2, \underline{2} & 0, \underline{4}, \underline{3} \\ \cline{3-5}
 & \xrightarrow{c} & 0, 0, 0 & 3, 0, \underline{3} & 2, \underline{1}, 0 \\ \cline{3-5}
 & & \multicolumn{3}{c}{\text{Player 3: } B}
\end{array}
$$

- For the matrix on the left, $BR_2(a, A) = z$, $BR_2(b, A) = x$, and $BR_2(c, A) = z$.
- For the matrix on the right, $BR_2(a, B) = \{y, z\}$, $BR_2(b, B) = z$, and $BR_2(c, B) = z$.

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- Finally, let's evaluate player 1's payoffs given player 2's strategy (fixing the column) in both matrices:

$$\begin{array}{cc|c|c|c|}
 & & & \text{Player 2} & \\
 & & \downarrow\ x & \downarrow\ y & \downarrow\ z \\ \hline
 & a & 2, 0, \underline{4} & \underline{1}, 1, 1 & 1, \underline{2}, \underline{3} \\ \hline
\text{Player 1} & b & \underline{3}, \underline{2}, \underline{3} & 0, 1, 0 & 2, 1, 0 \\ \hline
 & c & 1, 0, \underline{2} & 0, 0, \underline{3} & \underline{3}, \underline{1}, \underline{1} \\ \hline
\end{array}$$

Player 3: *A*

$$\begin{array}{cc|c|c|c|}
 & & & \text{Player 2} & \\
 & & \downarrow\ x & \downarrow\ y & \downarrow\ z \\ \hline
 & a & \underline{2}, 0, 3 & \underline{4}, \underline{1}, \underline{2} & 1, \underline{1}, 2 \\ \hline
\text{Player 1} & b & 1, 3, 2 & 2, 2, \underline{2} & 0, \underline{4}, \underline{3} \\ \hline
 & c & 0, 0, 0 & 3, 0, \underline{3} & \underline{2}, \underline{1}, 0 \\ \hline
\end{array}$$

Player 3: *B*

- For the matrix on the left, $BR_1(x, A) = b$, $BR_1(y, A) = a$, and $BR_1(z, A) = c$.
- For the matrix on the right, $BR_1(x, B) = a$, $BR_1(y, B) = a$, and $BR_1(z, B) = c$.

Harrington, Ch. 4 # 13

- There are three boxes which have all three payoffs underlined. Therefore, there are three Nash Equilibria to this game: (b, x, A), (c, z, A), and (a, y, B).

Player 2

		x	y	z
	a	$2, 0, \underline{4}$	$\underline{1}, 1, 1$	$1, \underline{2}, \underline{3}$
Player 1	b	$\underline{3}, \underline{2}, \underline{3}$	$0, 1, 0$	$2, 1, 0$
	c	$1, 0, \underline{2}$	$0, 0, \underline{3}$	$\underline{3}, \underline{1}, \underline{1}$

Player 3: A

Player 2

		x	y	z
	a	$\underline{2}, 0, 3$	$\underline{4}, \underline{1}, \underline{2}$	$1, \underline{1}, 2$
Player 1	b	$1, 3, 2$	$2, 2, \underline{2}$	$0, \underline{4}, \underline{3}$
	c	$0, 0, 0$	$3, 0, \underline{3}$	$\underline{2}, \underline{1}, 0$

Player 3: B

Harrington, Ch. 5 # 3

- It is the morning commute in Congestington, DC. There are 100 drivers, and each driver is deciding whether to take the toll road or take the back roads. The toll for the toll road is $10, while the back roads are free.
- In deciding on a route, each driver cares only about income, denoted y, and his travel time, denoted t. If a driver's final income is y and his travel time is t, then his payoff is assumed to be $y - t$ (where we have made the dollar amount of one unit of travel time equal to 1).
- A driver's income at the start of the day is $1,000.
- If m drivers are on the toll road, the travel time for a driver on the toll road is assumed to be m (in dollars). In contrast, if m drivers take the back roads, the travel time for those on the back roads is $2m$ (again, in dollars).
- Drivers make simultaneous decisions as to whether to take the toll road or the back roads.

Harrington, Ch. 5 # 3

a.) Derive each player's payoff function (i.e., the expression that gives us a player's payoff as a function of her strategy profile.)

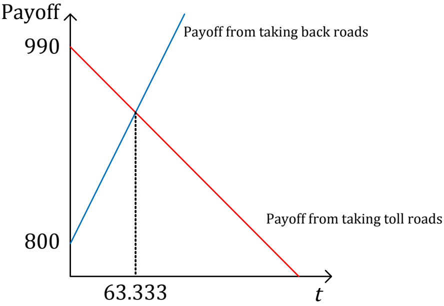
- If driver i takes back roads and the total number of drivers taking the toll road is t (so there are $100-t$ drivers on the back roads), then driver i's payoff is $1000-2(100-t)$. If driver i takes the toll road and the total number of drivers on the toll road is t, then driver i's payoff is $990-t$, which nets out the cost of the toll.

$$\begin{aligned} u_i(back) &= 1000-2m = 1000-2(100-t) = 800+2t \\ u_i(toll) &= 1000-10-t = 990-t \end{aligned}$$

Harrington, Ch. 5 # 3

b.) Find a Nash Equilibrium

- We can find a Nash Equilibrium by plotting both best response functions on the same set of axis. The intersection between the two responses will be the Nash Equilibrium.



Harrington, Ch. 5 # 3

- For those drivers taking the back roads, it must be true that:

$$\begin{aligned} u_i(back) &\geq u_i(toll) \\ 800 + 2t &\geq 990 - t \\ &\Longrightarrow t \geq \frac{190}{3} = 63.333 \cong 64 \end{aligned}$$

Harrington, Ch. 5 # 3

- For those drivers taking the toll roads, it must be true that:

$$
\begin{aligned}
& u_i(toll) \geq u_i(back) \\
& 990 - t \geq 800 - 2t \\
& \qquad \Longrightarrow \quad t \leq \frac{190}{3} = 63.333 \cong 63
\end{aligned}
$$

Harrington, Ch. 5 # 3

- Thus it must be true that $63 \leq t \leq 64$.
- There are then two Nash Equilibria:
  1.) A total of 36 drivers take the back roads and 64 take the toll road.
  2.) A total of 37 drivers take the back roads and 63 take the toll road.
    - This implies that no matter what, 36 drivers will take the back roads, and 63 will take the toll road. The final driver will be content with either route.
- This exercise shows the presence of congestion effects in roads. As in the "Applying for an internship" game, all players do not want to play the same strategy.

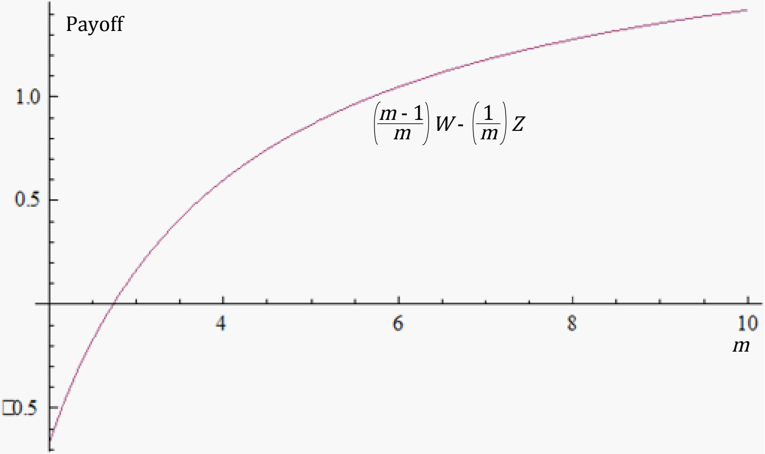
Harrington, Ch. 5 # 7

- There is a rough neighborhood with $n \geq 2$ residents. Each resident has to decide whether to engage in the crime of theft.
- If an individual chooses to be a thief and is not caught by the police, he receives a payoff of W. If he is caught by the police, he must pay a fine of Z (Jail time measured in dollars). If he chooses not to commit theft, he receives a zero payoff. Assume that $Z \geq W > 0$.
- All n residents simultaneously decide whether or not to commit theft. The probability of a thief being caught is $\frac{1}{m}$, where m is the number of residents who choose to engage in theft. Thus, the probability of being caught is lower when more crimes are committed and the police have more crimes to investigate.
- The payoff from being a thief, given that $m - 1$ other people have also chosen to be thieves, is then

$$\left(\frac{m-1}{m}\right) W - \left(\frac{1}{m}\right) Z.$$

Harrington, Ch. 5 # 7

- The figure below plots the case where $W = 2$ and $Z = 4$ (recall that $Z \geq W > 0$ by assumption).
- The figure shows that as m increases, the payoff from committing crime also increases.



Harrington, Ch. 5 # 7

- Find all Nash Equilibria
- In order for a value to be a Nash Equilibrium, it must be that of all those who have already chosen to be a thief, none of them wish to stop their lives of crime, or

$$\left(\frac{m-1}{m}\right) W - \left(\frac{1}{m}\right) Z \geq 0$$

since if this expression were negative, some individuals would prefer a zero payoff of a crimeless life rather than a negative expected payoff of committing theft.

Harrington, Ch. 5 # 7

- Likewise, in order for a value to be a Nash Equilibrium, it must be that all those who have chosen to not live a life of crime not want to become criminals. This would require that the payoff of adding one additional thief be non-positive, or

  $$\left(\frac{m}{m+1}\right)W - \left(\frac{1}{m+1}\right)Z \leq 0$$

  as a positive value for this expression would give individuals an incentive to become thieves.

Harrington, Ch. 5 # 7

- Let's compare these two expressions. Taking the ratio of the first terms, we have

$$\frac{\frac{m-1}{m}}{\frac{m}{m+1}} = \frac{m^2-1}{m^2} < 1 \implies \left(\frac{m-1}{m}\right) W < \left(\frac{m}{m+1}\right) W$$

- Likewise, comparing the second expressions,

$$\frac{\frac{1}{m}}{\frac{1}{m+1}} = \frac{m+1}{m} > 1 \implies -\left(\frac{1}{m}\right) Z < -\left(\frac{1}{m+1}\right) Z$$

- Combining these two ratios, we find

$$\left(\frac{m}{m+1}\right) W - \left(\frac{1}{m+1}\right) Z > \left(\frac{m-1}{m}\right) W - \left(\frac{1}{m}\right) Z$$

Harrington, Ch. 5 # 7

- This leads to an interesting problem. If we assume the first constraint (all thieves are content with being thieves) holds,

$$\left(\frac{m}{m+1}\right)W-\left(\frac{1}{m+1}\right)Z>\left(\frac{m-1}{m}\right)W-\left(\frac{1}{m}\right)Z\geq 0$$

  our second constraint cannot hold. Likewise, if we assume the second constraint (all non-thieves are content with not being thieves) holds,

$$\left(\frac{m-1}{m}\right)W-\left(\frac{1}{m}\right)Z<\left(\frac{m}{m+1}\right)W-\left(\frac{1}{m+1}\right)Z\leq 0$$

  our first constraint cannot hold.

Harrington, Ch. 5 # 7

- This implies that only two Nash Equilibria exist: one where everyone is a thief, and one where nobody is a thief.
  - All of the residents are thieves if

$$\left(\frac{n-1}{n}\right)W-\left(\frac{1}{n}\right)Z\geq 0\implies n\geq\frac{Z}{W}+1$$

  - None of the residents are thieves if

$$\left(\frac{n-1}{n}\right)W-\left(\frac{1}{n}\right)Z<0\implies n<\frac{Z}{W}+1$$