EconS 424 - Representation of Games and Strategies

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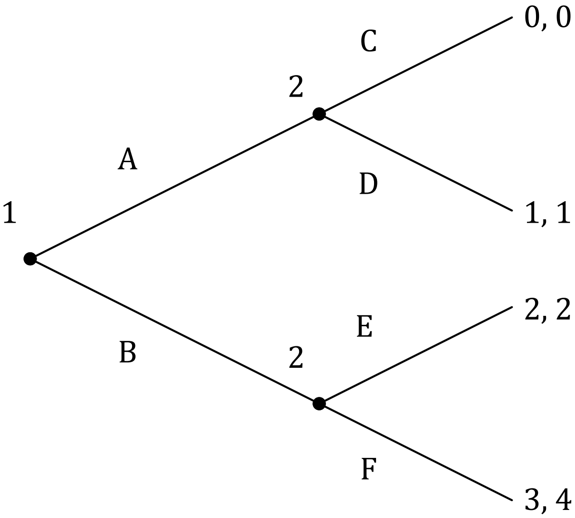
Exercise 1

Represent the extensive-form game depicted in Figure 1.1 using its normal-form (matrix) representation.

Exercise 1



Exercise 1

- We start identifying the strategy sets of all players in the game. The cardinality of these sets will determine the number of rows and columns in the normal-form representation of the game.
- Starting from the initial node (in the root of the game tree located on the left-hand side of the figure), Player 1 must select either strategy A or B, thus implying that the strategy space for player 1, S_1, is:

$$S_1 = \{A, B\}$$

Exercise 1

- In the next stage of the game, Player 2 conditions his strategy on player 1's choice, since player 2 observes such a choice before selecting his own. We need to consider that the strategy profile of player 2 (S_2) must be a complete plan of action (complete contingent plan) that includes all the possible outcomes of the game. Therefore, his strategy space becomes:

$$S_2 = \{CE, CF, DE, DF\}$$

where the first component of every strategy describes how player 2 responds upon observing that player 1 chose A, while the second component represents player 2's response after observing that player 1 selected B. For example, strategy CE describes that player 2 responds with strategy C after player 1 chooses A, but with strategy E after player 1 chooses B.

Exercise 1

- Using the strategy space for player 1, with only two availble strategies $S_1 = \{A, B\}$, and that of player 2, with four availbale strategies $S_2 = \{CE, CF, DE, DF\}$, we obtain the 2 x 4 payoff matrix

		Player 2			
		CE	CF	DE	DF
Player 1	A	0, 0	0, 0	1, 1	1, 1
	B	2, 2	3, 4	2, 2	3, 4

where, for instance, the payoffs associated with the strategy profile where player 1 chooses A and player 2 chooses C if A and E if B $\{A, CE\}$ is $(0, 0)$.

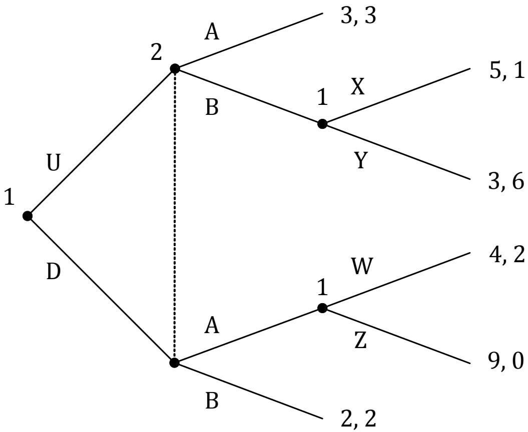
Exercise 1

- *Remark:* Note that, if player 2 could not observe player 1's action before selecting his own (either C or D), then player 2's strategy would be $S_2 = \{C, D\}$, implying that the normal form representation of the game would be a 2 x 2 matrix with A and B in rows for player 1, and C and D in columns for player 2.

Exercise 2

- Consider the extensive-form game on the next slide. Provide its normal form (matrix) representation.

Exercise 2



Exercise 2

- *Player 2.* From the extensive form game, we know player 2 only plays once and has two available choices, either A or B. The dashed line connecting the two nodes at which player 2 is called on to move indicates that player 2 cannot observe player 1's choice. Hence, he cannot condition his choice on player 1's previous action, ultimately implying that his strategy game is $S_2 = \{A, B\}$.

Exercise 2

- *Player 1.* Player 1, however, plays twice (in the root of the game tree, and after player 2 responds) and has multiple choices:
  - First, he must select either U or D, at the initial node of the tree, i.e., left hand side of the figure;
  - Then choose X or Y, in case that he played U at the beginning of the game. (Note that in this event he cannot condition his choice on player 2's choice, since he cannot observe whether player 2 selected A or B); and
  - Then choose W or Z, which only becomes available to player 1 in the event that player 2 responds with B after player 1 chose D.

Exercise 2

- Therefore, player 1's strategy space is composed of triplets, as follows,

$$S_1 = \{UXW, UXZ, UYW, UYZ, DXW, DXZ, DYW, DYZ\}$$

whereby the first component of every triplet describes player 1's choice at the beginning of the game (the root of the game tree), the second component represents his decision (X or Y) in the event that he chose U and afterwards player 2 responded with either A or B (something player 1 cannot observe), and the third component reflects his choice in the case that he chose D at the beginning of the game and player 2 responds with B.

Exercise 2

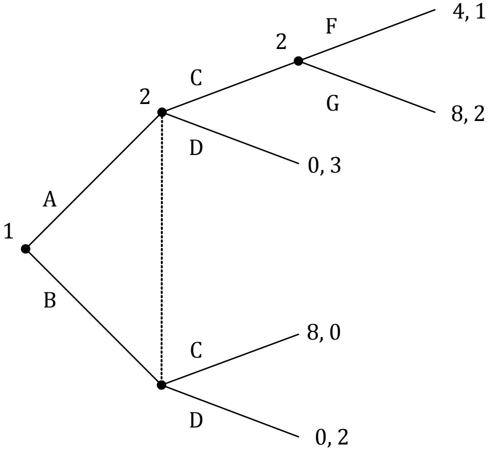
- As a consequence, the normal-form representation of the game is given by the following 8 x 2 matrix

		Player 2	
		A	B
Player 1	UXW	3, 3	5, 1
	UXZ	3, 3	5, 1
	UYW	3, 3	3, 6
	UYZ	3, 3	3, 6
	DXW	4, 2	2, 2
	DXZ	9, 0	2, 2
	DYW	4, 2	2, 2
	DYZ	9, 0	2, 2

Exercise 3

- In the extensive-form game pictured on the next slide, how many strategies does player 2 have?

Exercise 3



Exercise 3

- The strategy space for each player is:

$$\begin{aligned} S_1 &= \{A, B\} \\ S_2 &= \{(C, F), (C, G), (D, F), (D, G)\} \end{aligned}$$

- Player 1 selects A or B in the first period of the game. Player 2's strategy space is more involved. Indeed, for every pair in his strategy space, the first component denotes his choice after player 1 selects A or B (something he is not able to observe), while the second component denotes the action he chooses when he selected C and player 1 chose A in the first period of the game.

Exercise 4

- Consider a version of the *Cournot duopoly game*, which will be thoroughly analyzed in Chapter 10. Two firms (1 and 2) compete in a homogeneous goods market, where the firms produce exactly the same good. The firms simultaneously and independently select quantities to produce. The quantity selected by firm i is denoted q_i and must be greater than or equal to zero, for $i = 1, 2$. The market price is given by $p = 2 - q_1 - q_2$.
- For simplicity, assume that the cost to firm i of producing any quantity is zero. Further assume that each firm's payoff is defined as its profit. That is, firm i's payoff is pq_i, where j denotes firm i's opponent in the game. Describe the normal form of this game by expressing the strategy spaces and writing the payoffs as functions of the strategies.

Exercise 4

The normal form specifies player, strategy spaces, and payoff functions. Here $N = \{1, 2\}$. $S_i = [0, \infty)$. The payoff to player i is given by $u_i(q_i, q_j) = (2 - q_i - q_j)q_i$. Note that $S_i = [0, \infty)$ due to $q_i, q_j \geq 0$.

Exercise 5

- Consider a variation of the Cournot duopoly game in which the firms move sequentially rather than simultaneously. Suppose that firm 1 selects its quantity first. After observing firm 1's selection, firm 2 chooses its quantity.
  - This is called the *von Stackelberg duopoly* model.
- For this game, describe what a strategy of firm 2 must specify.

Exercise 5

- $N = \{1, 2\}.S_i = [0, \infty)$. Player 2's strategy must specify a choice of quantity for each possible quantity player 1 can choose. Thus, player 2's strategy space S_2 is the set of functions from $[0, \infty)$ to $[0, \infty)$.

Exercise 6

- Consider the normal-form game pictured below. Draw an extensive-form representation of this game. Can you think of other extensive forms that correspond to this normal-form game?

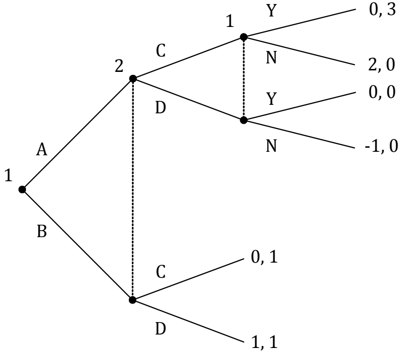
		Player 2	
		C	D
Player 1	AY	0, 3	0, 0
	AN	2, 0	-1, 0
	BY	0, 1	1, 1
	BN	0, 1	1, 1

Exercise 6

- This normal form game can be represented with different extensive form representations. We include two examples on the next two slides. Note that our only requirement is that, since the above matrix represents a simultaneous-move game, players 1 and 2 must choose their actions without knowing which action his opponent selects.

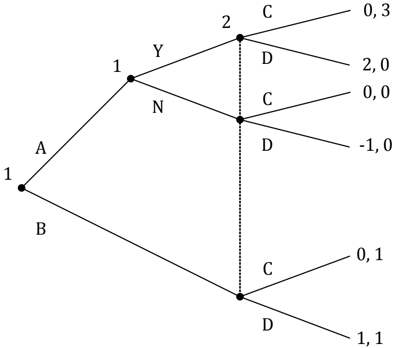
Exercise 6

- **First Extensive Form:** This property is satisfied in the first extensive form game tree, whereby player 2 doesn't observe which action player 1 chose before him, and similarly player 1 later on doesn't observe which action player 2 chose in the previous stage.



Exercise 6

- **Second Extensive Form**: This property is also satisfied in the second extensive form game: while player 1 observes he action he chose in the first stage (he has perfect recall about which actions he chose in the past), player 2 cannot observe which action player 1 chose before him that leads to player 2 being called on to move.



Exercise 7

- Evaluate the following payoffs for the game given by the normal form below.
  - [Remember, a mixed strategy for player 1 is $\sigma_1 \in \{U, M, D\}$, where $\sigma_1(U)$ is the probability that player 1 plays strategy U, and so forth. For simplicity, we write σ_1 as $(\sigma_1(U), \sigma_1(M), \sigma_1(D))$, and similarly for player 2.]

Player 2

		L	C	R
	U	10, 0	0, 10	3, 3
Player 1	M	2, 10	10, 2	6, 4
	D	3, 3	4, 6	6, 6

Exercise 7

- a. $u_2(M, R)$
  - The payoff that player 2 obtains when player 1 selects M and he chooses R is $u_2(M, R) = 4$.

		Player 2		
		L	C	R
Player 1	U	10, 0	0, 10	3, 3
	M	2, 10	10, 2	6, 4
	D	3, 3	4, 6	6, 6

Exercise 7

- $u_1(\sigma_1, R)$ for $\sigma_1 = (0.25, 0.5, 0.25)$
- The payoff that player 1 obtains when player 2 selects R and he randomizes between U, M and D according to σ_1 is

$$u_1(\sigma_1, R) = \frac{1}{4} * 3 + \frac{1}{2} * 6 + \frac{1}{4} * 6 = 5.25$$

		Player 2		
		L	C	R
Player 1	U	10, 0	0, 10	3, 3
	M	2, 10	10, 2	6, 4
	D	3, 3	4, 6	6, 6

- *Trick:* focus your attention in the third column of the matrix, where player 2 selects R, and then apply the randomization σ_1 in player 1's actions, which implies that player 1's payoff is either 3 (with probability $\frac{1}{4}$), 6 (with probability $\frac{1}{2}$), or 6 (with probability $\frac{1}{4}$).

Exercise 8

- Consider the simultaneous-move game depicted below, where two players choose between several strategies. Find which strategies survive the iterative deletion of strictly dominated strategies, IDSDS.

		Player 2			
		W	X	Y	Z
Player 1	U	3, 6	4, 10	5, 0	0, 8
	M	2, 6	3, 3	4, 10	1, 1
	D	1, 5	2, 9	3, 0	4, 6

Exercise 8

- Let us start by identifying the strategies of player 2 that are strictly dominated by other of his own strategies.
  - When player 2 chooses Z , in the fourth row, his payoff is either 8 (when player 1 chooses U), 1 (when player 1 chooses M) or 6 (when player 1 chooses D, in the third row). These payoffs are unambiguously lower than those in strategy X in the second row.
  - In particular, when player 1 chooses U (in the first row), player 2 obtains a payoff of 10 with X but only a payoff of 8 with Z; when player 1 chooses M, player 2 earns 3 with X but only 1 with Z; and when player 1 selects D, player 2 obtains 9 with X but 6 with Z.

Exercise 8

- Hence, player 2's strategy Z is strictly dominated by X, since the former yields a lower payoff than the latter regardless of the strategy that player 1 selects
    - (i.e., regardless of the row he uses).
- Thus, the strategies of player 2 that survive one round of the iterative deletion of strictly dominated strategies (IDSDS) are W, X and Y, as depicted in the payoff matrix below.

			Player 2	
		W	X	Y
	U	3, 6	4, 10	5, 0
Player 1	M	2, 6	3, 3	4, 10
	D	1, 5	2, 9	3, 0

Exercise 8

- Let us now turn to player 1 (by looking at the first payoff within every cell in the matrix).
    - In particular, we can see that strategy U strictly dominates both M and D, since it provides to player 1 a larger payoff than either M or D regardless of the strategy (column) that player 2 uses.
    - Specifically, when player 2 chooses W (left-most column), player 1 obtains a payoff of 3 by selecting U (in the top row) but only a payoff of 2 from choosing M (in the middle row) or 1 from choosing D (in the bottom row). Similarly, when player 2 chooses X (in the middle column), player 1 earns a payoff of 4 from U but only a payoff of 3 from M or a payoff of 2 from D. Finally, when player 2 selects Y (in the right-most column), player 1 obtains a payoff of 5 from U but only a payoff of 4 from M or a payoff of 3 from D.

Exercise 8

- Hence, strategy *U* yields player 1 a larger payoff independently of the strategy chosen by player 2, i.e., *U* strictly dominates both *M* and *D*, which allows us to delete strategies *M* and *D* from the payoff matrix. Thus, the strategy of player 1 that survive one additional round of the IDSDS is *U*, which helps us further reduce the payoff matrix to that below

Player 2

		W	X	Y
Player 1	U	3, 6	4, 10	5, 0

.

Exercise 8

- Now that the payoff matrix has been reduced to only one row, we look at player 2's strategies once again.
    - We can clearly see that strategy X strictly dominates both W and Y since it provides to player 2 a larger payoff than either W or Y. Since player 1 will only choose strategy U at this point, when player 2 chooses X he receives a payoff of 10 but only a payoff of 6 by selecting W or a zero payoff by selecting Z.
- Thus, strategy X strictly dominates both W and Y, which allows us to delete strategies W and Y from the payoff matrix. This, the strategy of player 2 that survives one additional round of the IDSDS is X, which reduces the payoff matrix to its Nash Equilibrium below.

		Player 2
		X
Player 1	U	4, 10

Exercise 9

- For the game depicted below, determine the best response of player 2, $BR_2(\theta_1)$, for $\theta_1 = (0, \frac{1}{3}, \frac{2}{3})$

			Player 2	
		L	C	R
	U	10, 0	0, 10	3, 3
Player 1	M	2, 10	10, 2	6, 4
	D	3, 3	4, 6	6, 6

Exercise 9

- We cannot find a single strictly dominant strategy, pure or mixed, for player 2. In order to show that, we must check that there is no strategy for player 2 that gives him a strictly higher payoff, given the randomization θ_1 that player 1 is using. We check this by computing player 2's expected payoff when he chooses L, C or R separately, given the randomization θ_1 of player 1, as follows.

$$\begin{aligned} u_2(L) &= 0*(0)+\frac{1}{3}*(10)+\frac{2}{3}*(3)=5.33 \\ u_2(C) &= 0*(10)+\frac{1}{3}*(2)+\frac{2}{3}*(6)=4.66 \\ u_2(R) &= 0*(3)+\frac{1}{3}*(4)+\frac{2}{3}*(6)=5.33 \end{aligned}$$

Exercise 9

- Thus, given the randomization θ_1 that player 1 uses, player 2 is indifferent between using L and R. Hence, while strategy C is dominated by L and R, we cannot identify a single strictly dominant strategy that yields a larger payoff for player 2. Therefore, the set of best responses for player 2, given the randomization θ_1 that player 1 uses, is L and R. More compactly, $BR_2(\theta_1) = \{L, R\}$.

Exercise 10

- Consider the following anti-coordination game on the next slide played by three potential entrants seeking to enter into a new industry, such as the development of software applications for smartphones. Every firm (labeled as A, B, and C) has the option of entering or staying out (i.e., remain in the industry they have been traditionally operating, e.g., software for personal computers). The normal form game on the next slide depicts the market share that each firm obtains, as a function of the entering decision of its rivals. Firms simultaneously and independently choose whether or not to enter. As usual in simultaneous-move games with three players, the triplet of payoffs describes the payoff for the row player (firm A) first, for the column player (firm B) second, and for the matrix player (firm C) third.
- Find the set of strategy profiles that survive the iterative deletion of strictly dominated strategies (IDSDS). Is the equilibrium you found using this solution concept unique?

Exercise 10

Firm C chooses Enter

		Firm B	
		Enter	Stay Out
Firm A	Enter	14,24,32	8,30,27
	Stay Out	30,16,24	13,12,50

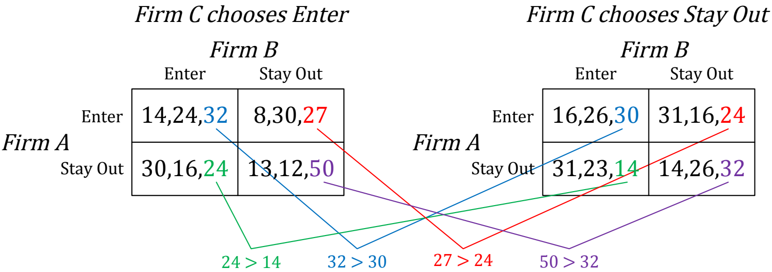
Firm C chooses Stay Out

		Firm B	
		Enter	Stay Out
Firm A	Enter	16,26,30	31,16,24
	Stay Out	31,23,14	14,26,32

Exercise 10

- We can start by looking at the payoffs for firm C (the matrix player).
  - [Recall that the application of IDSDS is insensitive to the deletion order. Thus, we can start deleting strictly dominated strategies for the row, column or matrix player, and still reach the same equilibrium result.]
- In particular, let us compare the third payoff of every cell across both matrices. The next slide provides you a visual illustration of how to do this pairwise comparison across matrices.

Exercise 10



Firm C chooses Enter

Firm A \ *Firm B*	Enter	Stay Out
Enter	14,24,32	8,30,27
Stay Out	30,16,24	13,12,50

Firm C chooses Stay Out

Firm A \ *Firm B*	Enter	Stay Out
Enter	16,26,30	31,16,24
Stay Out	31,23,14	14,26,32

$24 > 14$ $32 > 30$ $27 > 24$ $50 > 32$

Exercise 10

- We find that for firm C (matrix player), entering strictly dominates staying out, i.e., $u_C(s_A, s_B, E) > u_C(s_A, s_B, O)$ for any strategy of firm A, s_A, and firm B, s_B, $32 > 30$, $27 > 24$, $24 > 14$ and $50 > 32$ in the pairwise payoff comparison depicted in the previous slide.
- This allows us to delete the right-hand side matrix (corresponding to firm C choosing to stay out) since it would not be selected by firm C.
  - We can, hence, focus on the left-hand matrix alone (where firm C chooses to enter), which we reproduce on the next slide.

Exercise 10

		Firm B	
		Enter	Stay Out
Firm A	Enter	14,24,32	8,30,27
	Stay Out	30,16,24	13,12,50

Exercise 10

- We can now check that entering is strictly dominated for the row player (firm A), i.e., $u_A(E, s_B, E) < u_A(O, s_B, E)$ for any strategy of firm B, s_B, once we take into account that firm C selects its strictly dominant strategy of entering.
- Specifically, firm A prefers to stay out both when firm B enters (in the left-hand column, since $30 > 14$), and when firm B stays out (in the right-hand column, since $13 > 8$).
  - In other words, regardless of firm B's decision, firm A prefers to stay out.
- This allows us to delete the top row from the previous matrix, since the strategy "Enter" would never be used by firm A, which leaves us with a single row and two columns, as illustrated on the next slide.

Exercise 10

		Firm B	
		Enter	Stay Out
Firm A	Stay Out	30,16,24	13,12,50

Exercise 10

- Once we have done that, the game becomes an individual-decision making problem, since only one player (firm B) must select whether to enter or stay out. Since entering yields a payoff of 16 to firm B, while staying out only entails 12, firm B chooses to enter, given that it regards staying out as a strictly dominated strategy, i.e., $u_B(O, E, E) > u_B(O, O, E)$ where we fix the strategies of the other two firms at their strictly dominant strategies: staying out for firm A and entering for firm C.
- We can thus delete the column corresponding to staying out in the above matrix, as depicted on the next slide.

Exercise 10

		Firm B
		Enter
Firm A	Stay Out	30,16,24

- As a result, the only surviving cell (strategy profile) that survives the application of the iterative deletion of strictly dominated strategies (IDSDS) is that corresponding to (Stay Out, Enter, Enter), which predicts that firm A stays out, while both firms B and C choose to enter.