EconS 301

Homework #8 – *Answer key*

Exercise #1 – Monopoly

Consider a monopolist facing inverse demand function $p(q) = 15 - 3q$, where q denotes units of output. Assume that the total cost of this firm is $TC(q) = 5 + 4q$.

a) Find the monopolist's marginal revenue, and its marginal cost.
Answer:

$$MR = \frac{d(p(q) * q)}{dq} = \frac{d(15q - 3q^2)}{dq} = 15 - 6q \text{ and}$$

$$MC = \frac{d\big(TC(q)\big)}{dq} = \frac{d(5 + 4q)}{dq} = 4$$

b) Set marginal revenue equal to marginal cost, to find the optimal output for this monopolist q^M.
Answer: Let *MR*=*MC*, we obtain the optimal output for monopolist:

$$15 - 6q = 4$$
$$\Rightarrow q^M = \frac{11}{6}$$

c) Find the monopoly price, and profits. Find the consumer surplus, and total welfare.
Answer: Given the optimal output above, we plug into demand function to obtain the optimal price, that is

$$p^M = 15 - 3q^M = 15 - 3 * \frac{11}{6} = \frac{30}{2} - \frac{11}{2} = \frac{19}{2} = 9.5$$

Therefore, the consumer surplus is the triangular area between p^0 when $q = 0$ and p^M, that is

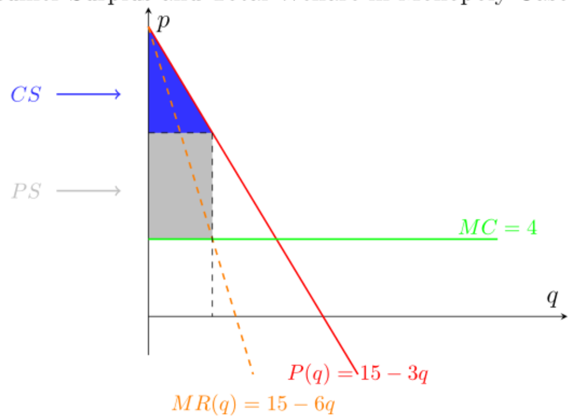
$$CS = \frac{1}{2}(p^0 - p^M) * q^M = \frac{1}{2}(15 - 9.5) * \frac{11}{6} = \frac{60.5}{12} \approx 5.04167$$

And finally, because MC curve is constant, meaning the producer surplus is rectangular the area between p^s when $p^s = MC$ and p^M,so

$$PS = (p^M - p^s) * q^M = (9.5 - 4) * \frac{11}{6} \approx 10.083$$

Therefore, the total welfare is $CS + PS = 10.083 + 5.04167 = 15.125$. See Figure 1 below.

Figure 1: Consumer Surplus and Total Welfare in Monopoly Case with Flat MC



d) Assume now that the market operated under perfect competition. Find the equilibrium output.
Answer: If market operated under perfect competition, we must have optimal price equals to marginal cost. That is, $p^M = MC = 4$. Then we plug it back into demand function and obtain the optimal output,

$$p^M = 15 - 3q = 4$$
$$\Rightarrow q^M = \frac{11}{3}$$

e) Find equilibrium price, profits, consumer surplus, and total welfare, under the perfectly competitive market you analyzed in part (e).
Answer: The consumer surplus is,

$$CS = \frac{1}{2}(p^0 - p^M) * q^M = \frac{1}{2}(15 - 4) * \frac{11}{3} = \frac{121}{6} \approx 20.167$$

And finally, because MC curve is a constant, meaning producer surplus is zero. So W = CS+0 = 20.167

f) Compare your results under a monopoly (part c) and those under a perfectly competitive market (part e).
Answer: Compared with the results in part c and part e, we can see that the monopoly captured more consumer surplus than in the perfectly competitive market, causing a deadweight loss.

$$DWL = W_{PC} - W_{Monopoly} = (CS_{PC} + PS_{PC}) - (CS_{Monopoly} + PS_{Monopoly})$$

Exercise #2 – Monopoly with constant elasticity demand curve

Consider a monopolist facing a constant elasticity demand curve $q(p) = 12p^{-3}$.

a) Assume that the total cost function is $TC(q) = 5 + 4q$. Use the inverse elasticity pricing rule (IEPR) to obtain the profit-maximizing price that this monopolist should charge.
Answer: Since given above is the constant elasticity demand curve, then we know that $\varepsilon_{Q,P} = -3$. Using IEPR rule, we obtain:

$$\frac{P - MC}{P} = -\frac{1}{-3} = \frac{1}{3}$$

And,

$$MC = \frac{d\big(TC(q)\big)}{dq} = \frac{d(5+4q)}{dq} = 4$$

Therefore,

$$\frac{P-4}{P} = \frac{1}{3}$$
$$\Rightarrow p^M = 6$$

b) How would your result in part (a) change if the demand curve changes to $q(p) = 12p^{-5}$, but still assuming the same cost function as in part (a)? Interpret.

Answer: Since given above is the constant elasticity demand curve, then we know that $\varepsilon_{Q,P} = -5$. Using IEPR rule, we obtain:

$$\frac{P-MC}{P} = -\frac{1}{-5} = \frac{1}{5}$$

And,

$$MC = \frac{d\big(TC(q)\big)}{dq} = \frac{d(5+4q)}{dq} = 4$$

Therefore,

$$\frac{P-4}{P} = \frac{1}{5} \Rightarrow p^M = 5$$

The results above can be interpreted as $\varepsilon_{Q,P}$ gets bigger the price that monopoly charge will become smaller. In other words, $\varepsilon_{Q,P}$ and p^M has negative relationship.

c) Assume that the total cost function is $TC(q) = 5 + 2(q)^2$. Use the inverse elasticity pricing rule (IEPR) to obtain the profit-maximizing price that this monopolist should charge.

Answer: Since given above is the constant elasticity demand curve, then we know that $\varepsilon_{Q,P} = -3$. Using IEPR rule, we obtain:

$$\frac{P-MC}{P} = -\frac{1}{-3} = \frac{1}{3}$$

And,

$$MC = \frac{d\big(TC(q)\big)}{dq} = \frac{d(5+2(q)^2)}{dq} = 4q$$

Therefore,

$$\frac{P-4q}{P} = \frac{1}{3} \Rightarrow 4P - 20q = 0$$

Solving for price P, we plug into demand function to obtain

$$P^4 - 72 = 0$$

Solving it,

$$p^M = 2.9130$$

d) How would your result in part (c) change if the demand curve changes to $q(p) = 12p^{-5}$, but still assuming the same cost function as in part (c)? Interpret.

Answer: Since given above is the constant elasticity demand curve, then we know that $\varepsilon_{Q,P} = -5$. Using IEPR rule, we obtain:

$$\frac{P - MC}{P} = -\frac{1}{-5} = \frac{1}{5}$$

And,

$$MC = \frac{d\big(TC(q)\big)}{dq} = \frac{d(5 + 2(q)^2)}{dq} = 4q$$

Therefore,

$$\frac{P - 4q}{P} = \frac{1}{5}$$

$$\Rightarrow 4P - 20q = 0$$

We plug demand function into above function and finally have,

$$4P - 20(12p^{-5}) = 0$$

$$\Rightarrow 4P^6 - 240 = 0$$

Solving it,

$$p^M = 1.9786$$

The results above can be interpreted as $\varepsilon_{Q,P}$ gets bigger the price that monopoly charge will become smaller. In other words, $\varepsilon_{Q,P}$ and p^M has negative relationship.

Exercise #3 - Multiplant monopoly

Consider a monopoly facing inverse demand function $p(Q) = 10 - Q$, where $Q = q_1 + q_2$ denotes the monopolist's production across two plants, 1 and 2. Assume that total cost in plant 1 is given by $TC_1(q_1) = (3 + 2q_1)q_1$, while that of plant 2 is $TC_2(q_2) = [3 + (2 + d)q_2]q_2$, where parameter $d \geq 0$ represents plant 2's inefficiency to plant 1. When $d = 0$, the total (and marginal) cost of both plants coincide; but when $d > 0$, plant 2 has a higher total and marginal cost than plant 1.

a) Write down the monopolist's joint profit maximization problem $\pi = \pi_1 + \pi_2$.

Answer: First, we find our profit for plant 1. In this setting, our profit can be defined as

$$\pi_1 = p(Q)q_1 - TC_1(q_1)$$

$$\pi_1 = (10 - q_1 - q_2)q_1 - (3 + 2q_1)q_1$$

Similarly, for plant 2,

$$\pi_2 = p(Q)q_2 - TC_2(q_2)$$

$$\pi_2 = (10 - q_1 - q_2)q_2 - (3 + (2 + d)q_2)q_2$$

To find the joint profit, we add our profit functions together

$$\pi = \pi_1 + \pi_2$$

$$\pi = (10 - q_1 - q_2)q_1 - (3 + 2q_1)q_1 + (10 - q_1 - q_2)q_2 - (3 + (2 + d)q_2)q_2$$

$$\pi = (10 - q_1 - q_2)(q_1 + q_2) - (3 + 2q_1)q_1 - (3 + (2 + d)q_2)q_2$$

b) Differentiate with respect to the output in plant 1, q_1, and then in plant 2, q_2. Set each expression equal to zero and use your results to find the optimal production in each plant.

Answer: Differentiating our profit function with respect to q_1 gives us

$$\frac{\partial \pi}{\partial q_1} = 10 - 2q_1 - q_2 - 3 - 4q_1 - q_2 = 0$$

Solving for q_1, we obtain

$$4q_1 + 2q_1 = 10 - 3 - q_2 - q_2$$

$$6q_1 = 7 - 2q_2$$

$$q_1 = \frac{7 - 2q_2}{6}$$

Next, we differentiate our profit function with respect to q_2,

$$\frac{\partial \pi}{\partial q_2} = -q_1 + 10 - q_1 - 2q_2 - 3 - 4q_2 - 2dq_2 = 0$$

Solving for q_2, we find

$$2q_2 + 4q_2 + 2dq_2 = 10 - 3 - q_1 - q_1$$

$$6q_2 + 2dq_2 = 7 - 2q_1$$

$$q_2(6 + 2d) = 7 - 2q_1$$

$$q_2 = \frac{7 - 2q_1}{6 + 2d}$$

To find our optimal consumption of q_1 and q_2, we will substitute $q_1 = \frac{7-2q_2}{6}$ that we found into our q_1 function. Thus,

$$q_1 = \frac{7}{6} - \frac{1}{3}\left(\frac{7 - 2q_1}{6 + 2d}\right)$$

We seek to solve for q_1. In order to do that, we first multiply both sides of the above expression by $(18 + 6d)$ to get rid of the fraction, as follows,

$$q_1(18 + 6d) = \frac{7}{6}(18 + 2d) - (7 - 2q_1)$$

$$18q_1 + 6dq_1 = 21 + 7d - 7 + 2q_1$$

$$q_1(16 + 6d) = 14 + 7d$$

$$q_1 = \frac{14 + 7d}{16 + 6d}$$

Next, we plug the optimal output in plant 1, $q_1 = \frac{14+7d}{16+6d}$, back into our $q_2 = \frac{7-2q_1}{6+2d}$ expression to find the optimal amount of production in plant 2.

$$q_2 = \frac{7 - 2\left(\frac{14 + 7d}{16 + 6d}\right)}{6 + 2d} = \frac{7 - 2\left(\frac{14 + 7d}{2(8 + 3d)}\right)}{6 + 2d}$$

$$= \frac{\frac{7(8+3d)-(14+7d)}{8+3d}}{6+2d} = \frac{42+14d}{(8+3d)(6+2d)} = \frac{7}{8+3d}$$

c) How does your result in part (2) change in the inefficiency of plant 2, d. What is the optimal production in each plant if $d = 0$?

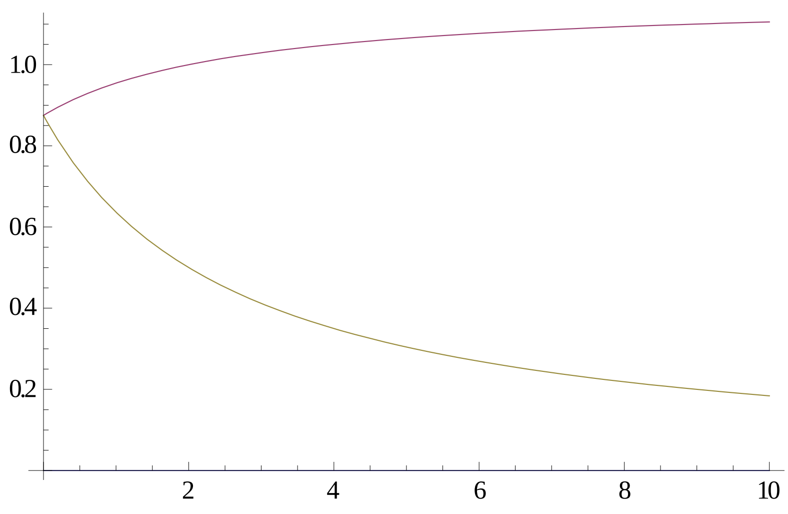
Answer: Since d is in the denominator for q_2, we know that as d increases, or our inefficiency in plant 2 increases, the quantity produced in plant 2 decreases. We also see that as d increases, our optimal production in q_1 is increasing. In summary, the monopolist produces more in plant 1 and less in plant 2 as the inefficiency of plant 2 increases. For illustration purposes, the next figure plots output q_1 (increasing red line) and q_2 (decreasing yellow line). When $d = 0$, our optimal production in plant 1, q_1, becomes

$$q_1 = \frac{14+7d}{16+6d} = \frac{14+(7*0)}{16+(6*0)} = \frac{14}{16} = \frac{7}{8}$$

which coincides with that in plant 2, since

$$q_2 = \frac{7}{8+3d} = \frac{7}{8+(3*0)} = \frac{7}{8}$$

Hence, when $d = 0$, both plants have the same total cost function, leading the monopolist to produce the same amount in plant 1 and 2. This is depicted in the vertical intercept of the figure where $d = 0$. However, when $d > 0$, production in plant 2 becomes more expensive than in plant 1, leading the monopolist to reduce its output on plant 2 but increase its output on plant 1.



As a curiosity, note that the aggregate production in both plants is

$$Q = q_1 + q_2 = \frac{7(4+d)}{16+6d}$$

which is decreasing in the inefficiency of firm 2, since its derivative with respect to d is

$$\frac{\partial Q}{\partial d} = -\frac{14}{(8+3d)^2}$$

which is negative for all values of d. In words, aggregate production decreases in the inefficiency of firm 2. That is, while an increase in d decreases q_2 and increases q_1, the decrease in q_2 dominates, producing a reduction in the total amount that this monopolist produces.

Exercise #4 - Monopsony with linear supply curve

Consider a firm who is the only employer in a small town (a labor monopsony). It faces an international price $p > 0$ for each unit of output that it produces, its production function is $q = L^\alpha$ where $\alpha > 0$, and the labor supply is given by $w(L) = A * L$ where $A > 0$. Intuitively, the firm's production function is concave (when $\alpha < 1$), linear (when $\alpha = 1$), or convex (when $\alpha > 1$).

a) Find the monopsonist marginal revenue product, and its marginal expenditure on labor.
Answer:

$$MRP_L = \frac{\partial TR}{\partial L} = \frac{\partial(p * L^\alpha)}{\partial L} = \alpha p L^{\alpha-1}$$

And,

$$ME_L = \frac{\partial TC}{\partial L} = w(L) + \frac{\partial(w(L))}{\partial L}L = AL + AL = 2AL$$

b) Set the marginal revenue product and marginal expenditure on labor equal to each other to find the optimal number of workers that the firm hires.
Answer: Let $MRP_L = ME_L$, we can find:

$$\alpha p L^{\alpha-1} = 2AL$$

$$\Rightarrow L^* = \left(\frac{\alpha p}{2A}\right)^{\frac{1}{2-\alpha}}$$

c) Find the equilibrium wage in this monopsony.
Answer: Given the optimal number of workers that firm hires, we can find:

$$w(L^*) = A * L^* = A * \left(\frac{\alpha p}{2A}\right)^{\frac{1}{2-\alpha}} = A^{\frac{1-\alpha}{2-\alpha}}\left(\frac{\alpha p}{2}\right)^{\frac{1}{2-\alpha}}$$

d) Assume now that the labor market was perfectly competitive. Find the optimal number of workers the firm hires, and the equilibrium wage in this setting.
Answer: If the labor market was perfectly competitive, then we have $MRP_L = w(L)$, so

$$\alpha p L^{\alpha-1} = A * L$$

$$\Rightarrow L^* = \left(\frac{\alpha p}{A}\right)^{\frac{1}{2-\alpha}}$$

$$\Rightarrow w(L^*) = A * \left(\frac{\alpha p}{A}\right)^{\frac{1}{2-\alpha}} = A^{\frac{1-\alpha}{2-\alpha}}(\alpha p)^{\frac{1}{2-\alpha}}$$

e) Find the deadweight loss of the monopsony.
Answer: The DWL is the area below the marginal revenue product and above the supply curve, between L^* and L^{PC} workers ($\alpha p L^{\alpha-1} = AL \Rightarrow L^{PC} = \left(\frac{\alpha p}{A}\right)^{\frac{1}{2-\alpha}}$), so

$$DWL = \int_{L^*}^{L^{PC}} MRP_L - w(L)\, dL = \int_{L^*}^{L^{PC}} \alpha p L^{\alpha-1} - AL\, dL$$
$$= pL^{\alpha} - \frac{AL^2}{2} \Big|_{L^*}^{L^{PC}}$$
$$= \left(1 - \left(\frac{1}{2}\right)^{\frac{\alpha}{2-\alpha}}\right) p \left(\frac{\alpha p}{A}\right)^{\frac{\alpha}{2-\alpha}} - \left(1 - \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}}\right) \frac{A}{2} \left(\frac{\alpha p}{A}\right)^{\frac{2}{2-\alpha}}$$

f) Evaluate your results in parts (a)-(e) at parameter values $A = 1$, and $p = \$2$. Then, evaluate your results at $\alpha = 1/2$, at $\alpha = 1$, and at $\alpha = 2$. How are your results affected as α increases (that is, as the firm's production function becomes more convex)

Answer:

1. $\alpha = 1/2, A = 1$, and $p = \$2$

$$MRP_L = \alpha p L^{\alpha-1} = \frac{1}{2} p L^{-\frac{1}{2}} = L^{-\frac{1}{2}} \text{ and } ME_L = 2AL = 2L;$$

When $MRP_L = ME_L$, $L^* = \left(\frac{\alpha p}{2A}\right)^{\frac{1}{2-\alpha}} = \left(\frac{p}{4A}\right)^{\frac{2}{3}} \approx 0.62996$ and optimum wage is

$$w(L^*) = A^{\frac{1-\alpha}{2-\alpha}} \left(\frac{\alpha p}{2}\right)^{\frac{1}{2-\alpha}} = A^{\frac{1}{3}} \left(\frac{p}{4}\right)^{\frac{2}{3}} \approx 0.62996;$$

When $MRP_L = w(L)$, $L^* = \left(\frac{\alpha p}{A}\right)^{\frac{1}{2-\alpha}} = \left(\frac{p}{2A}\right)^{\frac{2}{3}} = 1$ and optimum wage is

$$w(L^*) = A^{\frac{1-\alpha}{2-\alpha}} (\alpha p)^{\frac{1}{2-\alpha}} = A^{\frac{1}{3}} \left(\frac{p}{2}\right)^{\frac{2}{3}} = 1;$$

And finally, $DWL = \left(1 - \left(\frac{1}{2}\right)^{\frac{\alpha}{2-\alpha}}\right) p \left(\frac{\alpha p}{A}\right)^{\frac{\alpha}{2-\alpha}} - \left(1 - \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}}\right) \frac{A}{2} \left(\frac{\alpha p}{A}\right)^{\frac{2}{2-\alpha}}$

$$= \left(1 - \left(\frac{1}{2}\right)^{\frac{1}{3}}\right) p \left(\frac{p}{2A}\right)^{\frac{1}{3}} - \left(1 - \left(\frac{1}{2}\right)^{\frac{4}{3}}\right) \frac{A}{2} \left(\frac{p}{2A}\right)^{\frac{4}{3}}$$

$$\approx 0.206p \left(\frac{p}{2A}\right)^{\frac{1}{3}} - 0.30157A \left(\frac{p}{2A}\right)^{\frac{4}{3}} = 0.11043$$

2. $\alpha = 1, A = 1$, and $p = \$2$

$$MRP_L = \alpha p L^{\alpha-1} = p = 2 \text{ and } ME_L = 2AL = 2L;$$

When $MRP_L = ME_L$, $L^* = \left(\frac{\alpha p}{2A}\right)^{\frac{1}{2-\alpha}} = \left(\frac{p}{2A}\right)^{\frac{1}{2}} = 1$ and optimum wage is

$$w(L^*) = A^{\frac{1-\alpha}{2-\alpha}} \left(\frac{\alpha p}{2}\right)^{\frac{1}{2-\alpha}} = \frac{p}{2} = 1;$$

When $MRP_L = w(L)$, $L^* = \left(\frac{\alpha p}{A}\right)^{\frac{1}{2-\alpha}} = \frac{p}{A} = 2$ and optimum wage is

$$w(L^*) = A^{\frac{1-\alpha}{2-\alpha}} (\alpha p)^{\frac{1}{2-\alpha}} = p = 2;$$

And finally, $DWL = \left(1 - \left(\frac{1}{2}\right)^{\frac{\alpha}{2-\alpha}}\right) p \left(\frac{\alpha p}{A}\right)^{\frac{\alpha}{2-\alpha}} - \left(1 - \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}}\right) \frac{A}{2} \left(\frac{\alpha p}{A}\right)^{\frac{2}{2-\alpha}}$

$$= \left(1 - \frac{1}{2}\right) p \frac{p}{A} - \left(1 - \frac{1}{4}\right) \frac{A}{2} \left(\frac{p}{A}\right)^2$$

$$= 0.5\frac{p^2}{A} - 0.375\frac{p^2}{A} = 0.125\frac{p^2}{A} = 0.5$$

3. $\alpha = 2$, $A = 1$, and $p = \$2$

$$MRP_L = \alpha p L^{\alpha-1} = 2pL = 2L \text{ and } ME_L = 2AL = 2L;$$

When $MRP_L = ME_L$, $L^* = \left(\frac{\alpha p}{2A}\right)^{\frac{1}{2-\alpha}} = \left(\frac{p}{A}\right)^{\frac{2}{3}} = 1.5874$ and optimum wage is

$$w(L^*) = A^{\frac{1-\alpha}{2-\alpha}}\left(\frac{\alpha p}{2}\right)^{\frac{1}{2-\alpha}} = A^{\frac{1}{3}}\left(\frac{p}{4}\right)^{\frac{2}{3}} = 0.62996;$$

When $MRP_L = w(L)$, $L^* = \left(\frac{\alpha p}{A}\right)^{\frac{1}{2-\alpha}} = \left(\frac{p}{2A}\right)^{\infty} = 1$ and optimum wage is

$$w(L^*) = A^{\frac{1-\alpha}{2-\alpha}}(\alpha p)^{\frac{1}{2-\alpha}} = A^{\frac{1}{3}}\left(\frac{p}{2}\right)^{\infty} = 1;$$

And finally, $DWL = \left(1 - \left(\frac{1}{2}\right)^{\frac{\alpha}{2-\alpha}}\right) p \left(\frac{\alpha p}{A}\right)^{\frac{\alpha}{2-\alpha}} - \left(1 - \left(\frac{1}{2}\right)^{\frac{2}{2-\alpha}}\right)\frac{A}{2}\left(\frac{\alpha p}{A}\right)^{\frac{2}{2-\alpha}}$

$$= \left(1 - \left(\frac{1}{2}\right)^{\infty}\right) p \left(\frac{p}{2A}\right)^{\infty} - \left(1 - \left(\frac{1}{2}\right)^{\infty}\right)\frac{A}{2}\left(\frac{p}{2A}\right)^{\infty}$$

$$= p - \frac{A}{2} = 1.5$$

Therefore, as α increases,

a) MRP_L increases and ME_L is unchanged.
b) When $MRP_L = ME_L$, L^*, and wage $w(L^*)$ both increases if $1/2 \leq \alpha \leq 1$; but decrease if $1 \leq \alpha \leq 2$. When $MRP_L = w(L)$, L^* and wage $w(L^*)$ increase if $1/2 \leq \alpha \leq 1$ but decrease if $1 \leq \alpha \leq 2$.
c) DWL increases.

Exercise #5 - Cournot vs. Stackelberg models of quantity competition

Consider a market with two firms selling a homogeneous good, and competing in quantities. The inverse demand function is given by $p(q_1, q_2) = 1 - (q_1 + q_2)$, and both firms face a common marginal cost $1 > c > 0$.

a) Cournot model. Set up the profit-maximization problem of firm 1.
b) Differentiate with respect to q_1. Solve for q_1 to obtain firm 1's best response function $q_1(q_2)$.
c) Depict firm 1's best response function $q_1(q_2)$. Interpret.
d) Repeat steps (a)-(b) for firm 2, so you obtain firm 2's best response function $q_2(q_1)$.
e) Find the point where the above two best response functions cross each other.
f) What is the equilibrium price? And the equilibrium profits for each firm?
g) *Stackelberg model*. Consider now that firm 1 acts as a leader and firm 2 as the follower. Find the optimal production for firm 1 and 2 in this sequential competition (Stackelberg model). Then find the equilibrium price, and the profits that each firm makes.
h) *Comparison.* Compare the individual output for each firm, price, and profits in the Cournot model (where firms simultaneously select their output levels) and in the Stackelberg model (where firms sequentially choose their output).

Answer:

a) With Cournot, our profit function is

$$\pi_1 = p(Q)q_1 - cq_1 = (1 - q_1 - q_2)q_1 - cq_1$$

b) Taking the derivative with respect to q_1 gives us

$$\frac{\partial \pi_1}{\partial q_1} : 1 - 2q_1 - q_2 - c = 0$$

Solving for q_1 we find firm 1's best response function

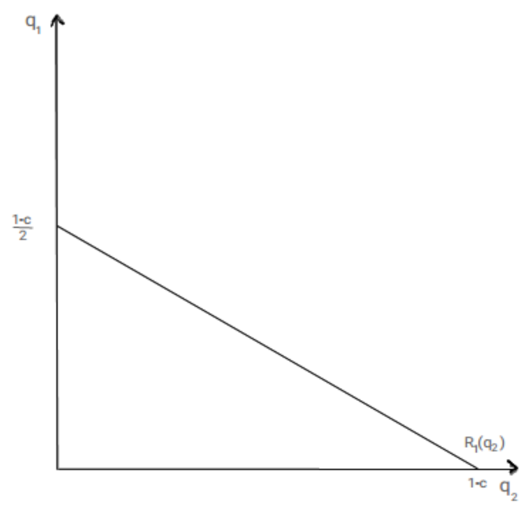
$$q_1(q_2) = \frac{1 - c - q_2}{2}$$

This best response function is often rearranged as follows

$$q_1(q_2) = \frac{1 - c}{2} - \frac{1}{2}q_2$$

The first term does not depend on q_2, and represents the vertical intercept of the best response function, while the second term represents the negative slope of the best response function, $-\frac{1}{2}$. In words, an increase in firm 2's production q_2 by one unit induces firm 1 to reduce its output q_1 by half a unit.

c) Firm 1's best response function can be depicted as:



In this setting, the best response function originates at $\frac{1-c}{2}$ (when $q_2 = 0$), and decreases in q_2 with a slope of $-\frac{1}{2}$.

d) Firm 2 has a symmetric profit function as firm 1 (only the subscripts change). Hence, firm 2's best response function is symmetric to that of firm 1 as follows:

$$q_2 = \frac{1 - c - q_1}{2}$$

e) Since firms are symmetric, we can say that, in equilibrium they must produce the same amount of output $q_1 = q_2$. Plugging this information in firm 1's best response function, we obtain

$$q_1 = \frac{1 - c - q_1}{2}$$

Rearranging, yields

$$2q_1 = 1 - c - q_1$$

Solving for q_1 gives us the equilibrium output for firm 1

$$q_1 = \frac{1 - c}{3}$$

Because $q_1 = q_2$, we can say that firm 2's equilibrium output coincides with that of firm 1

$$q_2 = \frac{1 - c}{3}$$

f) To obtain the equilibrium price, we insert the values of q_1 and q_2 that we found in part (e) into our price function to find the optimal price

$$p(Q) = 1 - q_1 - q_2 = 1 - \left(\frac{1 - c}{3}\right) - \left(\frac{1 - c}{3}\right) = \frac{3 - 1 + c - 1 + c}{3} = \frac{1 + 2c}{3}$$

We then plug this back into our profit function. In this setting:

$$\pi_1 = p(Q)q_1 - cq_1 = \left(\frac{1 + 2c}{3}\right)\left(\frac{1 - c}{3}\right) - c\left(\frac{1 - c}{3}\right)$$

$$= \left(\frac{(1 + 2c)(1 - c)}{9}\right) - \frac{(3c)(1 - c)}{9}$$

Simplifying:

$$\pi_1 = \frac{1 + c - 2c^2 - 3c + 3c^2}{9} = \frac{(1 - c)^2}{9}$$

By symmetry, the profit of firm 2 is also:

$$\pi_2 = \frac{(1 - c)^2}{9}$$

g) Using our best response function above from firm 2, $q_2 = \frac{1-c-q_1}{2}$, we plug it into firm 1's profit function to obtain

$$\pi_1 = \left(1 - q_1 - \frac{1 - c - q_1}{2}\right) q_1 - cq_1$$

Simplifying, yields

$$\pi_1 = \left(\frac{1 + c - q_1}{2}\right) q_1 - cq_1$$

Differentiating with respect to firm 1's output, q_1, we obtain

$$\frac{\partial \pi_1}{\partial q_1}: \quad \frac{1}{2} + \frac{c}{2} - q_1 - c = 0$$

Rearranging, and solving for q_1, we find the equilibrium output of firm 1

$$q_1 = \frac{1-c}{2}$$

We then plug this back into firm 2's best response function:

$$q_2 = \frac{1-c-q_1}{2} = \frac{1-c-\frac{1-c}{2}}{2}$$

Simplifying

$$q_2 = \frac{1}{2} - \frac{c}{2} - \frac{1}{2}\left(\frac{1-c}{2}\right) = \frac{2-2c-1+c}{4} = \frac{1-c}{4}$$

We insert this back into our price function to find the equilibrium price.

$$p = 1 - \frac{(1-c)}{2} - \frac{(1-c)}{4} = \frac{4-2+2c-1+c}{4} = \frac{1+3c}{4}$$

Finally, we plug these back into firm 1's profit function, to obtain firm 1's equilibrium profit

$$\pi_1 = \left(\frac{1+3c}{4}\right)\left(\frac{1-c}{2}\right) - c\left(\frac{1-c}{2}\right) = \frac{1-c+3c-3c^2}{8} - \frac{(c-c^2)}{2}$$

$$= \frac{1+2c-3c^2-4c+4c^2}{8} = \frac{(1-c)^2}{8}$$

Similarly, we find firm 2's profit in equilibrium

$$\pi_2 = \left(\frac{1+3c}{4}\right)\left(\frac{1-c}{4}\right) - c\left(\frac{1-c}{4}\right) = \frac{1-c+3c+3c^2}{16} - \frac{(c-c^2)}{4}$$

$$= \frac{1+2c+3c^2-4c+4c^2}{16} = \frac{(1-c)^2}{16}$$

h) First, we start with our quantity. From above we have the following:

$$Cournot: q_1 = q_2 = \frac{1-c}{3}$$

$$Stackelberg: q_1 = \frac{1-c}{2} \; and \; q_2 = \frac{1-c}{4}$$

Thus, we can see that the leader produces more in Stackelberg than in Cournot, whereas the follower produces more in Cournot than in Stackelberg.

Next, we compare our profits. From above:

$$Cournot: \pi_1 = \pi_2 = \frac{(1-c)^2}{9}$$

$$Stackelberg: \pi_1 = \frac{(1-c)^2}{8} \text{ and } \pi_2 = \frac{(1-c)^2}{16}$$

Again these profits follow the same pattern as above. The leader obtains a higher profit under Stackelberg than under Cournot, whereas the follower's profit is larger under Cournot than Stackelberg.

Finally, we compare price

$$Cournot: p = \frac{1+2c}{3}$$

$$Stackelberg: p = \frac{1+3c}{4}$$

To compare which is higher, we set up an inequality.

$$p_S < p_c$$

$$\frac{1+3c}{4} < \frac{1+2c}{3}$$

Rearranging,

$$3 + 9c < 4 + 8c$$

$$c < 1$$

Thus, as long as $c < 1$, the Stackelberg price is lower than the Cournot price. This condition must hold, as otherwise firms would be producing negative output levels.