EconS 301

Written Assignment #7 - *Answer key*

Exercise #1. Consider a firm with the following Cobb-Douglas product function for labor and capital

$$q = 8L^{1/3}K^{1/2}$$

As we showed in Homework #6, if input prices are $w = \$3$ and $r = \$2$, the total cost function of this firm is

$$TC = 0.3801q^{\frac{6}{5}},$$

which leaves the total cost as a function of output q alone. (Please see part g of Exercise #1 in the answer key to Homework #6 for more details.) Assume that the firm operates in a perfectly competitive market, meaning that the firm faces a price p for each unit of output, which is unaffected by its output decision.

a) Set up the firm's profit-maximization problem.
b) Differentiate with respect to q, set your result equal to zero, and solve for q.
c) *Long-run supply curve*. Use the above total cost function $TC(q)$ to find the average cost function, $AC(q) = \frac{TC(q)}{q}$. Find the minimum of the $AC(q)$ curve, which constitutes the "shut-down price" in a long-run setting.
    - *Hint*: Recall that you can find the minimum of the $AC(q)$ curve using two approaches: (1) obtaining the derivative of $AC(q)$ with respect to q, set it equal to zero, solve for q, and then evaluate $AC(q)$ at the output q you just found; or (2) set $AC(q) = MC(q)$, solve for q, and then evaluate $AC(q)$ at the output q you just found.
d) Use the above "shut-down price" in a long-run setting to describe the firm's long-run supply curve.

Answer:

a) Our profit maximization problem can be defined as $\pi = TR - TC$. In this setting, we defined profit as

$$\pi = pq - 0.3801q^{\frac{6}{5}}$$

b) Differentiating with respect to q gives us

$$\frac{\partial \pi}{\partial q}: p - \left(\frac{6}{5}\right)0.3801q^{\frac{1}{5}} = 0$$

Solving for q

$$0.45612q^{\frac{1}{5}} = p$$

$$q^{\frac{1}{5}} = \frac{p}{0.45612}$$

$$q = \left(\frac{p}{0.45612}\right)^5$$

c) We define average cost as $AC(q) = \frac{TC(q)}{q}$. Thus our average cost is

$$AC(q) = \frac{0.3801q^{\frac{6}{5}}}{q}$$
$$AC(q) = 0.3801q^{\frac{1}{5}}$$

To find the minimum of the average cost function, we set $AC(q) = MC(q)$ and solve for q. Since we already have our average cost, we need to calculate our marginal cost. We know that our marginal cost can be defined as $MC(q) = \frac{\partial TC}{\partial q}$. Thus,

$$MC(q) = \left(\frac{6}{5}\right)0.3801q^{\frac{1}{5}} = \ 0.45612q^{\frac{1}{5}}$$

Now, we set $AC(q) = MC(q)$, and solve for q. In this setting,

$$AC(q) = MC(q)$$
$$0.3801q^{\frac{1}{5}} = \ 0.45612q^{\frac{1}{5}}$$

In order for this inequality to hold, our $q = 0$. We then plug this back into our average cost function to find our shut down price. Plugging in $q = 0$, gives us

$$Min\ AC = 0.3801(0)^{\frac{1}{5}} = 0$$

This is our "shut-down price". If price is below this point, the firm will choose to shut down and not produce anything.

d) Since the firm is in a perfectly competitive market, we set $p = MC$ and solve for q to get our long-run supply. That is the optimal q to produce. As long as $p \ \geq AC$, the firm will choose to produce. If $p < AC$, the firm would choose to shut-down. From part (a), we found that $q = \left(\frac{p}{0.45612}\right)^5$ Thus, we can define our supply curve as

$$Supply\ (p) = \begin{cases} 0\ if\ p < 0 \\ \left(\frac{p}{0.45612}\right)^5\ if\ p \geq 0 \end{cases}$$

Since we know p is always positive, the firm will always produce.

Exercise #2. Repeat your analysis of Exercise #1 for the following linear production function

$$q = 7L + 4K$$

As we showed in Homework #6, if input prices are $w = \$3$ and $r = \$2$, the total cost function of this firm is

$$TC = \frac{3}{7}q = 0.42857q$$

Leaving the total cost as a function of output q alone. (Please see part g of Exercise #2 in the answer key to Homework #6 for more details.) Assume that the firm operates in a perfectly competitive market, meaning that the firm faces a price p for each unit of output, which is unaffected by its output decision.

Answer:

a) We defined our total profit as total revenue minus total cost. In this setting, our profit can be defined as

$$\pi = pq - \frac{3}{7}q$$

b) Taking the derivative with respect to q, gives us

$$\frac{\partial \pi}{\partial q} : p - \frac{3}{7} = 0$$

$$p = \frac{3}{7}$$

In this case, our marginal cost is constant $\left(\frac{3}{7}\right)$, so we don't have a q to solve for.

c) Our average cost can be defined as $AC(q) = \frac{TC(q)}{q}$. Thus our average cost is

$$AC(q) = \frac{\frac{3}{7}q}{q} = \frac{3}{7}$$

Setting this equal to our marginal cost gives us

$$AC(q) = MC(q)$$

$$\frac{3}{7} = \frac{3}{7}$$

Here we also see that there is no q to solve for because average cost and marginal cost are constant. They don't depend on q. Thus our shut-down price is $\frac{3}{7}$ in this case. If price is below $\frac{3}{7}$, the firm would choose to shut-down and produce nothing.

d) Since we can see that our $AC = MC = p$, we know that our demand supply curve will be a horizontal line. This means that as long as $p = MC$, the firm can choose any amount of q to produce. However, if our price drops below marginal cost (even by just a cent), the firm will choose to shut down. They would never produce anything if $p < MC$. Hence, we can define our supply curve as

$$Supply\ (p) = \begin{matrix} 0\ if\ p < \frac{3}{7} \\ \infty\ if\ p \geq \frac{3}{7} \end{matrix}$$

Exercise #3. Repeat your analysis of Exercise #1 for the following fixed-proportion production function

$$q = \min\{2L, 3K\}$$

As we showed in Homework #6, if input prices are $w = \$3$ and $r = \$2$, the total cost function of this firm is

$$TC = \frac{13}{6}q = 2.167q$$

which leaves the total cost as a function of output q alone. (Please see part g of Exercise #3 in the answer key to Homework #6 for more details.) Assume that the firm operates in a perfectly competitive market, meaning that the firm faces a price p for each unit of output, which is unaffected by its output decision.

Answer:

a) We can define our profit function as $\pi = TR - TC$. Thus,

$$\pi = pq - \frac{13}{6}q$$

b) Next, we take the derivative of our profit function and solve for q.

$$\frac{\partial \pi}{\partial q}: p - \frac{13}{6} = 0$$

$$p = \frac{13}{6}$$

We can see our marginal cost is constant because it does not depend on q. Hence, there is no q to solve for.

c) In this setting, our average cost is

$$AC = \frac{\frac{13}{6}q}{q} = \frac{13}{6}$$

To find the minimum of the $AC(q)$ curve, we set $AC(q) = MC(q)$ and solve for q.

$$\frac{13}{6} = \frac{13}{6}$$

Just as in question 2, we have no q to solve for because both our average cost and marginal cost are constant because they don't depend on q. Thus our shut down price is $\frac{13}{6}$. If price drops below $\frac{13}{6}$, the firm would choose to shut down.

d) Again we see that since $AC(q) = MC(q) = p$, our supply curve is completely horizon. This tells us that as long as $p \geq MC$, the firm can choose any amount of q to produce. However, if the price drops, $p < MC$, the firm would choose to shut down. In this setting, we define our supply function as

$$Supply\ (p) = \begin{cases} 0\ if\ p < \frac{13}{6} \\ \infty\ if\ p\ \geq \frac{13}{6} \end{cases}$$

Exercise #4. Consider a firm with total costs

$$TC = a + bq + cq^2$$

a) Identify the fixed cost FC, and the variable cost of this firm, $VC(q)$. (Each of them is just a part of the total cost.)
b) Find the average cost $AC(q)$, and the marginal cost $MC(q)$.
c) *Long-run supply curve*. Find the minimum of the $AC(q)$ curve, which constitutes the "shut-down price" in a long-run setting. Use this "shut-down price" to describe the firm's long-run supply curve.
d) Evaluate the long-run supply curve at $a = 10, b = 4$, and $c = 2$.
e) *Short-run supply curve*. Use your results from part (a) to find the average variable cost function, $AVC(q) = \frac{VC(q)}{q}$. Find the minimum of the $AVC(q)$ curve, which constitutes the "shut-down price" in a short-run setting. Use this "shut-down price" to describe the firm's short-run supply curve.
f) Evaluate the short-run supply curve at $a = 10, b = 4$, and $c = 2$.

Answer:

a) We know that the fixed cost is the cost the firm incurs no matter how much they choose to produce. In other words, our fixed cost is not a function of q. With our total cost function above, we know our fixed cost is a because it doesn't depend on q.

   On the other hand, the variable cost is a function of q because it will increase or decrease depending on how much the firm chooses to produce. In this case, our variable cost is $bq + cq^2$.

b) We define our average cost as $AC(q) = \frac{TC(q)}{q}$. In this setting, our average cost is

$$AC(q) = \frac{a + bq + cq^2}{q} = \frac{a}{q} + b + cq$$

   Similarly, we can define our marginal cost as $MC(q) = \frac{\partial TC(q)}{\partial q}$. In this setting, our marginal cost is

$$MC(q) = b + 2cq$$

c) To find the minimum of the $AC(q)$ curve, we set our average cost equal to our marginal cost and solve for q. Thus,

$$AC(q) = MC(q)$$

$$\frac{a}{q} + b + cq = b + 2cq$$

We multiply the entire equation by q to get rid of the fraction.

$$a + bq + cq^2 = bq + 2cq^2$$

Solving for q gives us

$$cq^2 = a$$

$$q = \left(\frac{a}{c}\right)^{\frac{1}{2}}$$

Next, we plug this optimal q into our average cost function to find the minimum AC.

$$Min\ AC = \frac{a}{q} + b + cq = \frac{a}{\left(\frac{a}{c}\right)^{\frac{1}{2}}} + b + c\left(\frac{a}{c}\right)^{\frac{1}{2}}$$
$$= (ac)^{\frac{1}{2}} + b + (ac)^{\frac{1}{2}} = 2(ac)^{\frac{1}{2}} + b$$

Thus, our shut down price is anytime that $p < AC$ which we can rewrite as

$$p < 2(ac)^{\frac{1}{2}} + b.$$

To find our optimal production at this shut-down price, we set $p = MC(q)$ and solve for q. In this setting,

$$p = b + 2cq$$

$$2cq = p - b$$

$$q = \frac{p-b}{2c}$$

Thus we can define our long-run supply as

$$Supply\ (p) = \begin{matrix} 0\ if\ p < 2(ac)^{\frac{1}{2}} + b \\ \left(\frac{p-b}{2c}\right) if\ p \geq 2(ac)^{\frac{1}{2}} + b \end{matrix}$$

d) First, we plug in $a = 10, b = 4,$ and $c = 2$ into our shut down price. Thus,

$$Min\ AC = 2(ac)^{\frac{1}{2}} + b = 2(10 * 2)^{\frac{1}{2}} + 4 = 12.944$$

We then plug in the same values into our optimal q found in part (c). Thus,

$$q = \frac{p-b}{2c} = \frac{p-4}{4}$$

Now, we can rewrite our supply function as

$$Supply\ (p) = \begin{cases} 0\ if\ p < 12.944 \\ \dfrac{p-4}{4}\ if\ p \geq 12.944 \end{cases}$$

e) From above, we know that our variable cost is $bq + cq^2$. Thus our average variable cost can be defined as

$$AVC(q) = \frac{AV(q)}{q} = \frac{bq + cq^2}{q} = b + cq$$

To find the minimum of the $AVC(q)$ curve, we set our $AVC = MC$ and solve for q. Thus,

$$b + cq = b + 2cq$$

$$cq = 0$$

In order for this equality to hold, $q = 0$. Plugging this value back into our $AVC(q)$ gives us

$$Min\ AVC = b + c(0) = b$$

Thus our "shut-down price" is anytime $p < AVC$ or $p < b$.

Next, we must find the quantity the firm produces if $p \geq AVC$. In order to find that quantity, we set $p = MC$ and solve for q. In this setting,

$$p = b + 2cq$$

$$2cq = p - b$$

$$q = \frac{p-b}{2c}$$

We can then define our supply function as

$$Supply\ (p) = \begin{cases} 0\ if\ p < b \\ \dfrac{p-b}{2c}\ if\ p \geq b \end{cases}$$

f) Next, we plug $a = 10, b = 4,$ and $c = 2$ into our shut down price. Thus,

$$Min\ AC = b = 4$$

We then plug in the same values into our optimal q found in part (c). Thus,

$$q = \frac{p-b}{2c} = \frac{p-4}{4}$$

Thus, we can define our supply function as

$$Supply\ (p) = \begin{cases} 0\ if\ p < 4 \\ \dfrac{p-4}{4}\ if\ p \geq 4 \end{cases}$$

Let's now assume that $c = 0$. In this setting, our total cost becomes $TC = a + bq$, and our marginal cost becomes $MC = b$. When our marginal cost is constant, our supply becomes completely horizontal at price b. In other words, our supply curve loses its convexity. This is the same thing that occurred in questions 2 and 3 above.

If we now assume that $b = 0$ and $c = 0$, our new total cost is defined as $TC = a$. When we take the derivative of total cost with respect to q, we get that $MC = 0$. Thus, our new marginal curve is again completely horizontal. However, because $MC = 0$, our marginal cost curve is now located completely on the q axis. The firm is only concerned with the marginal cost, so essentially their cost is 0.

Note that we still incur an average cost of $AC = \frac{a}{q}$. This will still influence our shut-down price in the long-run. If we set $AC(q) = MC(q)$,

$$AC(q) = MC(q)$$

$$\frac{a}{q} = 0$$

When we try to solve for q, we get that q is undefined. This means our average cost curve never crosses our marginal cost. Because average cost is greater than marginal cost, the firm never produces anything.

Finally assuming that $a = 0, b = 0,$ and $c = 0$, our new total cost is 0. The firm incurs no fixed cost and no variable cost. In this setting, our marginal cost is also 0. Because marginal cost is constant and equal to 0, we know our curve is flat and located completely on the q axis.

In this setting, our average cost can be defined as $\frac{TC}{q} = \frac{0}{q}$. Setting this equal to marginal cost to find our shut-down price gives us

$$AC(q) = MC(q)$$

$$\frac{0}{q} = 0$$

For this equality to hold, $q \neq 0$. Thus the firm would choose any positive amount of q to produce. Whatever q the firm chooses to produce will give them an average cost that is also 0. This means that the average cost curve will be flat and located on the q axis just like our marginal cost curve. Hence when the firm has no average cost and no marginal cost, they will choose to produce any amount of $q > 0$ and never shut down as long as $p > 0$.