EconS 301
Written Assignment #6

Answer Key

Exercise #1. Consider a firm with the following Cobb-Douglas product function for labor and capital

$$q = 8L^{1/3}K^{1/2}$$

a) Find the marginal rate of technical substitution, $MRTS$, for this firm.
b) Solve the firm's cost-minimization problem (CMP) for an output level of $q = 100$ units, and assuming that input prices are $w = \$3$ and $r = \$2$ for labor and capital, respectively. Which are the cost-minimizing amount of labor and capital that the firm hires?
   - [*Hint*: Set the $MRTS$ found in part (a) equal to the input price ratio. Then, insert your result into the firm's production function evaluated at $q = 100$ units. Last, solve for either L to find the cost-minimizing amount of labor, or for K to find the cost-minimizing amount of capital.]
c) What is the firm's total cost from hiring the optimal units of labor and capital you found in part (b)?
d) Replicate part (b), but now without assuming a specific output level q, or specific values for input prices w and r. Follow the same steps as in part (b), but notice that now you will find that the optimal workers hired, L, is not a number, but a function of q, w and r. This is what we refer as the firm's "demand for labor." A similar argument applies to the optimal amount of capital that the firm hires, K, which we refer as the firm's "demand for capital."
e) Confirm that if we evaluate the expressions you found in part (d) at the same parameter values considered in part (b) —that is an output of $q = 100$ units and input prices of $w = \$3$ and $r = \$2$— we obtain the same optimal units of labor and capital as in part (b).
f) What is the firm's total cost from hiring the optimal units of labor and capital you found in part (d)? As opposed to part (c) where your result was a dollar amount, your result now will be a function of q, w and r.
g) Confirm that if we evaluate the total cost found in part (f) at the same parameter values considered in part (c) —that is an output of $q = 100$ units and input prices of $w = \$3$ and $r = \$2$— we obtain the same total cost as in part (c).

Answer:

a) First, we find our MP_L and our MP_K.

$$MP_L = 8 * \frac{1}{3} L^{-\frac{2}{3}} K^{\frac{1}{2}} = \frac{8K^{\frac{1}{2}}}{3L^{\frac{2}{3}}}$$

$$MP_K = 8 * \frac{1}{2} L^{\frac{1}{3}} K^{-\frac{1}{2}} = \frac{4L^{\frac{1}{3}}}{K^{\frac{1}{2}}}$$

Thus our MRTS is

$$\frac{MP_L}{MP_K} = \frac{\frac{8K^{\frac{1}{2}}}{3L^{\frac{2}{3}}}}{\frac{4L^{\frac{1}{3}}}{K^{\frac{1}{2}}}} = \frac{2K}{3L}$$

b) First, we set our MRTS equal to our input price ratio

$$\frac{2K}{3L} = \frac{3}{2}$$

Solving for K, we obtain

$$4K = 9L$$

$$K = \frac{9}{4}L$$

We plug the K we found into our production function $q = 8L^{1/3}K^{1/2}$ when $q = 100$, that is, $100 = 8L^{1/3}K^{1/2}$, which entails

$$100 = 8L^{\frac{1}{3}}\left(\frac{9}{4}L\right)^{\frac{1}{2}}$$

which simplifies to $3L^{5/6} = 25$. Solving for labor L, we obtain

$$L = \left(\frac{25}{3}\right)^{6/5} = 12.73$$

Plugging this back into our K equation from above gives us

$$K = \frac{9}{4}(12.73) = 28.64$$

Thus our optimal input bundle is L = 12.73 and K = 28.64.

c) Our total cost is the amount spend on both L and K at prices w and r respectively. Thus our total cost can be written as

$$TC = wL + rK$$

Plugging in our values found in part (b) gives us

$$TC = 3(12.73) + 2(28.64) = 95.475$$

d) We take our MRTS from above, $\frac{2K}{3L}$, and set it equal to our new input price ratio, $\frac{w}{r}$.

$$\frac{2K}{3L} = \frac{w}{r}$$

Solving for K gives us

$$2Kr = 3Lw$$

$$K = \frac{3Lw}{2r}$$

Next, we plug this into our production function from above, $q = 8L^{1/3}K^{1/2}$, which yields

$$q = 8L^{\frac{1}{3}}\left(\frac{3Lw}{2r}\right)^{\frac{1}{2}}$$

Solving for L gives us

$$L = \left(\frac{q\left(2^{\frac{1}{2}}\right)\left(r^{\frac{1}{2}}\right)}{8\left(3^{\frac{1}{2}}\right)\left(w^{\frac{1}{2}}\right)}\right)^{6/5}$$

Thus our demand for labor is

$$L = \frac{1.515r^{\frac{3}{5}}q^{\frac{6}{5}}}{23.44w^{\frac{3}{5}}}$$

In order to find our demand for capital, we first plug the above demand for labor back into our equation from above, $K = \frac{3Lw}{2r}$, and solve for K.

$$K = \frac{3w}{2r}\left(\frac{q\left(2^{\frac{1}{2}}\right)\left(r^{\frac{1}{2}}\right)}{8\left(3^{\frac{1}{2}}\right)\left(w^{\frac{1}{2}}\right)}\right)^{6/5}$$

Simplifying gives us

$$K = \frac{3^{\frac{2}{5}}\left(w^{\frac{2}{5}}\right)\left(q^{\frac{6}{5}}\right)}{2^{\frac{2}{5}}\left(8^{\frac{6}{5}}\right)\left(r^{\frac{2}{5}}\right)}$$

Hence our demand for capital can be defined as

$$K = \frac{4.5463w^{\frac{2}{5}}q^{\frac{6}{5}}}{46.881r^{\frac{2}{5}}}$$

e) We start with our demand for labor equation and plug in w = 3, r = 2, and q = 100.

$$L = \frac{1.515r^{\frac{3}{5}}q^{\frac{6}{5}}}{23.44w^{\frac{3}{5}}} = \frac{1.515\left(2^{\frac{3}{5}}\right)\left(100^{\frac{6}{5}}\right)}{23.44\left(3^{\frac{3}{5}}\right)} = 12.73 \text{ units}$$

which coincides with the quantity we found in part (b).

Next, we use our demand for capital equation and plug in the same values.

$$K = \frac{4.5463w^{\frac{2}{5}}q^{\frac{6}{5}}}{46.881r^{\frac{2}{5}}} = \frac{4.5463\left(3^{\frac{3}{5}}\right)\left(100^{\frac{6}{5}}\right)}{46.881\left(2^{\frac{2}{5}}\right)} = 28.648 \text{ units}$$

which also coincides with our quantity from part (b).

f) We know our total cost is the amount of money we spend to produce q units of output, that is,

$$TC = wL + rK$$

Substituting in the demand for labor and capital that we found in part (d) gives us

$$TC = w\left(\frac{1.515r^{\frac{3}{5}}q^{\frac{6}{5}}}{23.44w^{\frac{3}{5}}}\right) + r\left(\frac{4.5463w^{\frac{2}{5}}q^{\frac{6}{5}}}{46.881r^{\frac{2}{5}}}\right)$$

Simplifying, we obtain

$$TC = \left(\frac{1.515r^{\frac{3}{5}}q^{\frac{6}{5}}w^{\frac{2}{5}}}{23.44}\right) + \left(\frac{4.5463w^{\frac{2}{5}}q^{\frac{6}{5}}r^{\frac{3}{5}}}{46.881}\right)$$

g) We take our total cost function found in part (f) and plug in the values from part (c), namely w = 3, r = 2, and q = 100. Thus we can define our total cost as

$$TC = \left(\frac{1.515r^{\frac{3}{5}}q^{\frac{6}{5}}w^{\frac{2}{5}}}{23.44}\right) + \left(\frac{4.5463w^{\frac{2}{5}}q^{\frac{6}{5}}r^{\frac{3}{5}}}{46.881}\right)$$

$$= \left(\frac{1.515\left(2^{\frac{3}{5}}\right)\left(100^{\frac{6}{5}}\right)\left(3^{\frac{2}{5}}\right)}{23.44}\right) + \left(\frac{4.5463\left(3^{\frac{2}{5}}\right)\left(100^{\frac{6}{5}}\right)\left(2^{\frac{3}{5}}\right)}{46.881}\right)$$

$$= 38.18753 + 57.29645 = \$95.484$$

Once again, we see that the value we found coincides with that in part (c).

If we were to evaluate this cost function using only w = 3 and r = 2, we would be left with total cost as a function of q, as follows

$$TC = = \left(\frac{1.515\left(2^{\frac{3}{5}}\right)\left(q^{\frac{6}{5}}\right)\left(3^{\frac{2}{5}}\right)}{23.44}\right) + \left(\frac{4.5463\left(3^{\frac{2}{5}}\right)\left(q^{\frac{6}{5}}\right)\left(2^{\frac{3}{5}}\right)}{46.881}\right)$$

$$= .15203q^{\frac{6}{5}} + .228101q^{\frac{6}{5}}$$

$$= .3801q^{\frac{6}{5}}$$

Exercise #2. Repeat your analysis of Exercise #1 for the following linear production function

$$q = 7L + 4K$$

Answer:

a) First, we find our MP_L and our MP_K.

$$MP_L = 7$$
$$MP_K = 4$$

Our MRTS can be defined as

$$\frac{MP_L}{MP_K} = \frac{7}{4}$$

We can see that in this setting, our MRTS is constant, since it is not a function of L or K.

b) We set our MRTS equal to our price ratio $\frac{w}{r}$

$$\frac{7}{4} \neq \frac{3}{2}$$

Since this equality doesn't hold, we know we have perfect substitutes. Thus we will check which input has the largest marginal utility per dollar (the highest "bang for the buck"), as follows

$$\frac{MP_L}{w} > \frac{MP_K}{r} \text{ since } \frac{7}{3} > \frac{4}{2}$$

Given that 2.333 > 2, we know that the firm will choose to only hire labor, L>0, but no capital, K=0.

Because we are only hiring units of L, we can rewrite our production function as

$$q = 7L$$
$$100 = 7L$$

Solving for L

$$L = \frac{100}{7} = 14.286$$

c) Since we are only hiring units of L, our total cost will be

$$TC = wL + rK = 3\ (14.286) + r0 = 42.858$$

d) We know that the firm only uses labor (*L*>0 but *K*=0) when the "bang for the buck" of hiring more units of labor exceeds that from hiring more capital, that is, $\frac{MP_L}{w} > \frac{MP_K}{r}$ which in this case is $\frac{7}{w} > \frac{4}{r}$. If the values of *w* and *r* satisfy this inequality, we obtain

$$q = 7L, \text{ or } L = \frac{q}{7}$$

If, instead, the values of *w* and *r* satisfy $\frac{7}{w} < \frac{4}{r}$, the firm only uses capital but no labor, entailing

$$q = 4K, \text{ or } K = \frac{q}{4}$$

e) Next, we will plug our values for w, r, and q into our equation above. This implies that condition $\frac{7}{w} > \frac{4}{r}$ must hold, which means that the firm only hires labor, namely,

$$L = \frac{q}{7} = \frac{100}{7} = 14.286$$

We see that this answer coincides with what we found above in part (b).

f) When $\frac{7}{w} > \frac{4}{r}$ the firm hires labor alone (no capital), yielding a total cost of

$$TC = wL + rK = w\left(\frac{q}{7}\right) + r\,(0) = \frac{wq}{7}$$

If, instead, $\frac{7}{w} < \frac{4}{r}$ holds, the firm hires capital alone, entailing a total cost of

$$TC = wL + rK = w\,(0) + r\left(\frac{q}{4}\right) = \frac{rq}{4}$$

g) Plugging in our values, we obtain that condition $\frac{7}{w} > \frac{4}{r}$ holds, and thus the firm hires labor alone. As a consequence, our above equation for the total cost $TC = \frac{wq}{7}$ becomes

$$TC = \frac{wq}{7} = \frac{3 * 100}{7} = 42.857$$

Again, we see this answer matches what we found in part (c).

If we were to evaluate this cost function using only w = 3 and r = 2, we would be left with total cost as a function of q, as follows

$$TC = \frac{wq}{7} = \frac{3}{7}q = 0.42857q$$

Exercise #3. Repeat your analysis of Exercise #1 for the following fixed-proportion production function

$$q = \min\{2L, 3K\}$$

Answer:

a) The $MRTS$ is not well defined since we have a kink in the isoquant. In particular, the kink occurs at the point where the two terms inside the “min” operator are equal to each other, that is, $2L = 3K$.

b) Because our MRTS is undefined at the kink, we cannot use our tangency condition. Since we know this is a fixed-proportion production, we know that the firm will produce at the kink of the isoquant, which implies that the two terms inside the “min” operator must be equal to each other, that is, $2L = 3K$. This implies that the production function can be rewritten as $q = 2L$, or as $q = 3K$ since $2L = 3K$.

To find our demand for labor, we set $q = 2L$. Since $q = 100$, that is, $100 = 2L$, and we solve for labor L, to obtain

$$L = \frac{100}{2} = 50$$

Similarly, we set $q = 3K$. Hence, our demand for capital (K) is

$$100 = 3K$$

$$K = \frac{100}{3} = 33.33$$

We can see this holds because if we plug this back into our production function.

$$q = \min\{2L, 3K\} = \min\{2 * 50, 3 * 33.33\} = \min\{100, 100\} = 100$$

c) Our total cost in this case is defined as

$$TC = wL + rK$$
$$TC = 3(50) + 2(33.33) = 216.67$$

d) Since we have a fixed proportion production plan, we know that $q = 2L = 3K$. Thus our demand for labor (L) is

$$q = 2L$$
$$L = \frac{q}{2}$$

Our demand for capital (K) is

$$q = 3K$$
$$K = \frac{q}{3}$$

e) Plugging in our values for L gives us

$$L = \frac{q}{2} = \frac{100}{2} = 50$$

Plugging in our values for K gives us

$$K = \frac{q}{3} = \frac{100}{3} = 33.33$$

We can see that these values coincide with what we found in part (b).

f) Substituting the values we found in part (d) into our total cost function gives us

$$TC = wL + rK = w\left(\frac{q}{2}\right) + r\left(\frac{q}{3}\right) = \frac{wq}{2} + \frac{rq}{3}$$

g) Plugging in our values for w, r, and q gives us

$$TC = \frac{wq}{2} + \frac{rq}{3} = \frac{3 * 100}{2} + \frac{2 * 100}{3} = 150 + 66.67 = 216.67$$

Again we see this matches the value we found in part (c).

If we only plug in w = 3 and r = 2, we can evaluate total cost, which becomes a function of q alone, as follows,

$$TC = \frac{wq}{2} + \frac{rq}{3} = \frac{3}{2}q + \frac{2}{3}q$$
$$= \frac{13}{6}q = 2.167q$$