EconS 301
Written Assignment #5

Answer key

Exercise #1. Consider a firm with the following Cobb-Douglas product function for labor and capital

$$q = L^{1/2}K^{1/2}$$

a) Consider an output level of $q = 100$. Find the expression of the isoquant for this output level. [*Hint*: Solve for capital, *K*.]
b) Find the marginal product of labor, MP_L. Is it increasing, decreasing, or constant in the units of labor, *L*, that the firm uses?
c) Find the marginal product of capital, MP_K. Is it increasing, decreasing, or constant in the units of capital, *K*, that the firm uses?
d) Use your result in parts (b)-(c) to find the marginal rate of technical substitution, *MRTS*, for this firm.
e) Is the *MRTS* increasing or decreasing in the units of labor, *L*? What does that imply about the shape of the isoquant?
f) Given your result in part (d), what can you say about the firm's ability to substitute one input for another?
g) Assume now that the firm were to increase all inputs by a common factor $\lambda > 0$. What happens to the output that the firm produces? [This question asks you to check if the firm's production function exhibits increasing, decreasing, or constant returns to scale.]

Solution:

a) The firm's production function becomes $100 = L^{1/2}K^{1/2}$. Squaring both sides, we obtain $10{,}000 = LK$. Solving for K yields the isoquant $K = \frac{10{,}000}{L}$.

b) Differentiating with respect to *L*, we obtain
$$MP_L = \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}} = \frac{\sqrt{K}}{2\sqrt{L}}.$$
which is decreasing in the units of labor L, which only show up in the denominator.

c) Differentiating with respect to *K*, we find
$$MP_K = \frac{1}{2}K^{-\frac{1}{2}}L^{\frac{1}{2}} = \frac{\sqrt{L}}{2\sqrt{K}}$$
which is also decreasing in the units of capital, *K*, which only show up in the denominator.

d) The *MRTS* is the ratio of marginal products found above, that is,
$$MRTS = \frac{MP_L}{MP_K} = \frac{\frac{\sqrt{K}}{2\sqrt{L}}}{\frac{\sqrt{L}}{2\sqrt{K}}} = \frac{\sqrt{K}}{2\sqrt{L}}\frac{2\sqrt{K}}{\sqrt{L}} = \frac{2\sqrt{K}\sqrt{K}}{2\sqrt{L}\sqrt{L}} = \frac{2K}{2L} = \frac{K}{L}.$$

e) The *MRTS* decreases in *L* (as we move rightward in the horizontal axis). Graphically, the isoquant becomes flatter as we move rightward, implying that the isoquant is bowed in from the origin.

f) The firm is willing to give up many units of capital to obtain one more unit of labor when capital is abundant but labor is scarce (points on the upper left-hand side of the isoquant). However, when labor becomes more abundant and capital more scarce, the firm is willing to give up few units of capital to obtain one more worker.

g) Increasing all inputs by a common factor λ, we obtain

$$(\lambda L)^{\frac{1}{2}}(\lambda K)^{\frac{1}{2}} = \lambda^{\frac{1}{2}}\lambda^{\frac{1}{2}}L^{\frac{1}{2}}K^{\frac{1}{2}} = \lambda(L^{\frac{1}{2}}K^{\frac{1}{2}}) = \lambda q$$

Therefore, output increases by λ as well, implying that the firm's production function exhibits constant returns to scale. Note that this result could be found immediately by noticing that in Cobb-Douglas production functions, the sum of the exponents indicates whether the firm exhibits constant returns to scale (if the sum is equal to 1), increasing returns to scale (if the sum is larger than 1), or decreasing returns to scale (if the sum is smaller than 1).

Exercise #2. Repeat your analysis of Exercise #1 for the following linear production function

$$q = 3L + 5K$$

Solution:

a) Since the firm seeks to produce 100 units, its production function becomes 100 = $3L + 5K$. Rearranging, and solving for K, we obtain the isoquant

$$K = 20 - \frac{3}{5}L$$

b) Differentiating with respect of L, we obtain the marginal product of labor $MP_L = 3$, which is constant.

c) Differentiating with respect of K, we obtain the marginal product of capital $MP_K = 5$, which is constant.

d) The $MRTS$ is the ratio of marginal products,

$$MRTS = \frac{MP_L}{MP_K} = \frac{3}{5}.$$

e) The $MRTS$ is a constant number, which implies the isoquant has a constant slope. That is, the isoquant is a straight line that crosses both the K-axis (vertical axis) and L-axis (horizontal axis).

f) The ability of this firm to substitute capital and labor is fixed since it has a constant $MRTS$. If the firm had one more unit of labor, it would need to give up $\frac{3}{5}$ units of capital.

g) Increasing all inputs by a common factor λ, we obtain

$$3(\lambda L) + 5(\lambda K) = \lambda(3L + 5K) = \lambda q$$

Thus, the firm exhibits constant returns to scale because the output increases by λ as well.