EconS 301

Homework #4 – Answer key

Exercise #1. Consider a consumer with the following Cobb-Douglas utility function for two goods, 1 and 2,

$$u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

Assume that the consumer faces a price of \$1 for good 2, and a total income of \$*I*. The price of good 1 is left unrestricted as p_1.

a) Find the marginal rate of substitution, *MRS*, for this consumer.
Answer:

$$MRS = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} = \frac{MU_1}{MU_2} = \frac{\frac{1}{3} x_1^{-2/3} x_2^{2/3}}{\frac{2}{3} x_1^{1/3} x_2^{-1/3}} = \frac{x_2}{2x_1}$$

b) Set up this consumer's utility maximization problem (UMP), and find the Walrasian demand.
[*Hint*: Write the tangency condition, solve for x_2, and insert your result into the consumer's budget line. Solving for x_1, you will obtain the demand for good 1. Recall that the expression you find should be a function of the price of good 1, p_1.]
Answer: We first write up the tangency condition as follows:

$$\frac{MU_1}{MU_2} = \frac{x_2}{2x_1} = \frac{p_1}{p_2} = p_1$$

Or

$$x_2 = 2x_1 \cdot p_1$$

Substitution of this equation above into the budget constraint gives

$$I = 2x_1 \cdot p_1 + p_1 x_1 = 3p_1 x_1$$

solving for x_1 yields the demand function for good 1:

$$x_1^* = \frac{I}{3p_1}$$

And finally using the tangency condition again, we obtain the demand curve for good 2

$$x_2^* = 2x_1 \cdot p_1 = 2\frac{I}{3p_1} \cdot p_1 = \frac{2I}{3}$$

c) Solve for income *I*, in order to obtain the Engel curve of good 1. Is the slope of the Engel curve positive? Interpret: is the good normal or inferior?.
Answer: Given demand function from part b, the Engel curve of good 1 is:

$$I = 3p_1 x_1$$

Therefore, the slope of the Engel curve is positive, implying it is a normal good.
Similarly, the Engel curve of good 2 is

$$I = \frac{3}{2} p_1 x_1$$

Therefore, the slope of the Engel curve is positive, implying it is also a normal good.

d) For the remainder of the exercise, you can assume an income of I=\$100. Set up this consumer's expenditure minimization problem (EMP), assuming that he seeks to reach a target utility level of $\bar{u}$. Find the Hicksian demand (also referred to as the "compensated" demand).
[*Hint*: Write the tangency condition, solve for x_2, and insert your result into the consumer's utility function. Solving for x_1, you will obtain the demand for good 1. Recall that the expression you find should be a function of the price of good 1, p_1.]
Answer: We first write up the tangency condition as follows:

$$\frac{MU_1}{MU_2} = \frac{x_2}{2x_1} = \frac{p_1}{p_2} = p_1$$

Or

$$x_2 = 2x_1 \cdot p_1$$

Substitution of this equation above into the utility constraint gives

$$\bar{u} = x_1^{\frac{1}{3}}(2p_1x_1)^{\frac{2}{3}}$$

solving for x_1 we obtain the Hicksian demand for good 1 (also known as the compensated demand):

$$x_1^e = (2p_1)^{-\frac{2}{3}}\ \bar{u}$$

And finally using the tangency condition, we find the Hicksian demand for good 2,

$$x_2^e = 2x_1 \cdot p_1 = 2\left((2p_1)^{-\frac{2}{3}}\ \bar{u}\right) \cdot p_1 = (2p_1)^{\frac{1}{3}}\ \bar{u}$$

e) Assume now that the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$. Find the increase in consumer surplus that this consumer enjoys from the price decrease.
Answer:

$$CS = \int_2^4 \frac{I}{3p_1} dp_1 = \int_2^4 \frac{100}{3p_1} dp_1 = \frac{100}{3}(\ln(4) - \ln(2)) = 23.105$$

f) Considering the same price decrease as in part (e), find the compensating variation (CV).
Answer: Recall that CV= Cost of bundle A – Cost of bundle B. With initial price and tangency condition, we have

$$\frac{MU_1}{MU_2} = \frac{x_2}{2x_1} = 4 \ or\ x_2 = 8x_1$$

And then plugging into budget constraint we have:

$$4x_1 + 8x_1 = 100$$

solving for x_1 yields:

$$x_1 = 8.33 \ \text{ and } \ x_2 = 8x_1 = 8 * 8.33 = 66.67$$

Which is the bundle A with initial prices. To find bundle B, given final prices we have:

$$U_A = (8.33)^{\frac{1}{3}}(66.67)^{\frac{2}{3}} = 33.33 = (x_1^B)^{\frac{1}{3}}(x_2^B)^{\frac{2}{3}}$$

In addition, bundle B has the indifference curve being tangent to the decomposition budget line (which has the final price ratio). Then,

$$\frac{MU_1}{MU_2} = \frac{x_2^B}{2x_1^B} = 2 \ or\ x_2^B = 4x_1^B$$

With two equations and two unknowns, we can solve for bundle B. In particular, inserting $x_2^B = 4x_1^B$ into the utility constraint found above, $(x_1^B)^{\frac{1}{3}}(x_2^B)^{\frac{2}{3}} = 33.33$, we obtain

$$(x_1^B)^{\frac{1}{3}}(4x_1^B)^{\frac{2}{3}} = 33.33$$

And solving for x_1^B, we obtain $x_1^B = 13.23$. Hence, $x_2^B = 4x_1^B = 4 * 13.23 = 52.91$ implying that the income that the individual needs to purchase bundle *B* is

$$I_B = 13.23 * 2 + 52.91 * 1 = 79.37$$

Therefore, we obtain compensating variation, that is:

$$CV = I_A - I_B = 100 - 79.37 = 20.63$$

g) Considering the same price decrease as in part (e), find the equivalent variation (EV).
Answer: Recall that EV= Cost of bundle E – Cost of bundle C. With final price and tangency condition, we have

$$\frac{MU_1}{MU_2} = \frac{x_2}{2x_1} = 2 \; or \; x_2 = 4x_1$$

And then plugging into budget constraint we have:

$$4x_1 + 4x_1 = 100$$

solving for x_1 yields:

$$x_1 = 12.5 \text{ and } x_2 = 4x_1 = 4 * 12.5 = 50$$

which is the bundle C. To find bundle E, we know that the individual reaches the same utility level as at bundle C, that is,

$$U_C = (12.5)^{\frac{1}{3}}(50)^{\frac{2}{3}} = 31.5 = (x_1^E)^{\frac{1}{3}}(x_2^E)^{\frac{2}{3}}$$

Furthermore, bundle E must have that the indifference curve is tangent to the budget line at the initial prices, that is,

$$\frac{MU_1}{MU_2} = \frac{x_2^E}{2x_1^E} = 4 \; or \; x_2^E = 8x_1^E$$

With two equations and two unknowns, we can solve for bundle B. In particular, we plug $x_2^E = 8x_1^E$ into the utility constratin $(x_1^E)^{\frac{1}{3}}(x_2^E)^{\frac{2}{3}} = 31.5$, to botain

$$(x_1^E)^{\frac{1}{3}}(8x_1^E)^{\frac{2}{3}} = 31.5$$

Solving for x_1^E, we find $x_1^E = 7.874$. Using the tangency condition, we get that $x_2^E = 8x_1^E = 8 * 7.8745 = 62.99$. Hence the income that the individual needs to purchase bundle *E* is

$$I_E = 7.8745 * 4 + 62.99 * 1 = 94.50$$

Therefore, we obtain equivalent variation, that is:

$$EV = I_E - I_C = 94.50 - 100 = -5.50$$

Exercise #2. Repeat your analysis of Exercise #1 for the following Stone-Geary utility function

$$u(x_1, x_2) = 3(x_1 - 1)^{1/2}{x_2}^{1/3}$$

Recall that in this utility function, the consumer requires 1 units of good 1 for his utility from this good to be positive (e.g., a glass of water to remain alive!). This type of utility function has been frequently used to describe addictive goods, such as drugs and alcohol, where the utility that the addicted individual obtains from good 1 (e.g., drugs) becomes negative unless he consumes a dose of the drug per day.

a) Find the marginal rate of substitution, *MRS*, for this consumer.
Answer:

$$MRS = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} = \frac{MU_1}{MU_2} = \frac{\frac{3}{2}(x_1 - 1)^{-\frac{1}{2}}{x_2}^{1/3}}{(x_1 - 1)^{\frac{1}{2}}{x_2}^{-\frac{2}{3}}} = \frac{3x_2}{2(x_1 - 1)}$$

b) Set up this consumer's utility maximization problem (UMP), and find the Walrasian demand.
[*Hint*: Write the tangency condition, solve for x_2, and insert your result into the consumer's budget line. Solving for x_1, you will obtain the demand for good 1. Recall that the expression you find should be a function of the price of good 1, p_1.]
Answer: We first write up the tangency condition as follows:

$$\frac{MU_1}{MU_2} = \frac{3x_2}{2(x_1 - 1)} = \frac{p_1}{p_2}$$

Rearrange it and yields,

$$\frac{3x_2}{2(x_1 - 1)} = \frac{p_1}{1}$$

Or

$$x_2 = \frac{2}{3} p_1 (x_1 - 1)$$

Substitution of this equation above into the budget $p_1 \mathrm{x}_1 + x_2 = I$ gives

$$I = p_1 x_1 + \tfrac{2}{3} p_1 (x_1 - 1)$$

solving for x_1 yields yields the demand curve for good 1:

$$x_1^* = \frac{3I + 2p_1}{5p_1}$$

And finally using the tangency condition again, we obtain the demand curve for good 2

$$x_2^* = \frac{2(I - p_1)}{5}$$

c) Solve for income *I*, in order to obtain the Engel curve of good 1. Is the slope of the Engel curve positive? Interpret: is the good normal or inferior?.
Answer: Given demand function from part b, the Engel curve of good 1 is:

$$I = \frac{5p_1 x_1 - 2p_1}{3}$$

Therefore, the slope of the Engel curve is positive, implying it is a normal good.
Similarly, the Engel curve of good 2 is

$$I = \frac{5x_2 + 2p_1}{2}$$

Therefore, the slope of the Engel curve is positive, implying it is also a normal good.

d) For the remainder of the exercise, you can assume an income of *I*=$100. Set up this consumer's expenditure minimization problem (EMP), assuming that he seeks to reach a target utility level of $\bar{u}$. Find the Hicksian demand (also referred to as the "compensated" demand).
[*Hint*: Write the tangency condition, solve for x_2, and insert your result into the consumer's utility function. Solving for x_1, you will obtain the demand for good 1. Recall that the expression you find should be a function of the price of good 1, p_1.]
Answer: We first write up the tangency condition as follows:

$$\frac{MU_1}{MU_2} = \frac{3x_2}{2(x_1 - 1)} = \frac{p_1}{p_2}$$

Rearranging it, yields

$$\frac{3x_2}{2(x_1 - 1)} = \frac{p_1}{1}$$

Or

$$x_2 = \frac{2}{3}p_1(x_1 - 1)$$

Substitution of this equation above into the utility constraint gives

$$\bar{u} = 3(x_1 - 1)^{1/2}\left(\frac{2}{3}p_1(x_1 - 1)\right)^{1/3}$$

solving for x_1 yields the Hicksian (or compensated) demand for good 1:

$$x_1^e = \left(\frac{\bar{u}}{(2p_1)^{\frac{1}{3}}(3)^{\frac{2}{3}}}\right)^{\frac{6}{5}} + 1$$

And finally using the tangency condition, we find the Hicksian demand for good 2

$$x_2^e = \left(\frac{(2p_1)^{\frac{3}{5}}(u)^{\frac{6}{5}}}{(3)^{\frac{9}{5}}}\right)$$

e) Assume now that the price of good 1 decreases from $p_1 = \$4$ to $p_1 = \$2$. Find the increase in consumer surplus that this consumer enjoys from the price decrease.
Answer:

$$CS = \int_2^4 \left(\frac{3I + 2p_1}{5p_1}\right) dp_1 = \int_2^4 \left(\frac{300 + 2p_1}{5p_1}\right) dp_1 = \frac{6}{5} + \log(2) = 42.789$$

f) Considering the same price decrease as in part (e), find the compensating variation (CV).
Answer: Recall that

CV= Cost of bundle A – Cost of bundle B.

With initial price and tangency condition, we have

$$\frac{MU_1}{MU_2} = \frac{3x_2}{2(x_1 - 1)} = 4 \text{ or } x_2 = \frac{8}{3}(x_1 - 1)$$

And then plugging into budget constraint we have:

$$4x_1 + \frac{8}{3}(x_1 - 1) = 100$$

solving for x_1 yields:

$$x_1 = 15.4 \; and \; x_2 = 38.4$$

Which is the bundle A with initial prices. To find bundle B, we first need to determine the utility level of bundle A

$$U_A = 3\,(15.4 - 1)^{\frac{1}{2}}(38.4)^{\frac{1}{3}} = 38.4 = 3(x_1^B - 1)^{\frac{1}{2}}(x_2^B)^{\frac{1}{3}}$$

In addition, bundle B has the indifference curve being tangent to the decomposition budget line (which has the final price ratio). Then,

$$\frac{MU_1}{MU_2} = \frac{3x_2^B}{2(x_1^B - 1)} = 2 \text{ or } x_2^B = \frac{4}{3}(x_1^B - 1)$$

With two equations and two unknowns, we can solve for bundle B. In particular, inserting $x_2^B = \frac{4}{3}(x_1^B - 1)$ into the utility constraint found above, $3(x_1^B - 1)^{\frac{1}{2}}(x_2^B)^{\frac{1}{3}} = 38.4$, we obtain

$$x_1^B = 15.4 \text{ and } x_2^B = 38.4$$

implying that the income that the individual needs to purchase bundle B is

$$I_B = 20.00 * 2 + 25.33 * 1 = 65.33.$$

Therefore, we obtain compensating variation, that is:

$$CV = I_A - I_B = 100 - 65.33 = 34.67$$

g) Considering the same price decrease as in part (e), find the equivalent variation (EV).
Answer: Recall that

EV= Cost of bundle E – Cost of bundle C.

With final price and tangency condition, we have

$$\frac{MU_1}{MU_2} = \frac{3x_2}{2(x_1 - 1)} = 2 \; or \; x_2 = \frac{4}{3}(x_1 - 1)$$

And then plugging into budget constraint we have:

$$2x_1 + \frac{4}{3}(x_1 - 1) = 100$$

solving for x_1 yields:

$$x_1 = 30.4 \text{ and } x_2 = 39.2$$

Which is the bundle C with the price changed. To find bundle E, we know that the individual reaches the same utility level as at bundle C, that is,

$$U_C = 3\,(30.4 - 1)^{\frac{1}{2}}(39.2)^{\frac{1}{3}} = 55.26 = 3(x_1^E - 1)^{\frac{1}{2}}(x_2^E - 1)^{\frac{1}{3}}$$

Furthermore, bundle E must have that the indifference curve is tangent to the budget line at the initial prices, that is,

$$\frac{MU_1}{MU_2} = \frac{3x_2^E}{2(x_1^E - 1)} = 4 \text{ or } x_2^E = \frac{8}{3}(x_1^E - 1)$$

With two equations and two unknowns, we can solve for bundle B. In particular, we plug $x_2^E = \frac{8}{3}(x_1^E - 1)$ into the utility constratin $3(x_1^E - 1)^{\frac{1}{2}}(x_2^E)^{\frac{1}{3}} = 55.26$, to obtain

$$x_1^E = 23.28 \text{ and } x_2^E = 59.42$$

Hence the income that the individual needs to purchase bundle *E* is

$$I_E = 23.28 * 4 + 59.42 * 1 = 152.54.$$

Therefore, we obtain equivalent variation, that is:

$$EV = I_E - I_C = 152.54 - 100 = 52.54$$