Reactions to Arrow’s impossibility theorem

- Introducing assumptions on individual preferences (section 21.D in MWG):
  - i.e., assuming single-peaked preferences for every individual).

- Using a different approach (section 6.3 in JR):
  - Aggregating the intensity of individual preferences (not only the ranking of alternatives for each individual) into a social welfare function that captures the intensity in social preferences.
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
  - Since preferences are defined as single-peaked with respect to a linear order in $X$, we first have to define what a linear order means (standard definition in math):
  - A binary relation $\geq$ on the set of alternatives $X$ is a **linear order** on $X$ if it is:
    - reflexive, i.e., $x \geq x$ for every $x \in X$,
    - transitive, i.e., $x \geq y$ and $y \geq z$ implies $y \geq z$, and
    - total, i.e., for any two distinct $x, y \in X$, we have that either $x \geq y$ or $y \geq x$, but not both.
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
  - If the set of alternatives is a subset of the real line, i.e., $X \subseteq \mathbb{R}$,
  - then the linear order $\geq$ is the natural "greater than or equal to" order of the real numbers.
Reactions to Arrow’s impossibility theorem

**Single-peaked preferences:**

The rational preference relation $\preceq$ is single peaked with respect to the linear order $\succeq$ on $X$ if there is an alternative $x \in X$ with the property that $\preceq$ is increasing with respect to $\succeq$ on the set of alternatives below $x$, $\{y \in X : x \succeq y\}$, and decreasing with respect to $\succeq$ on the set of alternatives above $x$, $\{y \in X : y \succeq x\}$. That is,

- If $x \succeq z > y$ then $z \succ y$, and
- If $y > z \succeq x$ then $y \succ z$,

**In words:** There is an alternative $x$ that represents a peak of satisfaction.

Moreover, satisfaction increases as we approach this peak (so there cannot be any other peak of satisfaction).
Single-peaked preferences:

- **Example 21.D.4:**
- Suppose a set of alternatives \( X = [a, b] \subset \mathbb{R} \).
- Then, a preference relation \( \preceq \) on \( X \) is single peaked if and only if it is strictly convex:
  - That is, if and only if, for every alternative \( w \in X \), we have that, for any two alternatives \( y \) and \( z \) weakly preferred to \( w \), i.e., \( y \preceq w \) and \( z \preceq w \) where \( y \neq z \), their linear combination is strictly preferred to \( w \),
    \[
    \alpha y + (1 - \alpha)z > w \quad \text{for all } \alpha \in (0, 1)
    \]
- Figures of utility functions satisfying/violating the single-peaked property.
Reactions to Arrow’s impossibility theorem
Reactions to Arrow’s impossibility theorem

Single-peaked preference

Preferences are not single peaked

Strict convexity of preferences holds:
If $u(y) \geq u(w)$ and $u(z) \geq u(w)$,
then $u(\alpha y + (1-\alpha)z) > u(w)$

Convexity of preferences does not necessarily hold:
$u(y) \geq u(w)$ and $u(z) \geq u(w)$,
but $u(\alpha y + (1-\alpha)z) < u(w)$
Reactions to Arrow’s impossibility theorem

Is the single-peaked property equivalently to strict concavity on the utility function?

**NO!**

Here we have a utility function which is convex, yet the single-peaked property holds: if \( u(y) \geq u(w) \) and \( u(z) \geq u(w) \), then \( u(\alpha y + (1-\alpha)z) < u(w) \)
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
  - We will now restrict our attention to settings in which all individuals have single-peaked preferences with respect to the same linear order $\succeq$.
  - Consider pairwise majority voting.
    - Formally, for any pair $\{x, y\} \subset X$, we say $x \succ F (\succeq^1, \succeq^2, \ldots, \succeq^I) y$ to be as "$x$ is socially at least as good as $y$", if the number of agents that strictly prefer $x$ to $y$ is larger or equal to the number of agents that strictly prefer $y$ to $x$, that is,
      
      $\text{if } \# \left\{ i \in I : x \succ^i y \right\} \geq \# \left\{ i \in I : y \succ^i x \right\}$
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
  - We will next show that, with single-peaked preferences, the social preferences arising from pairwise majority voting have maximal elements.
    - That is, there are alternatives that cannot be defeated by any other alternatives, i.e, the Condorcet paradox does not hold.
Reactions to Arrow’s impossibility theorem

Single-peaked preferences:

Before doing that, we need a few definitions:

Let $x_i$ denote the maximal alternative for individual $i$ according to his preference $\succeq^i$, i.e., his "peak".

Let us next define what we mean by a median agent:

Agent $h \in I$ is a median agent for the profile $(\succeq^1, \succeq^2, \ldots, \succeq^l)$ of single-peaked preferences with respect to the linear order $\geq$ if

$$\# \{i \in I : x_i \geq x_h \} \geq \frac{l}{2} \quad \text{and} \quad \# \{i \in I : x_h \geq x_i \} \geq \frac{l}{2}$$

That is, the number of individuals whose ideal point is larger than $h$’s ideal point is larger than half of the population.

Similarly, the number of individuals whose ideal point is smaller than $h$’s ideal point is larger than half of the population.
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
  - A natural conclusion of the definition of a median agent is that:
    - If there are no ties in peaks and the number of individuals is odd,
    - then there are exactly $\frac{l-1}{2}$ individuals with ideal points strictly smaller than $x_h$, and $\frac{l-1}{2}$ individuals with ideal points strictly larger than $x_h$.
    - That is, the median agent is unique.
Reactions to Arrow’s impossibility theorem

Example:
Determining the median agent in a group of size $I=5$.

$$\frac{I-1}{2} = \frac{5-1}{2} = \text{2 voters with ideal point lower than } x_3$$

Median voter

$$\frac{I-1}{2} = \frac{5-1}{2} = \text{2 voters with ideal point lower than } x_3$$
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
- We are now ready to claim the existence of a Condorcet winner in this setting, and to prove it.
  - Suppose that $\succeq$ is a linear order on $X$ and consider a profile of individual preferences $\left(\succeq^1, \succeq^2, \ldots, \succeq^I\right)$ where, for every individual $i$, $\succeq^i$ is single peaked with respect to $\succeq$.
  - Let $h \in I$ be a median agent with ideal point $x_h$.
  - Then, $x_h \hat{F} \left(\succeq^1, \succeq^2, \ldots, \succeq^I\right) y$ for every alternative $y \in X$. 

Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
- Interpretation of $\vec{x}_h \hat{F} (\succeq^1, \succeq^2, ..., \succeq^I) y$ for every alternative $y \in X$:
  - The ideal point of the median agent cannot be defeated by majority voting by any other alternative $y$.
  - When an alternative cannot be defeated by majority voting by any other alternative, we refer to it as a "Condorcet winner".
  - Hence, a Condorcet winner exists when the preferences of all agents are single peaked with respect to the same linear order.
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
- Proof of the Condorcet winner result:
  - Take any alternative $y \in X$ and suppose that the ideal point of the median agent, $x_h$, satisfies $x_h > y$ (the argument is analogous if $y > x_h$).
  - NTS that alternative $y$ cannot defeat $x_h$, that is,
    \[
    \# \left\{ i \in I : x_h \succ_i y \right\} \geq \# \left\{ i \in I : y \succ_i x_h \right\}
    \]
  - Consider now the set of individuals $S \subset I$ with ideal points to the right-hand side of $x_h$, that is
    \[
    \left\{ i \in I : x_i \geq x_h \right\}.
    \]
  - Then, $x_i \geq x_h > y$ for every individual $i \in S$. 


Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
- Proof (cont’d):
  - Hence, by single-peaked preferences, \( x_h \succeq^i y \) for every individual \( i \in S \).
    - That is, all individuals in \( S \) (i.e., all with ideal points to the right-hand side of \( x_h \)) will vote for the ideal point of the median agent, \( x_h \).
  - Finally, because agent \( h \) is a median agent, the number of individuals with ideal points to the right-hand side of \( x_h \), i.e., \( \#S \), satisfies \( \#S \geq \frac{l}{2} \).
  - Therefore,
    \[
    \# \left\{ i \in I : x_h \succ^i y \right\} \geq \# S \geq \frac{l}{2} \geq \# (I \setminus S) \geq \# \left\{ i \in I : y \succ^i x_h \right\}
    \]
Reactions to Arrow’s impossibility theorem

- **Single-peaked preferences:**
  - The existence of a Condorcet winner guarantees that we don’t run into cyclicality
    - That is, the order in which pairs of alternatives are confronted in pairwise majority voting does not affect the final outcome.
    - However, the previous assumptions don’t guarantee transitivity.
    - Let’s see one example in which a Condorcet winner exists, yet transitivity in the social preference relation is violated.
Reactions to Arrow’s impossibility theorem

- **Example of intransitive social preferences:**
- Consider a set of alternatives $X = \{x, y, z\}$ and $l = 4$ individuals.
- Consider the following profile of individual preferences:

  $x \succ^1 y \succ^1 z$ for individual 1,
  
  $z \succ^2 y \succ^2 x$ for individual 2,
  
  $x \succ^3 z \succ^3 y$ for individual 3, and
  
  $y \succ^4 x \succ^4 z$ for individual 4
Example of intransitive social preferences:

We thus have that

\[ \# \{ i \in I : x \succ^i y \} = \# \{ i \in I : y \succ^i x \} = \]
\[ \# \{ i \in I : z \succ^i y \} = \# \{ i \in I : z \succ^i y \} = 2 \]

which implies that \( x \) is socially indifferent to \( y \) and, similarly, \( y \) is socially indifferent to \( z \).

We can then write \( z \tilde{F} (\succsim^1, \succsim^2, \succsim^3, \succsim^4) y \) and \( y \tilde{F} (\succsim^1, \succsim^2, \succsim^3, \succsim^4) x \).
Example of intransitive social preferences:

For transitivity, we would need that $z \succsim F (\succsim 1, \succsim 2, \succsim 3, \succsim 4) x$.

Can we obtain this result? No:

- Since $\# \{ i \in I : x \succsim^i z \} = 3$ and $\# \{ i \in I : z \succsim^i x \} = 1$ implies
  
  $$x \succsim F (\succsim 1, \succsim 2, \succsim 3, \succsim 4) z,$$

  which is the opposite of what we NTS.

Hence, in this case majority voting fails to generate a transitive social welfare functional.
Reactions to Arrow’s impossibility theorem

- **Example of intransitive social preferences:**
- Social preferences are nonetheless acyclic since:
  - A pairwise majority voting between $x$ and $y$ yields a tie (2 vote for each);
  - A pairwise majority voting between $y$ and $z$ yields a tie (2 vote for each);
  - A pairwise majority voting between $z$ and $x$ yields alternative $z$ being the winner (the Condorcet winner) with three votes against one.

  Hence, alternative $z$ is the Condorcet winner:

  Trying to confront $z$ against another alternative, such as $x$ or $y$, yields either of these alternatives being defeated by $z$ (or a tie, but they never defeat $z$).
Reactions to Arrow’s impossibility theorem

- **How can we guarantee transitivity in the swf?**
- We need to impose two additional conditions:
  - Preference relation of every individual $i$ must be strict (no indifference between alternatives is allowed); and
  - The number of individuals $I$ is odd.
Reactions to Arrow’s impossibility theorem

- **Proof of transitivity in the swf:**
  - Consider a set \( X = \{x, y, z\} \), where \( x \sim_1 F^1 \sim_2 y \), ..., \( x \sim_I z \).
  - Then, \( x \) defeats \( y \), and \( y \) defeats \( x \).
  - Since individual preferences are strict and \( I \) is odd, there must be one alternative in \( X \) that is not defeated by any other alternative in \( X \).
  - Such alternative can be neither \( y \) (which is defeated by \( x \)) nor \( z \) (which is defeated by \( y \)).
  - Hence, such alternative has to be \( x \), and we can thus conclude that \( x \sim_1 F^1 \sim_2 y \), ..., \( x \sim_I z \), as required to prove transitivity.
Reactions to Arrow’s impossibility theorem

Hence, imposing the assumption of single-peaked preferences helped us obtain an acyclic social ranking, thus avoiding the Condorcet paradox.

But, our discussion considered that $X \subset \mathbb{R}$, i.e., the set of alternatives was unidimensional.

- What if, for instance, we are considering a policy issue in which individual preferences rank alternatives according to two dimensions?
- Bad news: cyclicalilty emerges again, even if we assume convex preferences.
Consider that the space of alternatives is bidimensional and, in particular, given by the unit square, i.e., $X = [0, 1]^2$.

A specific alternative is, hence, represented now by a pair $x = (x_1, x_2)$.

See next figure.
Reactions to Arrow’s impossibility theorem
Reactions to Arrow’s impossibility theorem

Consider three individuals with the following utility functions:

\[ u_1(x_1, x_2) = -2x_1 - x_2, \]
\[ u_2(x_1, x_2) = x_1 + 2x_2, \text{ and} \]
\[ u_3(x_1, x_2) = x_1 - x_2. \]

We can represent their indifference curves in the unit square, by solving for \( x_2 \), for a given utility level \( \bar{u} \),

\[ x_2 = -\bar{u} - 2x_1, \]
\[ x_2 = \frac{\bar{u} - x_1}{2}, \text{ and} \]
\[ x_2 = x_1 - \bar{u}. \]
Reactions to Arrow’s impossibility theorem

\[ u_1(x_1, x_2) = -2x_1 - x_2 \implies x_2 = -\bar{u} - 2x_1 \]

Since \( x_1 \) and \( x_2 \) enter negatively in individual 1's utility function
Reactions to Arrow’s impossibility theorem

\[ u_2(x_1, x_2) = x_1 + 2x_2 \Rightarrow x_2 = \frac{-\bar{u}}{2} - \frac{x_1}{2} \]

Since \( x_1 \) and \( x_2 \) enter positively in individual 2's utility function.
Reactions to Arrow’s impossibility theorem

\[ u_3(x_1, x_2) = x_1 - x_2 \Rightarrow x_2 = x_1 - \bar{u} \]

Since \( x_1 \) enters positively but \( x_2 \) negatively into individual 3’s utility function.
Reactions to Arrow’s impossibility theorem

- Preferences are all convex.
- Yet, no pair \( x = (x_1, x_2) \) can be Condorcet winner.
- To show that, we need to show that, for every pair \( x = (x_1, x_2) \), we can find another pair \( y = (y_1, y_2) \) which is preferred by at least two of the three individuals.
Reactions to Arrow’s impossibility theorem

Consider the following three cases:

Case 1: If \( x = (0, x_2) \), then a pair \( y = \left( \frac{1}{2}, x_2 \right) \) is preferred by agents 2 and 3 to \( x \).

\[
\begin{align*}
  u_1(x) &= -x_1 &> &u_1(y) &= -1 - x_1 \\
  u_2(x) &= 2x_2 &< &u_2(y) &= \frac{1}{2} + x_2 \\
  u_3(x) &= -x_2 &< &u_3(y) &= \frac{1}{2} - x_2
\end{align*}
\]
Consider the following three cases:

**Case 2:** If $x = (x_1, 1)$, then a pair $y = \left(x_1, \frac{1}{2}\right)$ is preferred by agents 1 and 3 to $x$.

\[
\begin{align*}
  u_1(x) &= -2x_1 - 1 < u_1(y) = -2x_1 - \frac{1}{2} \\
  u_2(x) &= x_1 + 2 > u_2(y) = x_1 + 1 \\
  u_3(x) &= x_1 - 1 < u_3(y) = x_1 - \frac{1}{2}
\end{align*}
\]
Consider the following three cases:

**Case 3:** If $x_1 > 0$ and $x_2 < 1$, then a pair $y = (x_1 - \varepsilon, x_2 + \varepsilon)$, where $\varepsilon > 0$, is preferred by agents 1 and 2 to $x$.

\[
\begin{align*}
  u_1(x) &= -2x_1 - x_2 < \\
  u_2(x) &= x_1 + 2x_2 < \\
  u_3(x) &= x_1 - x_2 >
\end{align*}
\]

\[
\begin{align*}
  u_1(y) &= -2(x_1 - \varepsilon) - (x_2 + \varepsilon) = \\
  &= -2x_1 - x_2 + \varepsilon \\
  u_2(y) &= (x_1 - \varepsilon) + 2(x_2 + \varepsilon) = \\
  &= x_1 + 2x_2 + \varepsilon \\
  u_3(y) &= (x_1 - \varepsilon) - (x_2 + \varepsilon) = \\
  &= x_1 - x_2 - 2\varepsilon
\end{align*}
\]
Reactions to Arrow's impossibility theorem

We have thus spanned the unit square:

Case 2:
\[ x = (x_1, 1) \]

Case 3:
\[ x = (x_1 > 0 \text{ and } x_2 < 1) \]

Case 2:
\[ x = (0, x_2) \]
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**

  Let’s not despair:

  - We have just found one counterexample in which the bidimensionality of the alternatives in $X$ yields cyclicality.
  - But, we can find other settings in which, despite alternatives being multidimensional, cyclicality doesn’t arise.
  - More generally, under which conditions can we guarantee that cyclicality does not emerge?
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**
- Consider alternatives with $n$-dimensions, $x \in \mathbb{R}^n$
- Individual preferences are represented by utility function

\[ u(y) = - \|y - x\| \]

where $x$ denotes the ideal point of this individual. Hence, the utility of vector $y$ is given by the Euclidean distance from his ideal point $x$.

- You can think about $x$ as the "peak" of the utility mountain of this individual, where the level sets of the mountain are circles.
- Figure for the case in which $X = \mathbb{R}^2$. 
Reactions to Arrow’s impossibility theorem

Euclidean preferences for alternatives $x \in R^2$

Since $u(y) = -\|y - x\|$, $u(x) = 0$

$u(y) < 0$ for all $y \neq x$

Utilitarian social welfare corresponds to the ideal point $x$. 
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**
- Hence, given two alternatives \( y \) and \( z \), an individual with ideal point \( x \) will prefer the one closer to \( x \) (where "closer to" is defined by the Euclidean distance).
  - Figure for the case in which \( X = \mathbb{R}^2 \).
Reactions to Arrow’s impossibility theorem

This individual prefers alternative y to z, since

\[ u(y) = \|y - x\| > u(z) = \|z - x\| \]

\[ -[(y_1 - x_1)^2 + (y_2 - x_2)^2]^{1/2} > -[(z_1 - x_1)^2 + (z_2 - x_2)^2]^{1/2} \]

Example: \(x=(1,1), y=(2,0.8), z=(3,3)\). Then

\[ -[(2-1)^2 + (0.8-1)^2]^{1/2} > -[(3-1)^2 + (3-1)^2]^{1/2} \iff -[1 + 0.04]^{1/2} > -\sqrt{8} \]

1.01 < 2.82
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**
  - Thus, the region peaks of different individuals that prefer $y$ to $z$ is
    \[ A(y, z) = \{ x \in \mathbb{R}^n : \|x - y\| < \|x - z\| \} \]
  - That is, all those individuals whose ideal points, $x$, are closer to $y$ than to $z$.
    - In the next figure, the boundary of $A(y, z)$ is given by a line (generally, it could be a hyperplane if $n > 2$)
    - This line is perpendicular to the segment connecting $y$ and $z$, and passing through its midpoint.
Reactions to Arrow’s impossibility theorem

The region of peaks (from different individuals) which prefer y to z.

This is just an example of any point in region $A(y,z)$. 
Reactions to Arrow’s impossibility theorem

• **Multidimensional alternatives:**
  
  Consider now a continuum of individuals, each of them with the above preferences.
  
  The ideal points, \( x \in \mathbb{R}^n \), are distributed with density function \( g(x) \).
  
  Then, for any two alternatives \( y \) and \( z \), the fraction of the population that prefers \( y \) to \( z \) is

  \[
  \int_{A(y,z)} g(z) \, dz \equiv m_g(y, z)
  \]
Reactions to Arrow’s impossibility theorem

Interpretation of $m_z(y, z)$ in one-dimensional alternatives, i.e., $n=1$

1) For simplicity, assume that $g(x)$ is uniformly distributed.

2) What is $g(x)$ if it is not uniformly distributed?

$m_z(y, z) = \text{Areas A+B}$

$m_z(z, y) = \text{Areas C+D}$
Reactions to Arrow’s impossibility theorem

**Multidimensional alternatives:**

Let us now show under which conditions there can be a Condorcet winner, i.e., an alternative $x^*$ that cannot be defeated by any other alternative $y$.

- **1st line of implication:**
  - If alternative $x^*$ is a median, then $x^*$ is a Condorcet winner.

- **2nd line of implication:**
  - If alternative $x^*$ is a Condorcet winner, then $x^*$ is a median.
Reactions to Arrow’s impossibility theorem

Multidimensional alternatives:

Let us now show under which conditions there can be a Condorcet winner.

- Suppose there is an alternative \( x^* \in \mathbb{R}^n \) such that any halfspace through \( x^* \) divides \( \mathbb{R}^n \) into two half-spaces, each having a total mass of \( \frac{1}{2} \) according to the density \( g(\cdot) \).
- See figure
Reactions to Arrow’s impossibility theorem

Half of the population is to the left of $x^*$ (shaded area), and half is to the right (unshaded area).
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**
- Point \( x^* \) then leaves exactly half of the population to the left (in the Euclidean sense) and the other half to the right:
  - As a consequence, point \( x^* \) is referred to as "median."
  - It coincides with the usual notion of median in the case of \( n = 1 \) (see next figure).
Interpretation of $m_z(y, z)$ in one-dimensional alternatives, i.e., $n=1$

1) For simplicity, assume that $g(x)$ is uniformly distributed.

2) What is $g(x)$ if it is not uniformly distributed?
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**
- A median $x^*$ in the above sense is a Condorcet winner:
  - Point $x^*$ cannot be defeated by any other alternative $y \neq x^*$.
  - In particular, $A(x^*, y)$ becomes larger than the half-space through $x^*$. (See next figure).
  - Therefore, $m_g(x^*, y) \geq \frac{1}{2}$, thus being defeated by point $x^*$. 
Reactions to Arrow’s impossibility theorem

In order to construct $A(x^*, y)$, which represents the set of individuals’ peaks that prefer $x^*$ to $y$, we first need to plot a line between $x^*$ and $y$, then we plot a hyper-plane (a line in this case) perpendicular to the segment connecting the area to the left of this hyper-plane represent $A(x^*, y)$ $x^*$ and $y$, and passing through its midpoint.
Reactions to Arrow’s impossibility theorem

Application to $n=1$

If $x^*$ is a median, then $x^*$ is a Condorcet winner.

Any other alternative $y \neq x^*$ would be different by $x^*$.

\[
m_g(x^*, y) > \frac{1}{2} \quad \text{and} \quad m_g(y, x^*) < \frac{1}{2}
\]
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**
  - A median $x^*$ in the above sense is a Condorcet winner:
    - Conversely, if $x^*$ is not a median, then it cannot be a Condorcet winner. $x^*$ is a median $\iff x^*$ is a Condorcet winner.
    - Specifically, we can move $x^*$ in any direction $q$ such that we give rise to a half-space larger than $\frac{1}{2}$.
    - More formally, there exists a direction $q \in \mathbb{R}^n$ such that the mass of the half-space
      \[
      \{ z \in \mathbb{R}^n : q \cdot z > q \cdot x^* \} \text{ is larger than } \frac{1}{2}
      \]
    - In other words, point $x^* + q\varepsilon$ defeats point $x^*$; see next figure.
      - That is, if $x^*$ is not a median, it cannot be a Condorcet winner.
Reactions to Arrow’s impossibility theorem

What if \( x^* \) is not a median?

Hence, \( m_q(x^* + q\varepsilon, x^*) = \int_{A(x^* + q\varepsilon, x^*)} q(z)dz > \frac{1}{2} \). That is, the region \( A(x^* + q\varepsilon, x^*) \) must contain more than half of the population if \( x^* \) was not a median.
Reactions to Arrow’s impossibility theorem

This result is easily illustrated in the more familiar setting of $n=1$ and $g(x)$ being uniformly distributed:

- If $x^*$ is not a median:

  $\frac{1}{2}$
  $\frac{1}{2}$

  Vote for $x^*$
  $m_\epsilon(x^*, x^*+\epsilon) < \frac{1}{2}$

  Hence $x^*$ cannot be a Condorcet winner.

- There are alternatives, such as $x^*+\epsilon$, which would defeat $x^*$:

  $m_\epsilon(x^*+\epsilon, x^*) > \frac{1}{2}$

  Vote for $x^*+\epsilon,$

  Vote for $x^*$

  However, if $x^*$=median, then we can not find alternatives to $x^*$ that would defeat $x^*$. 
Reactions to Arrow’s impossibility theorem

Multidimensional alternatives:

Notice what we just proved:

- Consider a density $g(\cdot)$ describing the probability distribution of ideal points for each individual in the population.
- If this density $g(\cdot)$ provides us with a median $x^*$ that divides the Euclidean space into two regions of equal area...
- then we can claim that such median is a Condorcet winner.

That’s ok, but the most demanding requirement is the second.

- We can prove how restrictive this result is, even if we assume a uniform distribution.
- Let’s consider two cases:
  - One that generates a median, and one that doesn’t.
Reactions to Arrow’s impossibility theorem

**Uniform distribution over a rectangle:**

Point \( x^* \) is the median, since every plane through \( x^* \) divides the rectangle into two equal areas.

**Uniform distribution over a triangle:**

There is no median (a point for which every plane through the point divides the triangle into two equal areas).
Reactions to Arrow’s impossibility theorem

- **Multidimensional alternatives:**
  - Caplin and Nalebuff (1988) tackled this problematic result and brought us the now famous "64% majority rule":
    - They showed that, for a uniform distribution (and, more generally, for any density function satisfying logarithmic concavity) there are always points (which they referred to as "generalized medians")...
    - with the property that a hyperplane through the point divides \( \mathbb{R}^n \) into two regions, each of them with a mass larger than \( \frac{1}{e} \approx 0.36 \).
  - What does that mean?
    - These points cannot be defeated by any other alternative if the majority required is not \( \frac{1}{2} \) of the votes, but any number larger than \( 1 - \frac{1}{e} \approx 0.64 \).
Second reaction:

Allowing for intensity of individual preferences to enter into social preferences.

- We will do that by using a social welfare function

\[ W \left( u^1(\cdot), u^2(\cdot), ..., u^I(\cdot) \right) \]

- We first need to impose two assumptions on \( W(\cdot) \):
  - Utility-level invariant, and
  - Utility-difference invariant.
Utility-level invariance:

Motivation: Consider that $u^1(x) > u^1(y)$ for individual 1, and $u^2(x) < u^2(y)$ for individual 2.

In addition, assume that $u^1(y) > u^2(x)$, i.e., individual 1 is better off at his least-preferred state than individual 2 is.

Then,

$$u^1(x) > u^1(y) > u^2(x)$$

where $u^2(y)$ must be larger than $u^2(x)$, but could rank above/below $u^1(x)$ or $u^1(y)$. (Figure.)
If $x, y \in R$, then:

- $u^1(x) > u^1(y)$ for individual 1
- $u^2(x) < u^2(y)$ for individual 2
- $u^1(y) > u^2(x)$ As depicted in point A and D

- However, $u^2(x)$ could rank above $u^1(y)$, and even above $u^1(x)$ Point C
  Point D
  Point B
Utility-level invariance:

Assume that, in this context, society seeks to make its least well off individual as well off as possible. That is,

\[
\max_{x, y} \left\{ \min \left\{ u^1(x), u^1(y) \right\}, \min \left\{ u^2(x), u^2(y) \right\} \right\}
\]

\[
= \max_{x, y} \left\{ u^1(y), u^2(x) \right\}
\]

and since \( u^1(y) > u^2(x) \), alternative \( y \) is socially preferred to \( x \).
Utility-level invariance:

Now, consider strictly increasing transformations $\psi_1(\cdot)$ and $\psi_2(\cdot)$ producing the same individual rankings $v_1(x) \equiv \psi_1(u_1(x)) > \psi_1(u_1(y)) \equiv v_1(y)$, and $v_2(x) \equiv \psi_2(u_2(x)) < \psi_2(u_2(y)) \equiv v_2(y)$

but altering the ranking across individuals, i.e., $v_1(y) < v_2(x)$.

In this setting, society would identify alternative $x$ as socially preferred to $y$.

But this new social ranking is troublesome: We have not changed the individual rankings over alternatives, yet the social ranking changed. (Figure.)

In order to avoid this possibility, we only need to avoid different monotonic transformations for individual 1 and 2. That's what utility-level invariance guarantees (i.e., $\psi_1 = \psi_2$).
Continuing with the above example:

\( \psi^1 \) shifts \( u^1(\cdot) \) downwards while \( \psi^2 \) shifts \( u^2(\cdot) \) upwards.

While the individual ranking is unaffected, i.e., \( v^1(x) > v^1(y) \) and \( v^2(x) < v^2(y) \), the ranking between \( v^1(y) \) and \( v^2(x) \) is affected.
Utility-level invariance:

**Definition:** A social welfare function $W(\cdot)$ is utility-level invariant if it is invariant to arbitrary, but common, strictly increasing transformations $\psi$ applied to every individual's utility function.

That is, for every profile of individual preferences

$u \equiv \left( u^1(\cdot), u^2(\cdot), \ldots, u^I(\cdot) \right)$, where

$u(x) \equiv \left( u^1(x), u^2(x), \ldots, u^I(x) \right)$ and

$u(y) \equiv \left( u^1(y), u^2(y), \ldots, u^I(y) \right)$ denote the profile of individual utility levels from any two alternatives $x \neq y$,

if $W(u(x)) > W(u(y))$ then $W(\psi(u(x))) > W(\psi(u(y)))$

under a common strictly increasing transformation $\psi(\cdot)$, where

$\psi(u(x)) \equiv \left( \psi(u^1(x)), \psi(u^2(x)), \ldots, \psi(u^I(x)) \right)$ and similarly for $\psi(u(y))$. 
Utility-difference invariance:

Let us now move to a second type of information often used in making social choices:

- The utility that each individual gains/losses when he moves from an alternative $y$ to another alternative $x$.
- That is, $u_1(x) - u_1(y)$ for individual 1, which in this example was considered positive, and
- $u_2(x) - u_2(y)$ for individual 2, which in this example is negative.
Utility-difference invariance:

A common comparison is then whether individual 1’s gain, $u^1(x) - u^1(y)$ of moving to $x$ is larger than individual 2’s loss, $u^2(y) - u^2(x)$.

$$u^1(x) - u^1(y) > u^2(y) - u^2(x)$$
Reactions to Arrow’s impossibility theorem - II

\[ u^1(x) - u^1(y) > 0 \]
\[ u^2(y) - u^2(x) > 0 \]

Comparing these differences,

\[ u^1(x) - u^1(y) > u^2(y) - u^2(x) \]
Utility-difference invariance:

For the swf to preserve this information, we need that monotonic transformations are linear, i.e.,

$$\psi^i \left( u^i(x) \right) = a^i + bu^i(x)$$

where $b > 0$ is common to all individuals.

Figure.
After applying the monotonic transformations, the difference $v^1(x) - v^1(y)$ is still larger than $v^2(y) - v^2(x)$. 

$\psi^1(u^1(\cdot)) = a^1 + b u^1(\cdot)$

$\psi^2(u^2(\cdot)) = a^2 + bu^2(\cdot)$
Utility-difference invariance:

*Definition:* A social welfare function $W(\cdot)$ is **utility-difference invariant** if it is invariant to strictly increasing transformations of the form

$$\psi \left( u^i(x) \right) = a^i + bu^i(x),$$

where $b > 0$ is common to all individuals.
Two more assumptions on the SWF:

- **Anonymity.** Let $u(x)$ and $\tilde{u}(x)$ be two utility vectors, where $\tilde{u}(x)$ has been obtained from $u(x)$ after a permutation of its elements. Then,

  $$W(u(x)) = W(\tilde{u}(x))$$

- **Interpretation:**
  - The social ranking of alternatives should not depend on the identity of the individuals involved, but only on the levels of utility each alternative entail.
Two more assumptions on the SWF:

Hammond Equity. Let $u(x)$ and $u(y)$ be the utility vectors of two distinct alternatives $x$ and $y$, where $u^k(x) = u^k(y)$ for every individual $k$ except for two individuals: $i$ and $j$. If

\[ u^i(x) < u^i(y) < u^j(y) < u^j(x) \]

then $W(u(y)) \geq W(u(x))$.

Interpretation:

- Society has a preference towards the alternative that produces the smallest variance in utilities across individuals (alternative $y$ in this case).
- Seems reasonable in some cases, but critizable in orders: for instance,

\[ u^i(x) = 1 < u^i(y) = 1.1 < u^j(y) = 1.2 < u^j(x) = 100. \]
Reactions to Arrow’s impossibility theorem - II

Hammond Equity

$u^1(x) < u^1(y) < u^2(y) < u^2(x)$
We can now show that some well-known SWF, such as the Rawlsian and the utilitarian, can be characterized by some of the properties we just mentioned:

- Utility-level invariance,
- Utility-difference invariance,
- Anonymity (A), and
- Hammond Equity (HE),
The Rawlsian SWF

- Welfare is given by that of the worst-off member, that is,
  \[ W(x) = \min \{ u^1(x), \ldots, u^I(x) \} \]

- *Theorem 6.2 in JR:*
  
  A strictly increasing and continuous swf \( W \) satisfies HE if and only if it can be represented with the Rawlsian form,
  \[ W(x) = \min \{ u^1(x), \ldots, u^I(x) \} \] .

- *As a corollary:*
  
  Moreover, \( W \) satisfies A and is utility-level invariant.

- Let’s prove these results.
The Rawlsian SWF

Proof:

1st line of implication:

- If $W$ is continuous, strictly increasing, and satisfies HE, then $W$ must be Rawlsian.

2nd line of implication:

- If $W$ is Rawlsian, then $W$ is continuous, strictly increasing, and satisfies HE.
The Rawlsian SWF

Proof: Suppose that $W$ is continuous, strictly increasing and satisfies HE.

We then NTS that $W$ takes the form

$$W(x) = \min \left\{ u^1(x), \ldots, u^l(x) \right\}$$

That is, $W(x) \geq W(y)$ if and only if

$$\min \left\{ u^1(x), \ldots, u^l(x) \right\} \geq \min \left\{ u^1(y), \ldots, u^l(y) \right\}$$

Consider the next figure.
Reactions to Arrow’s impossibility theorem - II

Point $\tilde{u}$ satisfies $u^2 < \tilde{u}^2 < \tilde{u}^1 < u^1 \rightarrow$ Indeed, $\tilde{u}^1 < u^1$ for individual 1, $u^2 < \tilde{u}^2$ for individual 2, and $\tilde{u}^2 < \tilde{u}^1$ across individuals (since $\tilde{u}$ lies below 45° line)
The Rawlsian SWF

Proof:

Here is what we are planning to do:

- The social indifference curve of a Rawlsian swf must be a right angle (and all kinks are crossed by a ray from the origin).
- We must then show that, starting from any arbitrary point $a$ on the 45-degree line:
  - All points in a horizontal ray starting from the 45-degree line, and
  - all points in a vertical ray starting from the 45-degree line,
  - must yield the same social welfare as in point $a$. 
Reactions to Arrow’s impossibility theorem - II

The Rawlsian SWF

- Consider the next figure.
- Choose an arbitrary point \( a \) on the 45-degree line, and point \( u \) on the ray extending from \( a \) to the right.
- We seek to show that \( W(u) = W(a) \).
- Define region I and II.
- Consider a point \( \tilde{u} \) in region I. Note that

\[
u^2 < \tilde{u}^2 < \tilde{u}_1 < u^1\]

- Graphically, note that point \( \tilde{u} \) is closer to the 45-degree line than \( u \) is, thus reducing utility dispersion across individuals; as depicted in the figure.
The Rawlsian SWF

Since point \( \tilde{u} \) implies a smaller utility dispersion than \( u \) society prefers, according to HE, point \( \tilde{u} \), i.e., \( W(\tilde{u}) \geq W(u) \).

This argument is true for any point \( \tilde{u} \) in region I, i.e., \( W(I) \geq W(u) \).

What about region II?

- We must have that \( W(II) < W(u) \) since \( W \) is strictly increasing and all points in region II are to the southwest of \( u \).
- Hence,

\[
W(I) \geq W(u) > W(II)
\]
The Rawlsian SWF

What about the points on the frontier between regions I and II, such as point $a$?

- By continuity of the swf $W$, since $W(I) \geq W(u)$ in region I and $W(u) > W(u)$ in region II, $W(u) = W(a)$, as we wished to show.

- We can extend the same argument, but now starting from a ray that extends from $a$ upwards (rather than rightwards).

  - That is, we have just examined the welfare at points below the 45-degree line, but a similar argument applies for points above the 45-degree line.

  - See figure.
Reactions to Arrow’s impossibility theorem - II

For individual 1, $\tilde{u}^1 > u^1$
For individual 2, $\tilde{u}^2 < u^2$
Across individuals, since $\tilde{u}$ lies above the 45°-line, $\tilde{u}^2 > \tilde{u}^1$, thus implying $u^2 > \tilde{u}^2 > \tilde{u}^1 > u^1$
The Rawlsian SWF

Because $W$ is strictly increasing, no other points can yield the same social welfare than a other than the two rays we just examined.

- That is, the union of the two rays provides us with the social indifference curve for $W$. (See figure.)
- Therefore, $W$ has the same indifference map as the function $\min \{ u^1(x), ..., u^I(x) \}$. 
Reactions to Arrow’s impossibility theorem - II
The Rawlsian SWF

Other direction: If \( W(x) = \min \{ u^1(x), \ldots, u^I(x) \} \) then HE holds.

- Let’s apply the definition of HE: if \( u^k(x) = u^k(y) \) for every individual \( k \) except for two individuals: \( i \) and \( j \), and assume that
  \[
  u^i(x) < u^i(y) < u^j(y) < u^j(x)
  \]

- Figure.
- We now NTS that the alternative with the smaller utility dispersion is socially preferred, i.e., \( W(u(y)) \geq W(u(x)) \).
Proving that HE holds in the Rawlsian SWF.

- \( \min \{ u^i(x), u^j(x) \} \)
- \( \min \{ u^i(y), u^j(y) \} \)

Region 1 \quad Region 2 \quad Region 3
The Rawlsian SWF

Then, $u^k(x) = u^k(y)$ lies in either of the following regions:

- **Region 1**, where $u^k(x) = u^k(y) < u^i(x)$.

  - Then $W(u(x)) = u^k(x)$ and $W(u(y)) = u^k(y)$, and
  - Society is indifferent between alternatives $y$ and $x$, i.e., $W(u(y)) = W(u(x))$, which is allowed according to the HE property (recall that we seek to show $W(u(y)) \geq W(u(x)))$. 
The Rawlsian SWF

Then, $u^k(x) = u^k(y)$ lies in either of the following regions:

- **Region 2**, where $u^i(x) < u^k(x) = u^k(y) < u^i(y)$.
  
  Then $W(u(x)) = u^i(x)$ and $W(u(y)) = u^k(y)$, and
  
  Society prefers alternative $y$ to $x$, i.e., $W(u(y)) > W(u(x))$, thus satisfying the HE property.

  Intuitively, alternative $y$ yields a smaller utility dispersion than $x$ does.
Reactions to Arrow’s impossibility theorem - II

- **The Rawlsian SWF**
- Then, \( u^k(x) = u^k(y) \) lies in either of the following regions:
  - **Region 3**, where \( u^i(y) < u^k(x) = u^k(y) \).
    - Then \( W(u(x)) = u^i(x) \) and \( W(u(y)) = u^i(y) \), and
    - Society prefers alternative \( y \) to \( x \), i.e., \( W(u(y)) > W(u(x)) \), thus satisfying the HE property.
    - Intuitively, alternative \( y \) yields a smaller utility dispersion than \( x \) does.
Reactions to Arrow’s impossibility theorem - II

**The Rawlsian SWF**

*Corollary:* $W(x) = \min \{ u^{1}(x), \ldots, u^{l}(x) \}$ satisfies anonymity, and is utility-level invariant.

- Anonymity is obvious. Take a utility vector $u^{1}(x), \ldots, u^{l}(x)$, where

  $\min \{ u^{1}(x), \ldots, u^{l}(x) \} = u^{k}(x)$

- Now perform a permutation on the identities of individuals, and apply the min on their utility levels again. The min is still $u^{k}(x)$.
Reactions to Arrow’s impossibility theorem - II

- **The Rawlsian SWF**

  - **Corollary:** \( W(x) = \min \{ u^1(x), ..., u^I(x) \} \) satisfies anonymity, and is utility-level invariant.

  - What about utility-level invariance?
    - Let’s first define what we need to show.
    - Consider a strictly increasing transformation common to all individuals \( \psi : \mathbb{R} \rightarrow \mathbb{R} \).
    - If \( W(u(x)) \geq W(u(y)) \) then the social ranking is preserved after applying a common strictly increasing transformation to all individuals' utility function, i.e.,

      \[
      W \left( \psi \left( u^1(x) \right), ..., \psi \left( u^I(x) \right) \right) \geq \psi \left( W \left( u^1(x), ..., u^I(x) \right) \right)
      \]
The Rawlsian SWF

Let us now show utility-level invariance.

Define a strictly increasing transformation common to all individuals $\psi : \mathbb{R} \to \mathbb{R}$. Then,

$$W \left( \psi \left( u^1(x) \right), \ldots, \psi \left( u^I(x) \right) \right) = \psi \left( W \left( u^1(x), \ldots, u^I(x) \right) \right)$$

Example: $\psi \left( u^i(x) \right) = \alpha + \beta u^i(x)$, then

$$\psi \left( W \left( u^1(x), \ldots, u^I(x) \right) \right) = \alpha + \beta \min \left\{ u^1(x), \ldots, u^I(x) \right\}$$
Reactions to Arrow’s impossibility theorem - II

- **The Rawlsian SWF**
- Let us now show utility-level invariance.
  - Therefore,
    \[
    W\left(\psi\left(u^1(x)\right), \ldots, \psi\left(u^I(x)\right)\right) \geq W\left(\psi\left(u^1(y)\right), \ldots, \psi\left(u^I(y)\right)\right)
    \]
    implies
    \[
    \psi\left(W\left(u^1(x), \ldots, u^I(x)\right)\right) \geq \psi\left(W\left(u^1(y), \ldots, u^I(y)\right)\right)
    \]
    which is equivalent to
    \[
    W\left(u^1(x), \ldots, u^I(x)\right) \geq W\left(u^1(y), \ldots, u^I(y)\right)
    \]
    as required by utility-level invariance.
The Rawlsian SWF

What about utility-difference invariance, UDI?

- It does not necessarily hold.
- To see this, consider a counterexample, where

\[
W(u(x)) = \min \{ u^1(x), u^2(x) \} = u^1(x) = 10, \text{ and }
\]

\[
W(u(y)) = \min \{ u^1(y), u^2(y) \} = u^2(y) = 5
\]

Hence, \( W(u(x)) > W(u(y)) \)
The Rawlsian SWF

What about utility-difference invariance, UDI?

We now apply the linear, but potentially asymmetric, strictly increasing transformation $\psi^i (u^i(x)) = a^i + bu^i(x)$, where $b > 0$.

Consider for instance $b = 1$, $a^1 = 1$ and $a^2 = 150$. We then obtain

$W(\psi^i (u(x))) = \min \left\{1 + u^1(x), 1 + u^2(x)\right\} = 1 + u^1(x) = 11,$

$W(\psi^i (u(y))) = \min \left\{150 + u^1(y), 150 + u^2(y)\right\}$

$= 150 + u^2(y) = 155$

which implies that the social ranking between alternatives $x$ and $y$ is reverted to $W(u(x)) > W(u(y))$.

Hence, UDI doesn’t necessarily hold for the Rawlsian swf.
Reactions to Arrow’s impossibility theorem - II

- The Utilitarian SWF
- This is probably the most commonly used swf in economics.

\[ W(x) = u^1(x) + u^2(x) + \ldots + u^l(x) = \sum_{i=1}^{l} u^i(x) \]
The Utilitarian SWF

Theorem 6.3 in JR:

A strictly increasing and continuous swf $W$ satisfies A and utility-difference invariance if and only if it can be represented with the utilitarian form, $W(x) = \sum_{i=1}^{I} u^i(x)$. 
Reactions to Arrow’s impossibility theorem - II

- **The Utilitarian SWF**
  - *Proof:*
  - Here is what we need to show:
    - 1st line of implication:
      - If $W$ is utilitarian, then A and UDI holds.
    - 2nd line of implication:
      - If A and UDI holds, then $W$ must be utilitarian.
The Utilitarian SWF

Proof:

- When $W$ takes the utilitarian form, $A$ holds since the utility level of each individual receives the same weight.
  
  That is, a permutation on the identities of individuals will not alter the social ranking of alternatives.
Reactions to Arrow’s impossibility theorem - II

- The Utilitarian SWF

**Proof:**

- When $W$ takes the utilitarian form, utility-difference invariance holds as well. In particular,

  
  \[
  W(x) = u^1(x) + u^2(x) \geq u^1(y) + u^2(y) = W(y),
  \]

  then

  \[
  \left( a^1 + bu^1(x) \right) + \left( a^2 + bu^2(x) \right) \\
  \geq \left( a^1 + bu^1(y) \right) + \left( a^2 + bu^2(y) \right)
  \]

  also needs to hold.
The Utilitarian SWF

Proof:

This inequality collapses to

\[ b \left[ u^1(x) + u^2(x) \right] \geq b \left[ u^1(y) + u^2(y) \right] \]

which is satisfied since \( u^1(x) + u^2(x) \geq u^1(y) + u^2(y) \), and \( b > 0 \) by definition.
The Utilitarian SWF

Proof:

- We now need to show the opposite line of implication: a strictly increasing and continuous SWF satisfying A and utility-difference invariance can only be represented with the utilitarian form.
- Consider the next figure.
- Take a point \( u \) on the 45-degree line.
Reactions to Arrow’s impossibility theorem - II
Reactions to Arrow’s impossibility theorem - II

The Utilitarian SWF

**Proof:**

- Sum the two components in point $\bar{u}$, i.e., $\bar{u}^1 + \bar{u}^2 \equiv \gamma$.
- Consider the set of points for which the sum of their two components, $u^1 + u^2$, yields exactly $\gamma$.

$$\Omega = \left\{ u^1 + u^2 \mid u^1 + u^2 = \gamma \right\}$$

- These are all the points in the line that crosses $\bar{u}$ and has a slope of -1.
Reactions to Arrow’s impossibility theorem - II

Line $\Omega = \{u^1 + u^2 | u^1 + u^2 = \gamma\}$ where $\gamma = \bar{u}^1 + \bar{u}^2$

$45^\circ$ - line

$\bar{u}_1$ and $\bar{u}_2$
The Utilitarian SWF

Proof:

Here is what we are planning to do:

- The social indifference curve of a utilitarian swf must be linear, i.e., $u^2 = W - u^1$.
- We must then show that all points in line $\Omega$ yield the same social welfare as in point $\overline{u}$.

$$W(\Omega) = W(\overline{u}).$$
The Utilitarian SWF

Proof:

- Choose any point in line $\Omega$, distinct from $\vec{u}$, such as $\tilde{u}$.
- Point $\tilde{u}^T$ is just a permutation of $\tilde{u}$, i.e., if $\tilde{u} = (\tilde{u}^1, \tilde{u}^2)$ point $\tilde{u}^T$ becomes $\tilde{u}^T = (\tilde{u}^2, \tilde{u}^1)$. 
Reactions to Arrow’s impossibility theorem - II
The Utilitarian SWF

Proof:

- By condition A, points $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}^T$ must be ranked the same way relative to $\mathbf{u}$.
- Note that we are not saying that societies with swf that satisfy A and UDI are indifferent between points $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}^T$; we don’t know that yet.
  - We only say that, if $W(\tilde{\mathbf{u}}) \geq W(\mathbf{u})$, then such social ranking is maintained for point $\tilde{\mathbf{u}}^T$, i.e., $W(\tilde{\mathbf{u}}^T) \geq W(\mathbf{u})$.
  - Likewise, if $W(\mathbf{u}) \geq W(\tilde{\mathbf{u}})$, then such social ranking is maintained for point $\tilde{\mathbf{u}}^T$, i.e., $W(\mathbf{u}) \geq W(\tilde{\mathbf{u}}^T)$.
The Utilitarian SWF

Proof:

1. Suppose that \( W(\tilde{u}) > W(\bar{u}) \).
2. Under UDI, this social ranking must be unaffected by linear transformations of the form \( \psi^i (u^i(\cdot)) = a^i + bu^i(\cdot) \).
3. Let \( b = 1 \) and \( a^i = \bar{u}^i - \tilde{u}^i \), i.e.,

\[
\psi^i \left( u^i(x) \right) = \underbrace{\bar{u}^i(x) - \tilde{u}^i(x)}_{a^i} + u^i(x)
\]

4. Applying this transformation to \( \tilde{u} \) yields

\[
\psi^i \left( \tilde{u}^i(x) \right) = \bar{u}^i(x) - \tilde{u}^i(x) + \tilde{u}^i(x) = \bar{u}^i(x), \text{ i.e.,}
\]

\[
\left( \psi^1 \left( \tilde{u}^1 \right), \psi^2 \left( \tilde{u}^2 \right) \right) = \bar{u}
\]
The Utilitarian SWF

Proof:

- Applying this transformation to $\bar{u}$ yields
  $$\psi^i \left( \bar{u}^i (x) \right) = \bar{u}^i (x) - \bar{u}^i (x) + \bar{u}^i (x) = 2\bar{u}^i (x) - \bar{u}^i (x)$$

- However, since point $\bar{u}$ lies on the 45-degree line,
  $$2\bar{u}^i (x) = \bar{u}^i (x) + \bar{u}^i (x).$$

- Using this property in our above result yields a transformation of

  $$\psi^i \left( \bar{u}^i (x) \right) = 2\bar{u}^i (x) - \bar{u}^i (x) = \left[ \bar{u}^i (x) + \bar{u}^i (x) \right] - \bar{u}^i (x) = \bar{u}^i (x)$$

- That is,

  $$\left( \psi^1 \left( \bar{u}^1 \right), \psi^2 \left( \bar{u}^2 \right) \right) = \bar{u}^T$$
Reactions to Arrow’s impossibility theorem - II

**The Utilitarian SWF**

**Proof:**

- Therefore, point $\tilde{u}$ is transformed into $\bar{u}$, and point $\bar{u}$ is transformed into $\tilde{u}^T$.
- Thus, if $W(\bar{u}) > W(\tilde{u})$, as we originally assumed, then UDI implies that $W(\tilde{u}^T) > W(\bar{u})$.
  - Hence, $W(\tilde{u}^T) > W(\bar{u})$ and $W(\bar{u}) > W(\tilde{u})$, which implies $W(\tilde{u}^T) > W(\bar{u})$, thus violating A.
  - Therefore, our initial assumption $W(\bar{u}) > W(\tilde{u})$ cannot hold.
The Utilitarian SWF

Proof:

- A similar argument applies if we, instead, start our proof assuming that $W(\tilde{u}) < W(\bar{u})$.
- We can therefore conclude that $W(\bar{u}) = W(\tilde{u})$ which, together with A, implies that

$$W(\bar{u}) = W(\tilde{u}) = W(\tilde{u}^T)$$

- Since point $\tilde{u}$ was chosen arbitrarily in the line $\Omega$, we can claim that the social welfare at point $\bar{u}$ is the same as any point along the line $\Omega$, i.e.,

$$W(\bar{u}) = W(\Omega)$$
The Utilitarian SWF

Note that dropping the requirement of A, we can expand our previous results to any "generalized utilitarian" swf of the form

\[ W(x) = \sum_{i=1}^{l} \alpha^i u^i(x) \]

where \( \alpha^i > 0 \) represents the weight society assigns to individual \( i \).

**Example:** For the case of two individuals, \( W = \alpha^1 u^1 + \alpha^2 u^2 \), which yields a social indifference curve of

\[ u^2 = \frac{W}{\alpha^2} - \frac{\alpha^1}{\alpha^2} u^1, \]

thus being still a straight, negatively sloped line, but the slope is now \( -\frac{\alpha^1}{\alpha^2} \).
Reactions to Arrow’s impossibility theorem - II

Social indifference curve if $\alpha_1 < \alpha_2$, so slope is $\frac{\alpha_1}{\alpha_2} > -1$

Social indifference curve if $\alpha_1 > \alpha_2$, so slope is $\frac{\alpha_1}{\alpha_2} < -1$

Social indifference curve if $\alpha_1 = \alpha_2$,
Reactions to Arrow’s impossibility theorem - II

- **Flexible form SWF**
  - In the analysis of certain policies, i.e., moving from $x$ to $y$, we might be interested in percentage change in utility for each individual, $\frac{u^i(x) - u^i(y)}{u^i(x)}$, and whether such a percentage is large for individual $i$ than for $j$.
  
  $$\frac{u^i(x) - u^i(y)}{u^i(x)} > \frac{u^j(x) - u^j(y)}{u^j(x)}$$

- If we seek to maintain the ranking of percentage changes across individuals invariant to monotonic transformations on the utility functions...
  
  - we need monotonic transformations to be *linear* and *common* among individuals, $\psi(u^i) = bu^i$, where $b > 0$ for all $i$.  


Flexible form SWF

Applying $\psi(u^i) = bu^i$, we obtain

$$\frac{bu^i(x) - bu^i(y)}{bu^i(x)} > \frac{bu^j(x) - bu^j(y)}{bu^j(x)}$$

which reduces to

$$\frac{u^i(x) - u^i(y)}{u^i(x)} > \frac{u^j(x) - u^j(y)}{u^j(x)}$$

Hence, when the swf is invariant to arbitrary, but linear and common, strictly increasing transformations of the form we say that the swf is utility-percentage invariant.
Flexible form SWF

As a consequence, if a swf satisfies utility-percentage invariance, it must also satisfy:

- Utility-level invariance, since for that we need that the strictly increasing transformations are common across individuals, i.e., \( \psi^i(\cdot) = \psi^j(\cdot) \) for any two individuals \( i \neq j \); and
- Utility-difference invariance, since for that we need that the strictly increasing transformation for each individual to be linear, i.e., \( \psi^i(u^i) = a^i + bu^i \) where \( b > 0 \).

That is, UPI is a special case of ULI and of UDI.
Flexible form SWF

UPI allows for whole class of swf, whereby the Rawlsian and utilitarian are just special cases.

Let’s start demonstrating that UPI yields homothetic social indifference curves.

Proof:
Consider the following figure.
Choose an arbitrary point $\bar{u}$. 
Reactions to Arrow’s impossibility theorem - II
Flexible form SWF

Since $W$ is strictly increasing, the social indifference curve must be negatively sloped.

Now choose a point through ray OA, i.e., $b\bar{u}$, where $b > 0$. 

Flexible form SWF

Select now another point, $\tilde{u}$, lying on the same social indifference curve, i.e., $W(\tilde{u}) = W(\bar{u})$.

- Following a similar argument as above, choose a point through ray OB, i.e., $b\tilde{u}$, where $b > 0$.
- By the UPI requirement, $W(b\tilde{u}) = W(b\bar{u})$, so points $b\tilde{u}$ and $b\bar{u}$ must lie on the same social indifference curve.
**Flexible form SWF**

- We NTS homotheticity of the social indifference curve:
  - The tangent at point $\bar{u}$ must coincide with that in point $b\bar{u}$, and
  - The tangent at point $\tilde{u}$ must coincide with that in point $b\tilde{u}$.

- The slope of chord CC approximates the slope of the tangent at $\bar{u}$, whereas
  - the slope of chord DD approximates the slope of the tangent at $b\bar{u}$.
  - (This, of course, happens when points $\bar{u}$ and $\tilde{u}$ are close.)
Reactions to Arrow’s impossibility theorem - II
Flexible form SWF

- Since points $b\bar{u}$ and $b\bar{u}$ have both been increased by the same factor $b$, the slope of chord CC coincides with that of DD.
- If we choose a point $\bar{u}$ closer and closer to $\bar{u}$, the slope of chords CC and DD still coincide,
  - but their slopes better approximates that of the tangent through each point.
- In the limit, the slope of the social indifference curve at point $\bar{u}$ coincides with that at point $b\bar{u}$, proving homotheticity.
Flexible form SWF

We have just showed that UPI yields homothetic social indifference curves.

But, what’s the effect of imposing other common assumptions on the swf in the shape of social indifference curves?

- **Anonymity**: Social indifference curves become "mirror images" above and below the 45-degree line.
- **Quasiconcavity**: Similarly as in consumer theory, this assumption on the swf implies that social indifference curves are bowed-in towards the origin.

  - Intuitively, society prefers "balanced" utility vectors to "unbalanced" ones, i.e., preference for equality.
Reactions to Arrow’s impossibility theorem - II

Quasiconcavity of the SWF

- At $u_A$ individual 2 is extremely well, relative to individual 1
- At $u_B$ individual 1 is extremely well, relative to individual 2
- At the linear combination of $u_A$ and $u_B$ society reaches a linear social welfare than the unequal utility vector $u_A$ or $u_B$ alone.

$u_C = \lambda u_A + (1 - \lambda) u_B$
Flexible form SWF

We can encompass all previous forms of swf into the following CES:

\[ W(x) = \sum_{i=1}^{l} \left[ \left( u^i(x) \right)^\rho \right]^{\frac{1}{\rho}} \]

where \( 0 < \rho < 1 \).

Hence, the constant elasticity of social substitution between the utility of any two individuals, \( \sigma \), can be expressed as

\[ \sigma = \frac{1}{1-\rho} \]

This swf satisfies three properties mentioned above (A, WP, and quasiconcavity).
Flexible form SWF

This swf also satisfies a property we discussed in EconS 501:

- **Strong separability:** The $MRS_{u^i,u^j}$ only depends on $u^i$ and $u^j$, but not on $u^k$ for any other individual $k \neq i, j$.

- In particular, $MRS_{u^i,u^j}$ of this CES swf is

\[
MRS_{u^i,u^j} = - \left( \frac{u^i}{u^j} \right)^{\rho-1}
\]
Flexible form SWF

Figures in next slide with three cases of CES swf, as parameter $\rho$ decreases:

- $\rho \to 1$ (linear social indifference curves, i.e., utilitarian swf),
- $-\infty < \rho < 1$ (curvy social indifference curves),
- $\rho \to -\infty$ (right-angel social indifference curves, i.e., Rawlsian swf).
Reactions to Arrow’s impossibility theorem - II

**CES social welfare function**

- $\rho \to 1$
  - Linear social indifference curves (Utilitarian SWF)
- $-\infty < \rho < 1$
  - Curvy social indifference curves (Cobb-Douglass type)
- $\rho \to -\infty$
  - Right-angle social indifference curves (Rawls SWF)